EXISTENCE AND UNIQUENESS
OF THE UNIVERSAL $W$-ALGEBRA

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ABSTRACT

We give a precise definition of the universal (classical) $W$-algebra for the $W_n$ series and prove its existence and uniqueness. The main observation is that there is a natural reduction from $W_n$ to $W_{n-1}$ which allows us to define the universal $W$-algebra as an inverse limit. This universal $W$-algebra is, in a sense, the smallest $W$-algebra from which all $W_n$ can be obtained by reduction. These results extend to other $W$-algebras obtained by reducing the Gel’fand-Dickey brackets, as well as to $W$-superalgebras obtained from the supersymmetric Gel’fand-Dickey brackets.

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§1 INTRODUCTION

The concept of a universal $W$-algebra, by which we roughly mean a $W$-algebra of which $W_n$ (for any $n$) is a reduction, has received a great deal of attention recently [1] [2] [3] [4] [5] [6]. It is a concept which, besides being of interest in its own right, finds applications in the matrix model approach to two-dimensional quantum gravity and string theory. Indeed the partition function of the $n$-matrix model can be almost uniquely specified by the $W_n$ constraints [7] [3]. It is therefore of interest to construct an abstract $W$-algebra whose constraints yield the $W_n$ constraints upon reduction and such that it is in a sense the smallest such algebra.

In this note we put forward a precise definition of what we mean by a universal $W$-algebra and prove its existence and uniqueness. We work with classical $W$-algebras as they appear in the hamiltonian treatment of the generalized KdV hierarchies. In other words, we work with Gel’fand-Dickey algebras. The idea of the construction is quite simple. If $W_{n-1}$ were embedded in $W_n$ we could hope to define a universal $W$-algebra as their inductive limit: i.e., something belongs to the inductive limit if it belongs to some $W_n$ for some $n$ and hence for all $m \geq n$. That would be an algebra containing all the $W_n$ and, in fact, the smallest such algebra. However the situation is not so paradisiacal and the relation between $W_{n-1}$ and $W_n$ is a more subtle one. In fact, we will show that $W_{n-1}$ is a reduction of $W_n$ in the following sense. Algebraically all this means is that there is a surjective homomorphism $W_n \rightarrow W_{n-1}$; but it also has a geometric interpretation if we think of the $W_n$ algebras as the fundamental Poisson brackets in a certain affine space of differential operators. Then the reduction $W_n \rightarrow W_{n-1}$ corresponds to the restriction of the fundamental Poisson brackets on an affine subspace defined by second-class constraints. Then the universal $W$-algebra can be constructed as the inverse limit of the $W_n$ via these reductions. In a sense, then, the universal $W$-algebra is the smallest $W$-algebra from which all the $W_n$ can be obtained by reduction.

The plan of this letter is as follows. First we briefly review $W$-algebras as they
appear in the context of integrable systems; that is, as Poisson brackets on certain spaces of one-dimensional differential operators. We introduce the Miura transformation, the Gel’fand-Dickey brackets and the Kupershmidt-Wilson theorem which relates them. We also set up the formalism that will allow us to make the above observations precise. This necessitates the introduction of Poisson algebras in a perhaps unusual setting. We illustrate these concepts with the Virasoro and \( W_3 \) algebras. We then make the observation that the Virasoro algebra is a reduction of \( W_3 \) and prove, using the Kupershmidt-Wilson theorem, that this persists for all \( n \); that is, that \( W_{n-1} \) is a reduction of \( W_n \) for any \( n \). This allows us to define the universal \( W \)-algebra as the inverse limit (see below) of the \( \{ W_n \} \). This is a universal object and such it is unique if it exists. We prove its existence by giving a model for it. This model, however, is not very explicit; in particular, it does not allow us to compute the algebra. Therefore we discuss the possible relation between the universal \( W \)-algebra and some of the candidates that have appeared in the literature, paying close attention, in particular, to the algebra of differential operators on the circle [1]. Finally we comment on the extension of these results to other \( W \)-algebras and to \( W \)-superalgebras.

\[ \text{§2} \quad W_n \text{ ALGEBRAS} \]

Let \( M_n \) denote the affine space of differential operators of the form \( L = \partial^n + \sum_{j=0}^{n-1} u_j \partial^j \), where the \( u_j \) are smooth (real- or complex-valued) functions on the circle. We think of the ring \( R_n \equiv R(u_0, u_1, \ldots, u_{n-1}) \) of differential polynomials in the \( u_j \) as the coordinate ring of \( M_n \). By a Poisson structure on \( M_n \) we mean a Poisson bracket on \( R_n \) induced by a Poisson bracket on the generators

\[
\{ u_i(x), u_j(y) \} = J(u_i, u_j) \cdot \delta(x - y),
\]

where \( J(u_i, u_j) \) is a differential operator\(^1\) with coefficients in \( R_n \) evaluated at the

\[^{1}\text{Strictly speaking this defines a local Poisson structure. Non-local structures,}

