EXISTENCE AND UNIQUENESS OF THE UNIVERSAL W-ALGEBRA

JOSÉ M. FIGUEROA-O'FARRILL[♮]

and

EDUARDO RAMOS[♯]

Instituut voor Theoretische Fysica, Universiteit Leuven
Celestijnenlaan 200 D, B–3001 Heverlee, BELGIUM

ABSTRACT

We give a precise definition of the universal (classical) W-algebra for the W_n series and prove its existence and uniqueness. The main observation is that there is a natural reduction from W_n to W_{n-1} which allows to us define the universal W-algebra as an inverse limit. This universal W-algebra is, in a sense, the smallest W-algebra from which all W_n can be obtained by reduction. These results extend to other W-algebras obtained by reducing the Gel'fand-Dickey brackets, as well as to W-superalgebras obtained from the supersymmetric Gel'fand-Dickey brackets.

♮ e-mail: jmf@iks.kuleuven.ac.be, fgbda11@blekul11.BITNET. Address after October 1991: Physikalisches Institut der Universität Bonn, Germany.

♯ e-mail: fgbda06@blekul11.BITNET, fgbda06@cc1.kuleuven.ac.be

§1 Introduction

The concept of a universal W-algebra, by which we roughly mean a W-algebra of which W_n (for any n) is a reduction, has received a great deal of attention recently **[1]** **[2]** **[3]** **[4]** **[5]** **[6]** . It is a concept which, besides being of interest in its own right, finds applications in the matrix model approach to two-dimensional quantum gravity and string theory. Indeed the partition function of the n-matrix model can be almost uniquely specified by the W_n constraints **[7]** **[3]** . It is therefore of interest to construct an abstract W-algebra whose constraints yield the W_n constraints upon reduction and such that it is in a sense the smallest such algebra.

In this note we put forward a precise definition of what we mean by a universal W-algebra and prove its existence and uniqueness. We work with classical W-algebras as they appear in the hamiltonian treatment of the generalized KdV hierarchies. In other words, we work with Gel'fand-Dickey algebras. The idea of the construction is quite simple. If W_{n-1} were embedded in W_n we could hope to define a universal W-algebra as their inductive limit: *i.e.* , something belongs to the inductive limit if it belongs to some W_n for some n and hence for all $m \geq n$. That would be an algebra containing all the W_n and, in fact, the smallest such algebra. However the situation is not so paradisiacal and the relation between W_{n-1} and W_n is a more subtle one. In fact, we will show that W_{n-1} is a reduction of W_n in the following sense. Algebraically all this means is that there is a surjective homomorphism $W_n \to W_{n-1}$; but it also has a geometric interpretation if we think of the W_n algebras as the fundamental Poisson brackets in a certain affine space of differential operators. Then the reduction $W_n \to W_{n-1}$ corresponds to the restriction of the fundamental Poisson brackets on an affine subspace defined by second-class constraints. Then the universal W-algebra can be constructed as the inverse limit of the W_n via these reductions. In a sense, then, the universal W-algebra is the smallest W-algebra from which all the W_n can be obtained by reduction.

The plan of this letter is as follows. First we briefly review W-algebras as they appear in the context of integrable systems; that is, as Poisson brackets on certain spaces of one-dimensional differential operators. We introduce the Miura transformation, the Gel'fand-Dickey brackets and the Kupershmidt-Wilson theorem which relates them. We also set up the formalism that will allow us to make the above observations precise. This necessitates the introduction of Poisson algebras in a perhaps unusual setting. We illustrate these concepts with the Virasoro and W_3 algebras. We then make the observation that the Virasoro algebra is a reduction of W_3 and prove, using the Kupershmidt-Wilson theorem, that this persists for all n; that is, that W_{n-1} is a reduction of W_n for any n. This allows us to define the universal W-algebra as the inverse limit (see below) of the $\{W_n\}$. This is a universal object and such it is unique if it exists. We prove its existence by giving a model for it. This model, however, is not very explicit; in particular, it does not allow us to compute the algebra. Therefore we discuss the possible relation between the universal W-algebra and some of the candidates that have appeared in the literature, paying close attention, in particular, to the algebra of differential operators on the circle **[1]** . Finally we comment on the extension of these results to other W-algebras and to W-superalgebras.

