EXTENDED SUPERCONFORMAL ALGEBRAS

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ABSTRACT

Solving the conformal bootstrap perturbatively, we undertake a systematic study of the possible extensions of the $N = 1$ super Virasoro algebra by a superprimary field of (half)integer spin $\frac{1}{2} \leq \Delta \leq \frac{7}{2}$. Besides extensions which exist only for specific values of the central charge, we find a non-linear algebra (super W_2), associative for all values of the central charge, and generated by a spin 2 superprimary.

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Introduction

Extended conformal and superconformal algebras have received a great deal of attention lately[1],[2],[3]. Their study is relevant for the classification problem of rational conformal field theories[4] (RCFTs) since every RCFT is by definition a minimal model of its chiral algebra (the operator subalgebra generated by its holomorphic fields) which, since it contains the Virasoro algebra as a subalgebra, extends it. As shown by Cardy[5], a conformal field theory (CFT) which is rational relative to the Virasoro algebra (*i.e.* , which contains a finite number of Virasoro primaries) must necessarily have $c < 1$. Similar arguments show that a superconformal field theory (SCFT) which is rational relative to the $N = 1$ super Virasoro algebra must have $c < \frac{3}{2}$. Therefore, in order to construct RCFTs for $c \geq 1$ (resp. rational SCFTs for $c \geq \frac{3}{2}$) one is lead to extended conformal (resp. superconformal) algebras.

There is by now a wealth of examples of extended conformal algebras. Among the best known ones are the affine Lie algebras (extensions of Virasoro by weight 1 primaries), the super Virasoro algebras (extensions of the Virasoro algebra by one or more primaries of weight $\frac{3}{2}$), and Zamolodchikov's W_3 algebra (the unique extension by a field of weight 3) which, in particular, has been shown to be the symmetry algebra of the 3-state Potts model at criticality[6]. The first results of a general nature were obtained by Bouwknegt in **[7]** , where he investigated the existence of extensions of the Virasoro algebra by a primary of integer or half-integer weight. Apart from finding new solutions for spins $\Delta > 3$, he argued based on group theoretic counting arguments that if one demands that the resulting mode algebra be associative for all values of the central charge, one can only have $\Delta = 1/2, 1, 3/2, 2, 3, 4, 6$. He also gave the value of the operator product coefficient C_{44}^4 for the spin 4 algebra. This algebra was later constructed in **[8]** , whereas in **[9]** we constructed the spin 6 algebra. These results have recently been confirmed in **[10]** by the study of the Jacobi identities.

The body of knowledge concerning the extended $N = 1$ superconformal algebras is smaller in comparison. The only hitherto known examples are the Kač-Todorov algebras[11] which are extensions by superprimaries (*i.e.* , primaries of the super Virasoro algebra) of spin $\frac{1}{2}$; the $N = 2$ and the small $N = 4$ super Virasoro algebras which are the unique extensions by one or three superprimary fields of spin 1; and the two algebras constructed in **[12]** : the extension by a spin 2 superprimary without self-coupling, which is associative for $c = -\frac{6}{5}$; and the one by a spin $\frac{5}{2}$ superprimary—called[1] super $W_{5/2}$—which is associative for $c = \frac{10}{7}$ and $c = -\frac{5}{2}$.

In this letter we report on our systematic investigations on the extensions of the $N = 1$ superconformal algebra by one or, in some cases, more than one superprimary fields in the framework of the conformal bootstrap **[16]** . A more detailed exposition can be found in **[17]** . Our main result is the explicit (quantum) construction of super W_2: a non-linear extended superconformal algebra associative for all values of the central charge, of which a classical version has been constructed from supersymmetric Toda field theory **[3]** .

There are known examples of two-dimensional statistical mechanics models displaying superconformal invariance [18],[19] at criticality as well as models with a non-linear extension of the conformal algebra as a symmetry[6]. This prompts the question of whether there are models displaying both. Our investigations are a first step in the construction of such supersymmetric conformal field theories. In particular, the non-linear algebra super W_2 invites the search for its (unitary) minimal models. On the other hand, recent advances in W-gravity make super W_2 a prime candidate for the construction of a W-supergravity theory, which could lead to new insights into the nature of superstring theory.

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[1] This algebra was originally termed super W_3 in **[12]** (*cf.* **[13]** and **[14]**); however, as noted in **[15]** , it is more natural to call the algebra generated by a spin Δ superprimary super W_Δ. We follow this terminology in this letter.

Extended Superconformal Algebras

Before turning to the explicit results let us briefly describe the superconformal bootstrap approach to extended superconformal algebras (see **[17]** for details). We investigate the possible extensions of the $N = 1$ super Virasoro algebra by a superprimary field $\phi_\Delta(z)$ of weight Δ. Since we are interested in algebras described by the (anti)commutators of modes, we restrict ourselves to (half)integral weights and one chiral sector.