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point $x$. Thus a Poisson structure on $M_n$ can be thought of as a bilinear map

$$ J : R_n \times R_n \rightarrow R_n[\partial] , $$

(2.2) such that the axioms of a Poisson bracket are obeyed by (2.1). These axioms can be translated with more or less effort into properties of the map $J$. For instance, the antisymmetry property translates into $J(a, b) = -J(b, a)^*$ for all $a, b \in R_n$ and where $*$ is the unique anti-involution in $R_n[\partial]$ obeying $\partial^* = -\partial$ and $a^* = a$, for $a \in R_n$. Another important property, and one which has important practical applications, is the derivation property which allows us to compute $J$ on any two elements of $R_n$ from the knowledge of $J$ on the generators. In terms of $J$, the derivation property translates into the following rules:

$$ J(a', b) = \partial \circ J(a, b) , \quad J(ab, c) = aJ(b, c) + bJ(a, c) , $$

(2.3) for $a, b, c \in R_n$ and where $\circ$ denotes the composition of differential operators. We shall not need to transcribe the Jacobi identity in terms of $J$.

We now proceed to define a Poisson structure on $R_n$. Let us formally factorize a given differential operator $L = \partial^n + \sum_{j=0}^{n-1} u_j \partial^j \in M_n$ as $L = (\partial + \phi_1)(\partial + \phi_2) \cdots (\partial + \phi_n)$. Comparing the two expressions for $L$ we find expressions for each $u_i$ as a differential polynomial of the $\phi_j$. In other words, this factorization induces an embedding of $R_n$ in the ring $R(\phi) \equiv R(\phi_1, \phi_2, \ldots, \phi_n)$ of differential polynomials of the $\phi_j$, which is known as the Miura transformation. On $R(\phi)$ we can define the following Poisson structure

$$ J(\phi_i, \phi_j) = \delta_{ij} \partial . $$

(2.4) Using the derivation property (2.3) we may compute $J(u_i, u_j)$ from the expressions where $J(u_i, u_j)$ is allowed to be an integral operator or, more generally, a pseudodifferential operator, play an important role in integrable systems and could, in principle, be considered within this formalism. We will, in fact, have to consider such extensions as an auxiliary device when we discuss reductions.
of the $u_i$ in terms of the $\phi_j$. In general, $J(u_i, u_j) \in R(\phi)[\partial]$, but remarkably [8] [9] they actually lie in $R_n[\partial]$, thus defining a Poisson structure in $R_n$. This result is known as the Kupershmidt-Wilson theorem and the Poisson structure on $M_n$, denoted $GD_n$, goes by the name of Gel’fand-Dickey bracket.

The Gel’fand-Dickey bracket can also be computed explicitly from the $u_i$ without having to resort to the Miura transformation. Let $X = \sum_{i=0}^{n-1} \partial^{-i-1}X_i$ be a pseudo-differential operator ($\Psi$DO). Define the Adler mapping [10] [11] $A(X)$ of $X$ as follows

$$A(X) \equiv (LX)_+L - L(XL)_+ = L(XL) - (LX)L,$$ (2.5)

where the + and − subscripts denote the differential and integral parts, respectively, of a $\Psi$DO. It is evident from its definition that it is a differential operator of order at most $n - 1$ depending linearly on the $X_i$. Indeed, one can show that

$$A(X) = - \sum_{i,j=0}^{n-1} (J(u_i, u_j) \cdot X_j) \partial^i.$$ (2.6)

Consider now the affine subspace of $M_n$ consisting of differential operators of the form $L = \partial^n + \sum_{j=0}^{n-2} u_j \partial^j$. It turns out that this is (formally) a symplectic subspace of $M_n$. In other words, this subspace can be described by the constraint $u_{n-1} = 0$ which is formally second-class. The induced Poisson structure is a classical realization of $W_n$ and is given simply by the Dirac bracket:

$$J_D(u_i, u_j) \equiv J(u_i, u_j) - J(u_i, u_{n-1}) \circ J(u_{n-1}, u_{n-1})^{-1} \circ J(u_{n-1}, u_j),$$ (2.7)

where everything is evaluated at $u_{n-1} = 0$. It is perhaps remarkable that despite the appearance of $J(u_{n-1}, u_{n-1})^{-1}$ the Dirac bracket is actually local.
As an example let us work out the cases \( n = 2, 3 \). A short calculation shows that (for \( n = 2 \))

\[
\begin{align*}
J(u_0, u_0) &= -\partial^3 - u_1 \partial^2 + \partial^2 u_1 - u_0 \partial - \partial u_0 + u_1 \partial u_1, \\
J(u_0, u_1) &= \partial^2 + u_1 \partial, \\
J(u_1, u_1) &= 2 \partial;
\end{align*}
\]

whence the Dirac bracket associated to the constraint \( u_1 = 0 \) is given by

\[
J_D(u_0, u_0) = -\frac{1}{2} \partial^3 - u_0 \partial - \partial u_0,
\]

which is nothing but a classical realization of the Virasoro algebra. A longer calculation shows that the Dirac brackets associated to the constraint \( u_2 = 0 \) in the \( n = 3 \) case are given by

\[
\begin{align*}
J_D(u_0, u_0) &= \frac{2}{3} (\partial^2 + u_1) \partial (\partial^2 + u_1) - u'_0 \partial - \partial u'_0, \\
J_D(u_0, u_1) &= - (\partial^2 + u_1) \partial^2 - 3 u_0 \partial - u_0, \\
J_D(u_1, u_1) &= - 2 \partial^3 - \partial u_1 - u_1 \partial;
\end{align*}
\]

which is a classical realization of \( W_3 \) where the spin 3 field is given by \( u_0 - \frac{1}{2} u'_1 \).