§2 W_n ALGEBRAS

Let M_n denote the affine space of differential operators of the form $L = \partial^n + \sum_{j=0}^{n-1} u_j \partial^j$, where the u_j are smooth (real- or complex-valued) functions on the circle. We think of the ring $R_n \equiv R(u_0, u_1, \ldots, u_{n-1})$ of differential polynomials in the u_j as the coordinate ring of M_n. By a Poisson structure on M_n we mean a Poisson bracket on R_n induced by a Poisson bracket on the generators

$$\{u_i(x)\, ,\, u_j(y)\} = J(u_i, u_j) \cdot \delta(x-y)\ , \tag{2.1}$$

where $J(u_i, u_j)$ is a differential operator[1] with coefficients in R_n evaluated at the

[1] Strictly speaking this defines a *local* Poisson structure. Non-local structures,

point x. Thus a Poisson structure on M_n can be thought of as a bilinear map

$$J : R_n \times R_n \to R_n[\partial] \ , \tag{2.2}$$

such that the axioms of a Poisson bracket are obeyed by (2.1) . These axioms can be translated with more or less effort into properties of the map J. For instance, the antisymmetry property translates into $J(a,b) = -J(b,a)^*$ for all $a,b \in R_n$ and where $*$ is the unique anti-involution in $R_n[\partial]$ obeying $\partial^* = -\partial$ and $a^* = a$, for $a \in R_n$. Another important property, and one which has important practical applications, is the derivation property which allows us to compute J on any two elements of R_n from the knowledge of J on the generators. In terms of J, the derivation property translates into the following rules:

$$J(a',b) = \partial \circ J(a,b) \ , \qquad J(ab,c) = aJ(b,c) + bJ(a,c) \ , \tag{2.3}$$

for $a,b,c \in R_n$ and where $\circ$ denotes the composition of differential operators. We shall not need to transcribe the Jacobi identity in terms of J.

We now proceed to define a Poisson structure on R_n. Let us formally factorize a given differential operator $L = \partial^n + \sum_{j=0}^{n-1} u_j \partial^j \in M_n$ as $L = (\partial + \phi_1)(\partial + \phi_2)\cdots(\partial + \phi_n)$. Comparing the two expressions for L we find expressions for each u_i as a differential polynomial of the ϕ_j. In other words, this factorization induces an embedding of R_n in the ring $R(\phi) \equiv R(\phi_1, \phi_2, \ldots, \phi_n)$ of differential polynomials of the ϕ_j, which is known as the Miura transformation. On $R(\phi)$ we can define the following Poisson structure

$$J(\phi_i, \phi_j) = \delta_{ij}\partial \ . \tag{2.4}$$

Using the derivation property (2.3) we may compute $J(u_i, u_j)$ from the expressions

where $J(u_i, u_j)$ is allowed to be an integral operator or, more generally, a pseudodifferential operator, play an important role in integrable systems and could, in principle, be considered within this formalism. We will, in fact, have to consider such extensions as an auxiliary device when we discuss reductions.

of the u_i in terms of the ϕ_j. In general, $J(u_i, u_j) \in R(\phi)[\partial]$, but remarkably **[8]** **[9]** they actually lie in $R_n[\partial]$, thus defining a Poisson structure in R_n. This result is known as the Kupershmidt-Wilson theorem and the Poisson structure on M_n, denoted GD_n, goes by the name of Gel'fand-Dickey bracket.

The Gel'fand-Dickey bracket can also be computed explicitly from the u_i without having to resort to the Miura transformation. Let $X = \sum_{i=0}^{n-1} \partial^{-i-1} X_i$ be a pseudo-differential operator (ΨDO). Define the Adler mapping **[10]** **[11]** $A(X)$ of X as follows

$$A(X) \equiv (LX)_+ L - L(XL)_+ = L(XL)_- - (LX)_- L \ , \tag{2.5}$$

where the $+$ and $-$ subscripts denote the differential and integral parts, respectively, of a ΨDO. It is evident from its definition that it is a differential operator of order at most $n-1$ depending linearly on the X_i. Indeed, one can show that

$$A(X) = - \sum_{i,j=0}^{n-1} \left(J(u_i, u_j) \cdot X_j \right) \partial^i \ . \tag{2.6}$$

Consider now the affine subspace of M_n consisting of differential operators of the form $L = \partial^n + \sum_{j=0}^{n-2} u_j \partial^j$. It turns out that this is (formally) a symplectic subspace of M_n. In other words, this subspace can be described by the constraint $u_{n-1} = 0$ which is formally second-class. The induced Poisson structure is a classical realization of W_n and is given simply by the Dirac bracket:

$$J_D(u_i, u_j) \equiv J(u_i, u_j) - J(u_i, u_{n-1}) \circ J(u_{n-1}, u_{n-1})^{-1} \circ J(u_{n-1}, u_j) \ , \tag{2.7}$$

where everything is evaluated at $u_{n-1} = 0$. It is perhaps remarkable that despite the appearance of $J(u_{n-1}, u_{n-1})^{-1}$ the Dirac bracket is actually local.