We now come to our main observation. If we impose the constraint \( u_0 = 0 \) on the algebra (2.10) and we compute the associated Dirac bracket, we recover the Virasoro algebra (2.9). In fact,

\[
\bar{J}(u_1, u_1) \equiv J_D(u_1, u_1) - J_D(u_1, u_0) \circ J_D(u_0, u_0)^{-1} \circ J_D(u_0, u_1) = -\frac{1}{2} \partial^3 - \partial u_1 - u_1 \partial.
\]

Therefore we see that the Virasoro algebra is a reduction of \( W_3 \). This is not an accident. In fact, it can be easily seen from the Kupershmidt-Wilson theorem that this extends to other \( W_n \). Consider the embedding of \( M_{n-1} \) into \( M_n \) given by
$L \mapsto L \circ \partial$. The image of this embedding is the subspace of $M_n$ defined by the constraint $u_0 = 0$ or, in terms of the Miura transformation, $\phi_n = 0$. In terms of the $\phi_i$, the constraint is clearly second-class and the Dirac bracket is the same as the old bracket except that $\phi_n$ never enters the picture again. And comparing the expression of the $u_i$ in terms of the $\phi_j$ we see that it is the same as the Miura transformation for $M_{n-1}$ except that the indices of the $u_i$ are shifted down by one. Thus after relabeling the $u_i$, the Poisson algebra they obey is precisely the one obtained from the Kupershmidt-Wilson theorem applied to $M_{n-1}$—in other words, $W_{n-1}$. In summary, $W_{n-1}$ is a reduction of $W_n$ for all $n$. Iterating the reduction, we see that for all $m < n$, $W_m$ is a reduction of $W_n$. In particular, since $W_2$ is the Virasoro algebra, the Virasoro algebra is a reduction of every $W_n$. It should be remarked that this reduction is not the restriction to the Virasoro subalgebra present in every $W_n$, since the central charge is different. Another remark is that the reduction from $W_n$ down to $W_{n-1}$ can also be obtained by embedding $M_{n-1}$ into $M_n$ by $L \mapsto \partial \circ L$, which, in the Miura description, corresponds to the second-class constraint $\phi_1 = 0$. It would be interesting to classify the reductions of $W_n$ obtained from linear constraints of the $\phi_i$.

How does this observation help us in defining a universal $W$-algebra? If $W_n$ were a subalgebra of $W_{n+1}$ we could then hope to define a universal algebra containing all $W_n$ as the inductive limit $\lim_{\rightarrow} W_n$. However this is not the case. Nevertheless, the next best thing occurs: the fact that $W_{n-1}$ is a reduction of $W_n$ does allow us to define their inverse limit or colimit. The resulting algebra would then have the property that all $W_n$ could be obtained from it by reduction. But in order to make these ideas precise we first need to introduce a few notions concerning limits of Poisson algebras as we have defined them.
§3 Some abstract nonsense

The notion of inverse limit or colimit is a useful notion in category theory and we refer the interested reader to the book [12] by Lang for a more general exposition than the one presented here. We will introduce only those concepts we need in as much (lack of) generality as necessary. But before talking about inverse limits we need to set up some formalism.

The basic objects we are dealing with are essentially Poisson algebras, but perhaps in a slightly eccentric guise. Let $R$ denote some differential ring: that is, a commutative ring with unit together with a derivation $\partial : R \to R$ denoted $r \mapsto r'$. We will let $R[\partial]$ denote the ring of differential operators with coefficients in $R$, with multiplication defined by the usual Leibnitz rule $\partial \circ r = r' + r \circ \partial$. The natural maps (morphisms) between differential rings are ring homomorphisms preserving the identity and commuting with the derivation. Every such map $\varphi : R \to S$ between differential rings induces a map—also denoted $\varphi$—between differential operators $\varphi : R[\partial] \to S[\partial]$ by acting on the coefficients. Now by a Poisson structure on a differential ring $R$ we mean a bilinear map $J : R \times R \to R[\partial]$ obeying the properties necessary for the bracket defined as in (2.1) to be a Poisson bracket. In particular, this implies the antisymmetry and derivation properties mentioned in the previous section. We call the pair $(R, J)$ a Poisson algebra; although when no confusion can result, the Poisson structure will simply be omitted. Morphisms between Poisson algebras are morphisms of differential rings which preserve the Poisson structures. In other words, if $(R, J)$ and $(S, K)$ are Poisson algebras, a differential ring morphism $\varphi : R \to S$ is a Poisson algebra morphism if $\varphi(J(a, b)) = K(\varphi(a), \varphi(b))$, for all $a, b \in R$.

We have already seen a few examples of Poisson algebras and of morphisms between them. Indeed the content of the Kupershmidt-Wilson theorem is that the Miura transformation $R_n \to R(\phi)$ is a Poisson (mono)morphism. Another example of Poisson morphism—albeit perhaps not so obvious as the Miura transformation—is the reduction from $GD_n$ to $W_n$.  

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Let \( \tilde{M}_n \) denote the subspace of \( M_n \) defined by the constraint \( u_{n-1} = 0 \). If \( I \) denotes the differential ideal generated by \( u_{n-1} \), then the coordinate ring of \( \tilde{M}_n \) is just \( R_n/I \) which is naturally isomorphic to \( R(u_0, u_1, \ldots, u_{n-2}) \). On this differential ring we have \( W_n \) as a Poisson structure. We would like to exhibit the reduction \( GD_n \to W_n \) as induced by a Poisson (epi)morphism \( R_n \to R_n/I \). However the natural surjection \( R_n \to R_n/I \), sending \( r \) to its class modulo \( I \), is not a Poisson morphism with respect to the Gel’fand-Dickey bracket: the reason being that \( I \) is not a Poisson ideal of \( R_n \) and thus the quotient does not inherit a Poisson structure. In fact, one way to think of the Dirac bracket is that it is the unique Poisson structure on \( R_n \) making the natural surjection \( R_n \to R_n/I \) into a Poisson morphism. But we are interested in keeping the original Poisson structure on \( R_n \) and thus we must come up with a Poisson morphism \( R_n \to R_n/I \) intertwining between \( GD_n \) and \( W_n \).

To motivate the construction let us look at the familiar case of the hamiltonian reduction of a symplectic manifold \( M \) by a set of (regular) second-class constraints \( \{\chi_i\} \). Since the constraints are second-class, the matrix \( \Omega_{ij} = \{\chi_i, \chi_j\} \) of their Poisson brackets is non-degenerate when restricted to the zero locus \( M_\circ \) of the \( \{\chi_i\} \). Let us denote its inverse by \( \Omega^{ij} \). The ring \( C^\infty(M_\circ) \) of smooth functions on \( M_\circ \) then inherits a Poisson structure as follows. Given \( f, g \in C^\infty(M_\circ) \), we extend them to smooth functions (also denoted \( f, g \)) on \( M \). Then their Poisson bracket is defined by the Dirac formula

\[
\{f, g\}_\circ \equiv \{f, g\} - \sum_{ij} \{f, \chi_i\} \Omega^{ij} \{\chi_j, g\}.
\]  

This is clearly independent of the extension since the difference between any two extensions is of the form \( \sum c_i \chi_i \), for \( c_i \) an arbitrary function and on \( M_\circ \), these functions have vanishing Dirac bracket with any function. Therefore the Dirac bracket induces a Poisson structure on the ring \( C^\infty(M)/I \cong C^\infty(M_\circ) \), where \( I \) is the ideal generated by the constraints or, equivalently, the ideal of functions vanishing on \( M_\circ \). Alternatively we consider the following equivalent construction.

Let $N(I)$ denote the normalizer of $I$ in $C^\infty(M)$; that is, $f \in N(I)$ if $\{f, I\} \subset I$. Any function $f \in C^\infty(M)$ can be decomposed as $f = f_N + f_I$, where $f_N \in N(I)$ and $f_I \in I$. Of course, the decomposition is not unique since there are functions that lie both in $I$ and in $N(I)$. In fact, $\sum c_{ij} \chi_i \chi_j$, for arbitrary functions $c_{ij}$, lies both in $I$ and in $N(I)$. These functions constitute the ideal $I^2$ and it is not hard to show that $N(I) \cap I = I^2$. A possible decomposition of a function $f$ is given by

$$f_I = \Omega^{ij} \{f, \chi_i\} \chi_j, \quad f_N = f - f_I. \quad (3.2)$$

One can check that $f_N \in N(I)$. A straightforward calculation now shows that the Poisson bracket $\{f_N, g_N\}$ agrees on $M_o$ with the Poisson bracket $\{f, g\}_o$. Therefore we have exhibited the reduction from $M$ to $M_o$ as a Poisson morphism $C^\infty(M) \to C^\infty(M_o)$.

We now do the same for our more formal situation. Since the constraint $u_{n-1}$ is second-class, the ideal $I$ it generates is such that $R_n$ admits a decomposition $R_n = N(I) + I$, where now $N(I)$ is defined as follows: $r \in N(I)$ if and only if $J(r, I) \subset I[\partial]$. As before the sum is not direct: the intersection is again $N(I) \cap I = I^2$. This sets up an isomorphism of differential rings $R_n/I \cong N(I)/I^2$, but $N(I)/I^2$ has in addition a natural Poisson structure since $N(I)$ is a Poisson subalgebra of $R_n$ containing $I^2$ as a Poisson ideal. Thus the Poisson morphism we are after is the composition of the natural maps:

$$R_n \cong N(I) + I \to N(I)/I^2 \cong R_n/I. \quad (3.3)$$

We can see this a bit more explicitly in terms of generators. $R_n$ is generated by $u_0, u_1, \ldots, u_{n-1}$ whereas $I$ is generated by $u_{n-1}$. It does not follow, however that $N(I)$ is generated by $u_0, u_1, \ldots, u_{n-2}$ but almost. One can define generators $\tilde{u}_i = u_i + \cdots$, for $i = 0, 1, \ldots, n - 2$ such that $\tilde{u}_i \in N(I)$ and hence generate it. Of course, the $\tilde{u}_i$ are not unique, since one can always add terms in $I^2$. But having chosen one such set of $\tilde{u}_i$, the map $\rho : R_n \to R_n/I$ is simply given by $\rho(\tilde{u}_i) = u_i \mod I$ for $i = 0, 1, \ldots, n - 2$ and $\rho(u_{n-1}) = 0$. 