As an example let us work out the cases $n = 2, 3$. A short calculation shows that (for $n = 2$)

$$\begin{aligned} J(u_0, u_0) =& -\partial^3 - u_1\partial^2 + \partial^2 u_1 - u_0\partial - \partial u_0 + u_1\partial u_1 \ , \\ J(u_0, u_1) =& \partial^2 + u_1\partial \ , \\ J(u_1, u_1) =& 2\partial \ ; \end{aligned} \tag{2.8}$$

whence the Dirac bracket associated to the constraint $u_1 = 0$ is given by

$$J_D(u_0, u_0) = -\frac{1}{2}\partial^3 - u_0\partial - \partial u_0 \ , \tag{2.9}$$

which is nothing but a classical realization of the Virasoro algebra. A longer calculation shows that the Dirac brackets associated to the constraint $u_2 = 0$ in the $n = 3$ case are given by

$$\begin{aligned} J_D(u_0, u_0) =& \frac{2}{3}(\partial^2 + u_1)\partial(\partial^2 + u_1) - u_0'\partial - \partial u_0' \ , \\ J_D(u_0, u_1) =& -(\partial^2 + u_1)\partial^2 - 3u_0\partial - u_0 \ , \\ J_D(u_1, u_1) =& -2\partial^3 - \partial u_1 - u_1\partial \ ; \end{aligned} \tag{2.10}$$

which is a classical realization of W_3 where the spin 3 field is given by $u_0 - \frac{1}{2}u_1'$.

We now come to our main observation. If we impose the constraint $u_0 = 0$ on the algebra (2.10) and we compute the associated Dirac bracket, we recover the Virasoro algebra (2.9) . In fact,

$$\begin{aligned} \widetilde{J}(u_1, u_1) \equiv& J_D(u_1, u_1) - J_D(u_1, u_0) \circ J_D(u_0, u_0)^{-1} \circ J_D(u_0, u_1) \\ =& -\frac{1}{2}\partial^3 - \partial u_1 - u_1\partial \ . \end{aligned} \tag{2.11}$$

Therefore we see that the Virasoro algebra is a reduction of W_3. This is not an accident. In fact, it can be easily seen from the Kupershmidt-Wilson theorem that this extends to other W_n. Consider the embedding of M_{n-1} into M_n given by

$L \mapsto L \circ \partial$. The image of this embedding is the subspace of M_n defined by the constraint $u_0 = 0$ or, in terms of the Miura transformation, $\phi_n = 0$. In terms of the ϕ_i, the constraint is clearly second-class and the Dirac bracket is the same as the old bracket except that ϕ_n never enters the picture again. And comparing the expression of the u_i in terms of the ϕ_j we see that it is the same as the Miura transformation for M_{n-1} except that the indices of the u_i are shifted down by one. Thus after relabeling the u_i, the Poisson algebra they obey is precisely the one obtained from the Kupershmidt-Wilson theorem applied to M_{n-1}—in other words, W_{n-1}. In summary, W_{n-1} is a reduction of W_n for all n. Iterating the reduction, we see that for all $m < n$, W_m is a reduction of W_n. In particular, since W_2 is the Virasoro algebra, the Virasoro algebra is a reduction of every W_n. It should be remarked that this reduction is not the restriction to the Virasoro subalgebra present in every W_n, since the central charge is different. Another remark is that the reduction from W_n down to W_{n-1} can also be obtained by embedding M_{n-1} into M_n by $L \mapsto \partial \circ L$, which, in the Miura description, corresponds to the second-class constraint $\phi_1 = 0$. It would be interesting to classify the reductions of W_n obtained from linear constraints of the ϕ_i.

How does this observation help us in defining a universal W-algebra? If W_n were a subalgebra of W_{n+1} we could then hope to define a universal algebra containing all W_n as the inductive limit $\varinjlim_n W_n$. However this is not the case. Nevertheless, the next best thing occurs: the fact that W_{n-1} is a reduction of W_n does allow us to define their inverse limit or colimit. The resulting algebra would then have the property that all W_n could be obtained from it by reduction. But in order to make these ideas precise we first need to introduce a few notions concerning limits of Poisson algebras as we have defined them.

§3 Some abstract nonsense

The notion of inverse limit or colimit is a useful notion in category theory and we refer the interested reader to the book **[12]** by Lang for a more general exposition than the one presented here. We will introduce only those concepts we need in as much (lack of) generality as necessary. But before talking about inverse limits we need to set up some formalism.

The basic objects we are dealing with are essentially Poisson algebras, but perhaps in a slightly eccentric guise. Let R denote some differential ring: that is, a commutative ring with unit together with a derivation $\partial : R \to R$ denoted $r \mapsto r'$. We will let $R[\partial]$ denote the ring of differential operators with coefficients in R, with multiplication defined by the usual Leibnitz rule $\partial \circ r = r' + r \circ \partial$. The natural maps (morphisms) between differential rings are ring homomorphisms preserving the identity and commuting with the derivation. Every such map $\varphi : R \to S$ between differential rings induces a map—also denoted φ—between differential operators $\varphi : R[\partial] \to S[\partial]$ by acting on the coefficients. Now by a Poisson structure on a differential ring R we mean a bilinear map $J : R \times R \to R[\partial]$ obeying the properties necessary for the bracket defined as in (2.1) to be a Poisson bracket. In particular, this implies the antisymmetry and derivation properties mentioned in the previous section. We call the pair (R, J) a Poisson algebra; although when no confusion can result, the Poisson structure will simply be omitted. Morphisms between Poisson algebras are morphisms of differential rings which preserve the Poisson structures. In other words, if (R, J) and (S, K) are Poisson algebras, a differential ring morphism $\varphi : R \to S$ is a Poisson algebra morphism if $\varphi(J(a, b)) = K(\varphi(a), \varphi(b))$, for all $a, b \in R$.