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As an example consider the reduction from $GD_2$ to the Virasoro algebra. We impose the constraint $u_1 = 0$. Now, as can be seen in (2.8), $u_0$ does not lie in the normalizer of the ideal generated by $u_1$. However, $\tilde{u}_0 = u_0 - \frac{1}{2} u_1'$ obeys $J(\tilde{u}_0, u_1) = u_1 \partial$, and hence generates the normalizer. Moreover, $J(\tilde{u}_0, \tilde{u}_0) = J_D(u_0, u_0) \mod I[\partial]$, so that they agree on the subspace $u_1 = 0$.

In fact, one does not need to be clever to find the $\tilde{u}_i$. Notice that, in this example, $\tilde{u}_0 = u_0 - J(u_0, u_1) \circ J(u_1, u_1)^{-1} \cdot u_1 \mod I^2$, which is to be compared with the definition of $f_N$ above. In general, to reduce $GD_n$ down to $W_n$, we define, for $i = 0, 1, \ldots, n-2$,

$$\tilde{u}_i = u_i - J(u_i, u_{n-1}) \circ J(u_{n-1}, u_{n-1})^{-1} \cdot u_{n-1} \mod I^2.$$  \hfill (3.5)

One can then show that $J(u_i, u_{n-1}) \circ J(u_{n-1}, u_{n-1})^{-1}$ is a differential operator and, using (2.3), that $J(\tilde{u}_i, u_{n-1}) \in I[\partial]$, so that $\tilde{u}_i \in N(I)$. Furthermore, computing the Gel’fand-Dickey brackets of the $\tilde{u}_i$ one finds precisely the Dirac brackets of the $u_i$:

$$J(\tilde{u}_i, \tilde{u}_j) = J_D(u_i, u_j) \mod I[\partial].$$  \hfill (3.6)

§4 The universal $W$-algebra as an inverse limit

The reduction $W_n \to W_{n-1}$ goes along similar lines, except that in order to describe the normalizer of the ideal generated by the constraint $u_0 = 0$ we have to extend the formalism slightly to encompass not just differential polynomials of the $u_i$ but also more formal objects obtained from them by the action of $\partial^{-1}$. Let us illustrate this with an example: the reduction of $W_3$ down to Virasoro. Analogously to the reduction $GD_n \to W_n$, we define $\tilde{u}_1 = u_1 - J_D(u_1, u_0) \circ J_D(u_0, u_0)^{-1} \cdot u_0 \mod I^2$. Only that now, $J_D(u_1, u_0) \circ J_D(u_0, u_0)^{-1}$ is no longer a differential operator (even putting $u_0 = 0$) so that $\tilde{u}_1$ is not a a differential polynomial of the $u_i$; in fact, on $u_0 = 0$, $J_D(u_1, u_0) \circ J_D(u_0, u_0)^{-1} = \frac{3}{2}(\partial^2 + u_1)^{-1} \partial$. 

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Thus we need to extend the differential ring $R_n$ in such a way that the action of $\partial^{-1}$ is defined. This involves introducing symbols $\partial^{-1} \cdot u_i$, $\partial^{-2} \cdot u_i$, ... obeying $(\partial^{-1} \cdot u_i)' = \partial^{-1} \cdot u_i' = u_i$, ... The operator $\partial^{-1}$ is not a derivation but obeys the following rule:

$$\partial^{-1} \cdot (ab) = \sum_{j=0}^{\infty} (-1)^j a^{(j)} \partial^{-j-1} \cdot b ,$$  \hspace{1cm} (4.1)

whence we can define $\partial^{-1}$ everywhere from a knowledge of $\partial^{-1}$ on the generators. Notice, by the way, that this necessitates introducing formal infinite sums of monomials. We will still denote the extended ring by $R_n$ and we will let $R_n((\partial^{-1}))$ denote the ring of pseudodifferential operators with coefficients on $R_n$. Then $R_n((\partial^{-1}))$ acts naturally on $R_n$. We now extend the Poisson bracket to the extended ring $R_n$ as a bilinear map

$$J : R_n \times R_n \to R_n((\partial^{-1})) .$$  \hspace{1cm} (4.2)

We have to extend the derivation property to compute $J(\partial^{-1} \cdot a, b)$ from $J(a, b)$ so that, again, it is sufficient to know $J$ on generators. Since $(\partial^{-1} \cdot a)' = a$, using (2.3) we find that $J(\partial^{-1} \cdot a, b) = \partial^{-1} \circ J(a, b)$. The antisymmetry property is the same, where $\ast$ is now the unique extension of $\ast$ to an anti-involution on the extended ring.

Now we look at the reduction $GD_n \to GD_{n-1}$ obtained by putting $u_0 = 0$. Let $I$ denote the pseudodifferential ideal generated by $u_0$. For $i = 1, 2, \ldots, n-1$ we define $\tilde{u}_i = u_i - K_i \cdot u_0$, where $K_i \in R_n((\partial^{-1}))$ is defined so that $J(\tilde{u}_i, u_0) \in I((\partial^{-1}))$. This fixes $K_i = J(u_i, u_0) \circ J(u_0, u_0)^{-1} \mod I((\partial^{-1}))$. Thus,

$$\tilde{u}_i = u_i - J(u_i, u_0) \circ J(u_0, u_0)^{-1} \cdot u_0 \mod I^2 .$$  \hspace{1cm} (4.3)

A straightforward calculation now shows that

$$J(\tilde{u}_i, \tilde{u}_j) = J(u_i, u_j) - J(u_i, u_0) \circ J(u_0, u_0)^{-1} \circ J(u_0, u_j) \mod I((\partial^{-1})) .$$  \hspace{1cm} (4.4)

We can then define morphisms $\rho_n : R_n \to R_{n-1}$ by $\rho_n(\tilde{u}_i) = u_i - 1$, $\rho_n(u_0) = 0$ which induce Poisson morphisms $GD_n \to GD_{n-1}$. More invariantly, these morphisms can
be defined as the composition of the following natural maps

\[ R_n \cong N(I) + I \rightarrow N(I)/I^2 \cong R_n/I \cong R_{n-1}. \]  

(4.5)

We will also denote by \( \rho_n : W_n \rightarrow W_{n-1} \) the Poisson morphism induced by the above map under the \( GD \rightarrow W \) reduction.

We have therefore a set of Poisson algebras \( \{W_n\} \) and Poisson morphisms \( \rho_n : W_n \rightarrow W_{n-1} \). This is called an inverse system of Poisson algebras and one can define its colimit or inverse limit \( \lim_{\leftarrow n} W_n \) as follows. A colimit \( \lim_{\leftarrow n} W_n \) consists of a Poisson algebra \( U \) together with Poisson morphisms \( \pi_n : U \rightarrow W_n \) obeying \( \rho_n \circ \pi_n = \pi_{n-1} \), which enjoy the following (universal) property: given any other Poisson algebra \( P \) and morphisms \( \mu_n : P \rightarrow W_n \) obeying \( \rho_n \circ \mu_n = \mu_{n-1} \) there exists a unique Poisson morphism \( \varphi : P \rightarrow U \) such that for all \( n \)

\[ \mu_n = \pi_n \circ \varphi; \]  

(4.6)

in other words, the maps \( \mu_n \) all factor through \( U \).

As usual with universal objects, if they exist they are unique up to a unique isomorphism. In fact, suppose that \( (\tilde{U}, \{\tilde{\pi}_n\}) \) is another such colimit. Then by the universal property of \( U \) there is a unique morphism \( \tilde{U} \rightarrow U \) satisfying (4.6) , and by the universal property of \( \tilde{U} \) there is a unique such morphism \( U \rightarrow \tilde{U} \) as well. Now, composing them we get two morphisms \( U \rightarrow U \) and \( \tilde{U} \rightarrow \tilde{U} \) which are unique. But certainly the identity maps also satisfy the above properties, hence, by uniqueness, the compositions \( U \rightarrow \tilde{U} \rightarrow U \) and \( \tilde{U} \rightarrow U \rightarrow \tilde{U} \) are precisely the identity maps. In other words, \( U \cong \tilde{U} \) for a unique isomorphism.

We still have to show that the colimit exists. Since Poisson algebras are pseudodifferential rings and Poisson morphisms are, in particular, morphisms of pseudodifferential rings, we can appeal to the construction of a model for the colimit in the category of pseudodifferential rings and just check that the extra structure (i.e., the Poisson structure) is preserved.
Let $R \equiv \prod_n R_n$ denote the product of the pseudodifferential rings $R_n$. That is, $R$ consists of all sequences $(r_n)$ with $r_n \in R_n$. $R$ becomes a pseudodifferential ring in a natural way by simply defining the operations entry-wise: $(r_n)' = (r'_n)$, \[ \partial^{-1}\cdot (r_n) = (\partial^{-1}\cdot r_n), \] and $(r_n)(s_n) = (r_ns_n)$. Let $U$ denote the subset of $R$ defined by

$$U = \{(u_n) \in R \mid u_{n-1} = \rho_n(u_n)\}.$$ \hspace{1cm} (4.7)

It is easy to check that $U$ is a pseudodifferential subring of $R$ and that it has the universal property of the colimit if we define $\pi_n : U \rightarrow R_n$ as the composition $U \rightarrow R \rightarrow R_n$ where the first map is the inclusion and the second is the natural projection onto the $n$th factor. In fact, if $S$ is another pseudodifferential ring with morphisms $\sigma_n : S \rightarrow R_n$, then the map $S \rightarrow U$ is given by $s \mapsto (\sigma_n(s))$ which is easily checked to obey the conditions (4.6). If we now turn on the Poisson structure on each $R_n$, then $R$ becomes a Poisson algebra defining the bracket entry-wise, $U$ becomes a Poisson subalgebra, and all maps in sight are Poisson morphisms. Thus $U$ becomes a colimit in the category of Poisson algebras.