We have already seen a few examples of Poisson algebras and of morphisms between them. Indeed the content of the Kupershmidt-Wilson theorem is that the Miura transformation $R_n \to R(\phi)$ is a Poisson (mono)morphism. Another example of Poisson morphism—albeit perhaps not so obvious as the Miura transformation—is the reduction from GD_n to W_n.

Let $\widetilde{M}_n$ denote the subspace of M_n defined by the constraint $u_{n-1} = 0$. If I denotes the differential ideal generated by u_{n-1}, then the coordinate ring of $\widetilde{M}_n$ is just R_n/I which is naturally isomorphic to $R(u_0, u_1, \ldots, u_{n-2})$. On this differential ring we have W_n as a Poisson structure. We would like to exhibit the reduction $GD_n \to W_n$ as induced by a Poisson (epi)morphism $R_n \to R_n/I$. However the natural surjection $R_n \to R_n/I$, sending r to its class modulo I, is not a Poisson morphism with respect to the Gel'fand-Dickey bracket: the reason being that I is not a Poisson ideal of R_n and thus the quotient does not inherit a Poisson structure. In fact, one way to think of the Dirac bracket is that it is the unique Poisson structure on R_n making the natural surjection $R_n \to R_n/I$ into a Poisson morphism. But we are interested in keeping the original Poisson structure on R_n and thus we must come up with a Poisson morphism $R_n \to R_n/I$ intertwining between GD_n and W_n.

To motivate the construction let us look at the familiar case of the hamiltonian reduction of a symplectic manifold M by a set of (regular) second-class constraints $\{\chi_i\}$. Since the constraints are second-class, the matrix $\Omega_{ij} = \{\chi_i\,,\,\chi_j\}$ of their Poisson brackets is non-degenerate when restricted to the zero locus M_o of the $\{\chi_i\}$. Let us denote its inverse by Ω^{ij}. The ring $C^\infty(M_o)$ of smooth functions on M_o then inherits a Poisson structure as follows. Given $f, g \in C^\infty(M_o)$, we extend them to smooth functions (also denoted f, g) on M. Then their Poisson bracket is defined by the Dirac formula

$$\{f\,,\,g\}_o \equiv \{f\,,\,g\} - \sum_{ij} \{f\,,\,\chi_i\}\Omega^{ij}\{\chi_j\,,\,g\}\ . \tag{3.1}$$

This is clearly independent of the extension since the difference between any two extensions is of the form $\sum c_i\chi_i$, for c_i an arbitrary function and on M_o, these functions have vanishing Dirac bracket with any function. Therefore the Dirac bracket induces a Poisson structure on the ring $C^\infty(M)/I \cong C^\infty(M_o)$, where I is the ideal generated by the constraints or, equivalently, the ideal of functions vanishing on M_o. Alternatively we consider the following equivalent construction.

Let $N(I)$ denote the normalizer of I in $C^\infty(M)$; that is, $f \in N(I)$ if $\{f\,,\,I\} \subset I$. Any function $f \in C^\infty(M)$ can be decomposed as $f = f_N + f_I$, where $f_N \in N(I)$ and $f_I \in I$. Of course, the decomposition is not unique since there are functions that lie both in I and in $N(I)$. In fact, $\sum_{ij} c_{ij}\chi_i\chi_j$, for arbitrary functions c_{ij}, lies both in I and in $N(I)$. These functions constitute the ideal I^2 and it is not hard to show that $N(I) \cap I = I^2$. A possible decomposition of a function f is given by

$$f_I = \Omega^{ij}\{f\,,\,\chi_i\}\chi_j\ , \qquad f_N = f - f_I\ . \tag{3.2}$$

One can check that $f_N \in N(I)$. A straightforward calculation now shows that the Poisson bracket $\{f_N\,,\,g_N\}$ agrees on M_o with the Poisson bracket $\{f\,,\,g\}_o$. Therefore we have exhibited the reduction from M to M_o as a Poisson morphism $C^\infty(M) \to C^\infty(M_o)$.

We now do the same for our more formal situation. Since the constraint u_{n-1} is second-class, the ideal I it generates is such that R_n admits a decomposition $R_n = N(I) + I$, where now $N(I)$ is defined as follows: $r \in N(I)$ if and only if $J(r, I) \subset I[\partial]$. As before the sum is not direct: the intersection is again $N(I) \cap I = I^2$. This sets up an isomorphism of differential rings $R_n/I \cong N(I)/I^2$, but $N(I)/I^2$ has in addition a natural Poisson structure since $N(I)$ is a Poisson subalgebra of R_n containing I^2 as a Poisson ideal. Thus the Poisson morphism we are after is the composition of the natural maps:

$$R_n \cong N(I) + I \to N(I)/I^2 \cong R_n/I\ . \tag{3.3}$$

We can see this a bit more explicitly in terms of generators. R_n is generated by $u_0, u_1, \ldots, u_{n-1}$ whereas I is generated by u_{n-1}. It does not follow, however that $N(I)$ is generated by $u_0, u_1, \ldots, u_{n-2}$ but almost. One can define generators $\widetilde{u}_i = u_i + \cdots$, for $i = 0, 1, \ldots, n-2$ such that $\widetilde{u}_i \in N(I)$ and hence generate it. Of course, the $\widetilde{u}_i$ are not unique, since one can always add terms in I^2. But having chosen one such set of $\widetilde{u}_i$, the map $\rho : R_n \to R_n/I$ is simply given by $\rho(\widetilde{u}_i) = u_i \bmod I$ for $i = 0, 1, \ldots, n-2$ and $\rho(u_{n-1}) = 0$.

As an example consider the reduction from GD_2 to the Virasoro algebra. We impose the constraint $u_1 = 0$. Now, as can be seen in (2.8) , u_0 does not lie in the normalizer of the ideal generated by u_1. However, $\widetilde{u}_0 = u_0 - \frac{1}{2}u_1'$ obeys $J(\widetilde{u}_0, u_1) = u_1\partial$, and hence generates the normalizer. Moreover, $J(\widetilde{u}_0, \widetilde{u}_0) = J_D(u_0, u_0) \bmod I[\partial]$, so that they agree on the subspace $u_1 = 0$.

In fact, one does not need to be clever to find the $\widetilde{u}_i$. Notice that, in this example,

$$\widetilde{u}_0 = u_0 - J(u_0, u_1) \circ J(u_1, u_1)^{-1} \cdot u_1 \bmod I^2 , \tag{3.4}$$

which is to be compared with the definition of f_N above. In general, to reduce GD_n down to W_n, we define, for $i = 0, 1, \ldots, n-2$,

$$\widetilde{u}_i = u_i - J(u_i, u_{n-1}) \circ J(u_{n-1}, u_{n-1})^{-1} \cdot u_{n-1} \bmod I^2 . \tag{3.5}$$

One can then show that $J(u_i, u_{n-1}) \circ J(u_{n-1}, u_{n-1})^{-1}$ is a differential operator and, using (2.3) , that $J(\widetilde{u}_i, u_{n-1}) \in I[\partial]$, so that $\widetilde{u}_i \in N(I)$. Furthermore, computing the Gel'fand-Dickey brackets of the $\widetilde{u}_i$ one finds precisely the Dirac brackets of the u_i:

$$J(\widetilde{u}_i, \widetilde{u}_j) = J_D(u_i, u_j) \bmod I[\partial] . \tag{3.6}$$

§4 The universal W-algebra as an inverse limit

The reduction $W_n \to W_{n-1}$ goes along similar lines, except that in order to describe the normalizer of the ideal generated by the constraint $u_0 = 0$ we have to extend the formalism slightly to encompass not just differential polynomials of the u_i but also more formal objects obtained from them by the action of ∂^{-1}. Let us illustrate this with an example: the reduction of W_3 down to Virasoro. Analogously to the reduction $GD_n \to W_n$, we define $\widetilde{u}_1 = u_1 - J_D(u_1, u_0) \circ J_D(u_0, u_0)^{-1} \cdot u_0 \bmod I^2$. Only that now, $J_D(u_1, u_0) \circ J_D(u_0, u_0)^{-1}$ is no longer a differential operator (even putting $u_0 = 0$) so that $\widetilde{u}_1$ is not a a differential polynomial of the u_i; in fact, on $u_0 = 0$, $J_D(u_1, u_0) \circ J_D(u_0, u_0)^{-1} = \frac{3}{2}(\partial^2 + u_1)^{-1}\partial$.

Thus we need to extend the differential ring R_n in such a way that the action of ∂^{-1} is defined. This involves introducing symbols $\partial^{-1} \cdot u_i, \partial^{-2} \cdot u_i, \ldots$ obeying $(\partial^{-1} \cdot u_i)' = \partial^{-1} \cdot u_i' = u_i, \ldots$ The operator ∂^{-1} is not a derivation but obeys the following rule:

$$\partial^{-1} \cdot (ab) = \sum_{j=0}^{\infty} (-1)^j a^{(j)} \partial^{-j-1} \cdot b \ , \tag{4.1}$$

whence we can define ∂^{-1} everywhere from a knowledge of ∂^{-1} on the generators. Notice, by the way, that this necessitates introducing formal infinite sums of monomials. We will still denote the extended ring by R_n and we will let $R_n((\partial^{-1}))$ denote the ring of pseudodifferential operators with coefficients on R_n. Then $R_n((\partial^{-1}))$ acts naturally on R_n. We now extend the Poisson bracket to the extended ring R_n as a bilinear map

$$J : R_n \times R_n \to R_n((\partial^{-1})) \ . \tag{4.2}$$

We have to extend the derivation property to compute $J(\partial^{-1} \cdot a, b)$ from $J(a, b)$ so that, again, it is sufficient to know J on generators. Since $(\partial^{-1} \cdot a)' = a$, using (2.3) we find that $J(\partial^{-1} \cdot a, b) = \partial^{-1} \circ J(a, b)$. The antisymmetry property is the same, where $*$ is now the unique extension of $*$ to an anti-involution on the extended ring.

Now we look at the reduction $GD_n \to GD_{n-1}$ obtained by putting $u_0 = 0$. Let I denote the pseudodifferential ideal generated by u_0. For $i = 1, 2, \ldots, n-1$ we define $\widetilde{u}_i = u_i - K_i \cdot u_0$, where $K_i \in R_n((\partial^{-1}))$ is defined so that $J(\widetilde{u}_i, u_0) \in I((\partial^{-1}))$. This fixes $K_i = J(u_i, u_0) \circ J(u_0, u_0)^{-1} \bmod I((\partial^{-1}))$. Thus,

$$\widetilde{u}_i = u_i - J(u_i, u_0) \circ J(u_0, u_0)^{-1} \cdot u_0 \bmod I^2 \ . \tag{4.3}$$

A straightforward calculation now shows that

$$J(\widetilde{u}_i, \widetilde{u}_j) = J(u_i, u_j) - J(u_i, u_0) \circ J(u_0, u_0)^{-1} \circ J(u_0, u_j) \bmod I((\partial^{-1})) \ . \tag{4.4}$$

We can then define morphisms $\rho_n : R_n \to R_{n-1}$ by $\rho_n(\widetilde{u}_i) = u_{i-1}$, $\rho_n(u_0) = 0$ which induce Poisson morphisms $GD_n \to GD_{n-1}$. More invariantly, these morphisms can be defined as the composition of the following natural maps

$$R_n \cong N(I) + I \to N(I)/I^2 \cong R_n/I \cong R_{n-1} \ . \tag{4.5}$$

We will also denote by $\rho_n : W_n \to W_{n-1}$ the Poisson morphism induced by the above map under the $GD \to W$ reduction.

We have therefore a set of Poisson algebras $\{W_n\}$ and Poisson morphisms $\rho_n : W_n \to W_{n-1}$. This is called an inverse system of Poisson algebras and one can define its colimit or inverse limit $\varprojlim_n W_n$ as follows. A colimit $\varprojlim_n W_n$ consists of a Poisson algebra U together with Poisson morphisms $\pi_n : U \to W_n$ obeying $\rho_n \circ \pi_n = \pi_{n-1}$, which enjoy the following (universal) property: given any other Poisson algebra P and morphisms $\mu_n : P \to W_n$ obeying $\rho_n \circ \mu_n = \mu_{n-1}$ there exists a *unique* Poisson morphism $\varphi : P \to U$ such that for all n

$$\mu_n = \pi_n \circ \varphi \ ; \tag{4.6}$$

in other words, the maps μ_n all factor through U.

As usual with universal objects, if they exist they are unique up to a unique isomorphism. In fact, suppose that $(\widetilde{U}, \{\widetilde{\pi}_n\})$ is another such colimit. Then by the universal property of U there is a unique morphism $\widetilde{U} \to U$ satisfying (4.6) , and by the universal property of $\widetilde{U}$ there is a unique such morphism $U \to \widetilde{U}$ as well. Now, composing them we get two morphisms $U \to U$ and $\widetilde{U} \to \widetilde{U}$ which are unique. But certainly the identity maps also satisfy the above properties, hence, by uniqueness, the compositions $U \to \widetilde{U} \to U$ and $\widetilde{U} \to U \to \widetilde{U}$ are precisely the identity maps. In other words, $U \cong \widetilde{U}$ for a unique isomorphism.

We still have to show that the colimit exists. Since Poisson algebras are pseudodifferential rings and Poisson morphisms are, in particular, morphisms of pseudodifferential rings, we can appeal to the construction of a model for the colimit in the category of pseudodifferential rings and just check that the extra structure (*i.e.* , the Poisson structure) is preserved.

Let $R \equiv \prod_n R_n$ denote the product of the pseudodifferential rings R_n. That is, R consists of all sequences (r_n) with $r_n \in R_n$. R becomes a pseudodifferential ring in a natural way by simply defining the operations entry-wise: $(r_n)' = (r'_n)$, $\partial^{-1} \cdot (r_n) = (\partial^{-1} \cdot r_n)$, and $(r_n)(s_n) = (r_n s_n)$. Let U denote the subset of R defined by

$$U = \{(u_n) \in R \mid u_{n-1} = \rho_n(u_n)\} \ . \tag{4.7}$$

It is easy to check that U is a pseudodifferential subring of R and that it has the universal property of the colimit if we define $\pi_n : U \to R_n$ as the composition $U \to R \to R_n$ where the first map is the inclusion and the second is the natural projection onto the n^{th} factor. In fact, if S is another pseudodifferential ring with morphisms $\sigma_n : S \to R_n$, then the map $S \to U$ is given by $s \mapsto (\sigma_n(s))$ which is easily checked to obey the conditions (4.6) . If we now turn on the Poisson structure on each R_n, then R becomes a Poisson algebra defining the bracket entry-wise, U becomes a Poisson subalgebra, and all maps in sight are Poisson morphisms. Thus U becomes a colimit in the category of Poisson algebras.