Summarizing, we have shown the existence and uniqueness of a universal $W$-algebra for the $\{W_n\}$ series. The universal property was defined above and shall not be repeated, but we mention some consequences. First of all, since the maps $\rho_n : W_n \rightarrow W_{n-1}$ are surjective, so are the maps $\pi_n : U \rightarrow W_n$, whence we can think of them as reductions which, due to the fact that $\rho_n \circ \pi_n = \pi_{n-1}$, are compatible with the reductions $W_n \rightarrow W_{n-1}$ and, in a sense, extend them. In other words, it does not matter how we reduce $U$ down to $W_n$. We can either first reduce $U$ to $W_m$ (for $m > n$) and then reduce $W_m \rightarrow W_n$ or reduce $U$ to $W_n$ directly. Suppose now that someone hands us a Poisson algebra $P$ such that all $W_n$ are obtained from it by (compatible) reductions; i.e., there are Poisson epimorphisms $\mu_n : P \rightarrow W_n$ obeying $\rho_n \circ \mu_n = \mu_{n-1}$. Then these reductions factor through $U$ in the sense that we can first map $P$ to $U$ and then reduce $U$ to the $W_n$. Loosely speaking then, $U$ is the smallest $W$-algebra which yields all $W_n$ by reduction.
The model given above for the universal $W$-algebra suffices for a proof of existence and uniqueness, but does not really give us an algorithm to compute this algebra explicitly. From a constructive point of view, the above model is therefore not very satisfactory. It thus behooves us to elucidate the possible relations between some of the candidates for “universal” $W$-algebras that have appeared in the literature and the universal $W$-algebra described here. Since the only thing we know about our algebra is the universal property it obeys, it seems that the way to proceed is to probe the candidates to see if they obey this property—hence concluding that they are isomorphic to our algebra—or, on the contrary, that they are not universal. Two classes of candidates of which we will not be able to say much are the following: several versions of $W_\infty$-type algebras \cite{2} \cite{13} ; and algebras which appear as a hamiltonian structure for the KP hierarchy \cite{14} \cite{5} \cite{6} \cite{15}. Both of these proposals make sense heuristically. Since the $W_n$ algebras are realized as the second hamiltonian structure of the $n^{th}$ order KdV hierarchy and this hierarchy is a reduction of KP, it seems reasonable to expect that the second hamiltonian structure of the KP hierarchy should yield all the $W_n$ algebras upon reduction. We have tried to prove this reduction explicitly, but so far without success. As for the $W_\infty$-type algebras, the rationale seems to be the following: for any fixed $n$, $W_n$ is the smallest algebra from which all the $W_m$ for $m \leq n$ can be obtained by reduction. It thus seems reasonable that the model for the universal $W$-algebra should be the limit $n \to \infty$ of $W_n$. The question is how to make this limit precise. In this paper we have given one such possibility that seems to be the natural one, since we know of no other relationship between the $W_n$ algebras but the fact that one can get $W_{n-1}$ from $W_n$ by reduction. It is a very interesting open problem to see if any one of the $W_\infty$-type algebras in the literature is indeed universal. This seems to be supported by the experimental fact \cite{4} that, at least for a particular value ($c = -2$) of the central charge, $W_n$ can be obtained as a reduction of $W_\infty$. As a side remark, notice also that the first hamiltonian structure for the KP hierarchy is, in fact, $W_{1+\infty}$ \cite{16}. But since it is the second structure
for the KdV hierarchies that corresponds to $W_n$, we are not sure if this is not just a macabre coincidence.

The one proposal for universal $W$-algebra that we will look at in some detail is the algebra $D$ of differential operators on the circle, put forward by Radul [1]. It should be remarked, however, that $D$ is not a Poisson algebra but only a Lie algebra. Eventually, of course, one is interested in quantizing $W$ algebras. Since quantization does not respect the Poisson structure but just the Lie structure, this does not represent a drawback.

To understand the maps $D \to GD_n$ we have to introduce some minor notation. We refer the reader to the forthcoming book by Dickey [17] for a more detailed discussion, or to our paper [18] for a less brief summary. Given $L \in M_n$, the tangent space $T_L M_n$ to $M_n$ at $L$ is naturally identified with the differential operators of order at most $n - 1$. Similarly the cotangent space $T_L^* M_n$ is naturally identified with the space $\mathcal{P}^- / \partial^{-n} \mathcal{P}^-$, where $\mathcal{P}$ is the ring of pseudodifferential operators and $\mathcal{P}^-$ the subring of integral operators. The pairing between $T_L M_n$ and $T_L^* M_n$ is given by the Adler trace [10]. The Adler map (2.5) is a linear map $T_L^* M_n \to T_L M_n$. The Lie bracket of vector fields gives a Lie algebra structure to $T_L M_n$ relative to which the image of the Adler map is a subalgebra. Thus one can pull back this structure to give $T_L^* M_n$ a Lie algebra structure. Explicitly, if $X, Y \in T_L^* M_n$, their Lie bracket is given by

$$[X, Y] \equiv \partial_A(X)Y + X(LY)_- - ((XL)_+ Y)_- - (X \leftrightarrow Y),$$

where $\partial_A(X)$ represents the Lie derivative of the vector field defined by $A(X) \in T_L M_n$. One can check that

$$[A(X), A(Y)] = A([X, Y]).$$

This Lie bracket $[,]$ is essentially the Poisson bracket defined by the Adler map; indeed for $G, H$ functions on $M_n$, it follows that $d\{G, H\} = [dG, dH]$. 