Summarizing, we have shown the existence and uniqueness of a universal W-algebra for the $\{W_n\}$ series. The universal property was defined above and shall not be repeated, but we mention some consequences. First of all, since the maps $\rho_n : W_n \to W_{n-1}$ are surjective, so are the maps $\pi_n : U \to W_n$, whence we can think of them as reductions which, due to the fact that $\rho_n \circ \pi_n = \pi_{n-1}$, are compatible with the reductions $W_n \to W_{n-1}$ and, in a sense, extend them. In other words, it does not matter how we reduce U down to W_n. We can either first reduce U to W_m (for $m > n$) and then reduce $W_m \to W_n$ or reduce U to W_n directly. Suppose now that someone hands us a Poisson algebra P such that all W_n are obtained from it by (compatible) reductions; *i.e.* , there are Poisson epimorphisms $\mu_n : P \to W_n$ obeying $\rho_n \circ \mu_n = \mu_{n-1}$. Then these reductions factor through U in the sense that we can first map P to U and then reduce U to the W_n. Loosely speaking then, U is the smallest W-algebra which yields all W_n by reduction.

§5 In search of an explicit model

The model given above for the universal W-algebra suffices for a proof of existence and uniqueness, but does not really give us an algorithm to compute this algebra explicitly. From a constructive point of view, the above model is therefore not very satisfactory. It thus behooves us to elucidate the possible relations between some of the candidates for "universal" W-algebras that have appeared in the literature and the universal W-algebra described here. Since the only thing we know about our algebra is the universal property it obeys, it seems that the way to proceed is to probe the candidates to see if they obey this property—hence concluding that they are isomorphic to our algebra—or, on the contrary, that they are not universal. Two classes of candidates of which we will not be able to say much are the following: several versions of W_∞-type algebras **[2]** **[13]** ; and algebras which appear as a hamiltonian structure for the KP hierarchy **[14]** **[5]** **[6]** **[15]** . Both of these proposals make sense heuristically. Since the W_n algebras are realized as the second hamiltonian structure of the n^{th} order KdV hierarchy and this hierarchy is a reduction of KP, it seems reasonable to expect that the second hamiltonian structure of the KP hierarchy should yield all the W_n algebras upon reduction. We have tried to prove this reduction explicitly, but so far without success. As for the W_∞-type algebras, the rationale seems to be the following: for any fixed n, W_n is the smallest algebra from which all the W_m for $m \leq n$ can be obtained by reduction. It thus seems reasonable that the model for the universal W-algebra should be the limit $n \to \infty$ of W_n. The question is how to make this limit precise. In this paper we have given one such possibility that seems to be the natural one, since we know of no other relationship between the W_n algebras but the fact that one can get W_{n-1} from W_n by reduction. It is a very interesting open problem to see if any one of the W_∞-type algebras in the literature is indeed universal. This seems to be supported by the experimental fact **[4]** that, at least for a particular value ($c = -2$) of the central charge, W_n can be obtained as a reduction of W_∞. As a side remark, notice also that the first hamiltonian structure for the KP hierarchy is, in fact, $W_{1+\infty}$ **[16]** . But since it is the second structure for the KdV hierarchies that corresponds to W_n, we are not sure if this is not just a macabre coincidence.

The one proposal for universal W-algebra that we will look at in some detail is the algebra $\mathcal{D}$ of differential operators on the circle, put forward by Radul **[1]** . It should be remarked, however, that $\mathcal{D}$ is not a Poisson algebra but only a Lie algebra. Eventually, of course, one is interested in quantizing W algebras. Since quantization does not respect the Poisson structure but just the Lie structure, this does not represent a drawback.

To understand the maps $\mathcal{D} \to GD_n$ we have to introduce some minor notation. We refer the reader to the forthcoming book by Dickey **[17]** for a more detailed discussion, or to our paper **[18]** for a less brief summary. Given $L \in M_n$, the tangent space $T_L M_n$ to M_n at L is naturally identified with the differential operators of order at most $n-1$. Similarly the cotangent space $T^*_L M_n$ is naturally identified with the space $\mathcal{P}_-/\partial^{-n}\mathcal{P}_-$, where $\mathcal{P}$ is the ring of pseudodifferential operators and $\mathcal{P}_-$ the subring of integral operators. The pairing between $T_L M_n$ and $T^*_L M_n$ is given by the Adler trace **[10]** . The Adler map (2.5) is a linear map $T^*_L M_n \to T_L M_n$. The Lie bracket of vector fields gives a Lie algebra structure to $T_L M_n$ relative to which the image of the Adler map is a subalgebra. Thus one can pull back this structure to give $T^*_L M_n$ a Lie algebra structure. Explicitly, if $X, Y \in T^*_L M_n$, their Lie bracket is given by

$$[\![X, Y]\!] \equiv \partial_{A(X)} Y + X(LY)_- - ((XL)_+ Y)_- - (X \leftrightarrow Y) , \tag{5.1}$$

where $\partial_{A(X)}$ represents the Lie derivative of the vector field defined by $A(X) \in T_L M_n$. One can check that

$$[A(X) , A(Y)] = A([\![X, Y]\!]) . \tag{5.2}$$

This Lie bracket $[\![,]\!]$ is essentially the Poisson bracket defined by the Adler map; indeed for G, H functions on M_n, it follows that $d\{G , H\} = [\![dG, dH]\!]$.

Now Radul defines a map $\pi_n : \mathcal{D} \to T^*_L M_n$ by $\pi_n(E) = -(EL^{-1})_-$. One can easily show that π_n is a homomorphism of Lie algebras, where $\mathcal{D}$ is given a Lie algebra structure via the commutator. In other words,

$$[\![\pi_n(E), \pi_n(F)]\!] = \pi_n([E\,,\,F])\ . \tag{5.3}$$

Therefore this induces a map $\mathcal{D} \to GD_n$, which we also denote π_n. It is interesting to notice that since $\pi_n((EL^{-1})_+L) = 0$, only $E-(EL^{-1})_+L = (EL^{-1})_-L$ matters, which is a differential operator of order $n-1$.

However, it is easy to see that the maps $\pi_n : \mathcal{D} \to GD_n$ are incompatible with the reduction $GD_n \to GD_{n-1}$. To see this notice simply that the reduction $GD_n \to GD_{n-1}$ is induced by the embedding $M_{n-1} \to M_n$ given by $L \mapsto L\partial$. Now, there is a natural embedding $T^*_{L\partial}M_{n-1} \subset T^*_{L\partial}M_n$ defined as those 1-forms whose image under the Adler map are tangent vectors to M_{n-1}. Then these 1-forms inherit a Lie bracket $[\![,]\!]$ simply from the one $T^*_{L\partial}M_n$. The reduction is compatible if for $L\partial \in M_n$, the image of the map π_n lies in $T^*_{L\partial}M_{n-1}$; in other words, if $A(\pi_n(E)) \in T_{L\partial}M_n$. Now $T_{L\partial}M_n$ corresponds to the infinitesimal deformations of $L\partial$, which in this case are nothing but the differential operators of order at most $n-1$ without free term. An explicit computation shows that $A(\pi_n(E)) = (L\partial E\partial^{-1}L^{-1})_-L\partial$ which can be rewritten as $L\partial E - (L\partial E\partial^{-1}L^{-1})_+L\partial$ which, since E has in general a free term, is not tangent to $M_{n-1} \subset M_n$.

This discards $(\mathcal{D}, \pi_n)$ as a model for the universal W-algebra.

§6 Some extensions of our results

Finally we mention two extensions of our results. Notice that the universal W-algebra for $\{W_n\}$ described here is a reduction of the universal Gel'fand-Dickey algebra. There are other W-algebras besides W_n which arise as reductions of GD_n. In particular, for $n = 2\ell + 1$ odd, one can construct a W-algebra as the hamiltonian reduction of the Gel'fand-Dickey bracket with respect to the second-class constraint $L^* = -L$. This subspace of antisymmetric Lax operators is a (formally) symplectic submanifold of $M_{2\ell+1}$ and the Dirac brackets induce the W-algebra[2] $DS(B_\ell)$ associated à la Drinfel'd-Sokolov [**20**] to the simple Lie algebra B_ℓ. Similarly for $n = 2\ell$ even, the subspace defined by the second-class constraint $L^* = L$ inherits a Poisson structure which defines the W-algebra $DS(C_\ell)$ associated to C_ℓ. The reduction $GD_n \to GD_{n-2}$ induces a reduction $DS(B_\ell) \to DS(B_{\ell+1})$ for n odd, and $DS(C_\ell) \to DS(C_{\ell+1})$ for n even, giving rise to universal W-algebras as the corresponding inverse limits.

Another straightforward extension of these results is the existence and uniqueness of universal W-superalgebras for the W-superalgebras obtained from the supersymmetric Gel'fand-Dickey brackets $\{SGD_n\}$ constructed in [**18**] . In particular, the infinite series of $N = 1$ [**21**] and $N = 2$ [**22**] W-superalgebras obtained as reductions of the supersymmetric Gel'fand-Dickey brackets give rise, upon taking the relevant inverse limits, to universal W-superalgebras for those series. The proof of the existence and uniqueness of these universal W-superalgebras is a straightforward supersymmetrization of the proof given here: the reduction $SGD_n \to SGD_{n-1}$ being proven using the supersymmetric Kupershmidt-Wilson theorem of [**18**] .

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[2] It should be remarked that these algebras are not the WB_ℓ algebras [**19**] . The same comment holds in the C_ℓ case to be discussed below.

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