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Now Radul defines a map \( \pi_n : \mathcal{D} \to T^*_L M_n \) by \( \pi_n(E) = -(EL^{-1})_- \). One can easily show that \( \pi_n \) is a homomorphism of Lie algebras, where \( \mathcal{D} \) is given a Lie algebra structure via the commutator. In other words,

\[
[\pi_n(E), \pi_n(F)] = \pi_n([E, F]).
\]

Therefore this induces a map \( \mathcal{D} \to GD_n \), which we also denote \( \pi_n \). It is interesting to notice that since \( \pi_n((EL^{-1})_+ L) = 0 \), only \( E - (EL^{-1})_+ L = (EL^{-1})_- L \) matters, which is a differential operator of order \( n - 1 \).

However, it is easy to see that the maps \( \pi_n : \mathcal{D} \to GD_n \) are incompatible with the reduction \( GD_n \to GD_{n-1} \). To see this notice simply that the reduction \( GD_n \to GD_{n-1} \) is induced by the embedding \( M_{n-1} \to M_n \) given by \( L \mapsto L\partial \). Now, there is a natural embedding \( T^*_{L\partial} M_{n-1} \subset T^*_L M_n \) defined as those 1-forms whose image under the Adler map are tangent vectors to \( M_{n-1} \). Then these 1-forms inherit a Lie bracket \([,]\) simply from the one \( T^*_L M_n \). The reduction is compatible if for \( L\partial \in M_n \), the image of the map \( \pi_n \) lies in \( T^*_{L\partial} M_{n-1} \); in other words, if \( A(\pi_n(E)) \in T_{L\partial} M_{n} \). Now \( T_{L\partial} M_n \) corresponds to the infinitesimal deformations of \( L\partial \), which in this case are nothing but the differential operators of order at most \( n - 1 \) without free term. An explicit computation shows that \( A(\pi_n(E)) = (L\partial E\partial^{-1} L^{-1})_- L \partial \) which can be rewritten as \( L\partial E - (L\partial E\partial^{-1} L^{-1})_+ L \partial \) which, since \( E \) has in general a free term, is not tangent to \( M_{n-1} \subset M_n \).

This discards \((\mathcal{D}, \pi_n)\) as a model for the universal \( W \)-algebra.

\section{Some extensions of our results}

Finally we mention two extensions of our results. Notice that the universal \( W \)-algebra for \( \{W_n\} \) described here is a reduction of the universal Gel’fand-Dickey algebra. There are other \( W \)-algebras besides \( W_n \) which arise as reductions of \( GD_n \). In particular, for \( n = 2\ell + 1 \) odd, one can construct a \( W \)-algebra as the hamiltonian reduction of the Gel’fand-Dickey bracket with respect to the second-class constraint \( L^* = -L \). This subspace of antisymmetric Lax operators is a
(formally) symplectic submanifold of $M_{2\ell+1}$ and the Dirac brackets induce the $W$-algebra $DS(B_\ell)$ associated à la Drinfel’d-Sokolov [20] to the simple Lie algebra $B_\ell$. Similarly for $n = 2\ell$ even, the subspace defined by the second-class constraint $L^* = L$ inherits a Poisson structure which defines the $W$-algebra $DS(C_\ell)$ associated to $C_\ell$. The reduction $GD_n \to GD_{n-2}$ induces a reduction $DS(B_\ell) \to DS(B_{\ell+1})$ for $n$ odd, and $DS(C_\ell) \to DS(C_{\ell+1})$ for $n$ even, giving rise to universal $W$-algebras as the corresponding inverse limits.

Another straightforward extension of these results is the existence and uniqueness of universal $W$-superalgebras for the $W$-superalgebras obtained from the supersymmetric Gel’fand-Dickey brackets $\{SGD_n\}$ constructed in [18]. In particular, the infinite series of $N = 1$ [21] and $N = 2$ [22] $W$-superalgebras obtained as reductions of the supersymmetric Gel’fand-Dickey brackets give rise, upon taking the relevant inverse limits, to universal $W$-superalgebras for those series. The proof of the existence and uniqueness of these universal $W$-superalgebras is a straightforward supersymmetrization of the proof given here: the reduction $SGD_n \to SGD_{n-1}$ being proven using the supersymmetric Kupershmidt-Wilson theorem of [18].

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2 It should be remarked that these algebras are not the $WB_\ell$ algebras [19]. The same comment holds in the $C_\ell$ case to be discussed below.
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