In $N = 1$ superspace, a superprimary $\phi_\Delta(z)$ assembles, together with its superpartner, into a primary superfield $\Phi_\Delta(Z) = \phi_\Delta(z) + \theta\psi_{\Delta+1/2}(z)$, which allows us to write manifestly supercovariant expressions for the operator algebra. Just as in Virasoro CFT, here the local fields assemble themselves into superconformal families, into which the operator product expansion of any two fields can be decomposed. In particular, given two superprimary fields their operator product expansion can be written as follows:

$$\phi_\Delta(z)\phi_{\Delta'}(w) = \sum_{\Delta''} C_{\Delta\Delta'}^{\Delta''} \left[\phi_{\Delta''}\right](z|w) \ , \tag{1}$$

where the contribution $\left[\phi_{\Delta''}\right](z|w)$ from the superconformal family of $\phi_{\Delta''}$ is completely determined by superconformal covariance and can be written, for generic values of the central charge, in terms of the inverse of the Šapovalov form of the Verma module of the NS algebra corresponding to the highest weight state created by $\phi_{\Delta''}$ on the superprojective invariant vacuum.

Hence the operator algebra of a SCFT is fixed by superconformal covariance up to a few parameters: the dimensions of the superprimary fields and the operator product coefficients $C_{\Delta\Delta'}^{\Delta''}$. The (super)conformal bootstrap consists of fixing these parameters by demanding duality of the correlators. Duality of any correlator follows from duality of the general four point functions and duality of these in turn follows from that of the four point functions involving primaries only. These four point functions can then be computed in terms of (super)conformal blocks, for which a perturbative expansion exists. The requirement of duality translates into a set of conditions for the operator product coefficients and the central charge which can be imposed perturbatively generalizing a group theoretical argument due to Bouwknegt **[7]** .

Let us now turn to the explicit construction of extended $N = 1$ superconformal algebras. We will mainly focus on those algebras which can be obtained by extending the $N = 1$ super Virasoro algebra by one superprimary ϕ_Δ of spin $\frac{1}{2} \leq \Delta \leq \frac{7}{2}$, except for the cases $\Delta \leq \frac{3}{2}$ where we allow for more than one field of dimension Δ. The OPE is of the following form

$$\phi_\Delta \times \phi_\Delta = C^0_{\Delta\Delta}[\phi_0] + C^\Delta_{\Delta\Delta}[\phi_\Delta] + \text{regular terms} . \tag{2}$$

In general, the self-coupling $C^\Delta_{\Delta\Delta}$ can only be nonzero for $\Delta = 2n$ or $2n + \frac{3}{2}$, with $n \in \mathbb{N}_0$, unless, of course, we consider more than one field. We leave open the possibility that further superprimaries may appear in the regular terms of the above OPE so that their presence may not be detectable in the mode algebra.

Our results are summarised in the following table:

Δ	Algebra	Allowed c values
$\frac{1}{2}$	Kač-Todorov	all c
1	$N = 2$ and Small $N = 4$	all c
$\frac{3}{2}$	Direct Product of $N = 1$	all c
2	Super W_2	all c
$\frac{5}{2}$	Super $W_{5/2}$	$c = \frac{10}{7},\ -\frac{5}{2}$
3	Super W_3	$c = \frac{5}{4},\ -\frac{45}{2},\ -\frac{27}{7}$
$\frac{7}{2}$	Super $W_{7/2}$	$c = \frac{7}{5},\ -\frac{17}{11}$

Let us briefly discuss our results case by case. For the first two values of Δ we recover well-known results: the Kač-Todorov (also known as super Kač-Moody) algebras **[11]** for $\Delta = \frac{1}{2}$, and the $N = 2$ superconformal algebra **[20]** (for one extra field) or the small $N=4$ superconformal algebra **[20]** (for three extra fields) when $\Delta=1$. The case $\Delta=\frac{3}{2}$ is completely analogous to the extension of the Virasoro algebra by weight 2 primaries treated in **[21]** .

The case $\Delta=2$ is the first non-trivial case, since the resulting algebra is non-linear. The simpler case of no self-coupling was studied in **[12]** where it was found that the algebra is only associative for $c=-\frac{6}{5}$. When the self-coupling is added, the restriction on the central charge is lifted, resulting in an algebra associative for all values of the central charge.

Apart from the super energy-momentum tensor this algebra contains a spin 2 primary superfield $\Phi_2(Z)=\frac{i}{\sqrt{5}}W(z)+\theta U(z)$. The nontrivial (anti)commutators are given by

$$
\begin{aligned}
[W_m\,,\,W_n] =&(m-n)(L_{m+n}+\kappa W_{m+n})+\frac{c}{12}(m^3-m)\delta_{m+n,0}\ , \qquad (3)\\
[U_r\,,\,W_m] =&\frac{54i}{\sqrt{5}(4c+21)}(LG)_{m+r}+\frac{54i\kappa}{\sqrt{5}(5c+6)}(GW)_{m+r}\\
&+\frac{i}{4\sqrt{5}}\left[r^2+3m^2-2mr-\frac{9}{4}-\frac{81}{4c+21}(m+r+\frac{3}{2})(m+r+\frac{5}{2})\right]G_{m+r}\\
&+\frac{\kappa}{5}\left[2r-3m+\frac{108}{5c+6}(m+r+\frac{5}{2})\right]U_{m+r}\ , \qquad (4)\\
\{U_r\,,\,U_s\} =&\frac{108\kappa}{5(5c+6)}(LW)_{r+s}+\frac{108}{5(4c+21)}(LL)_{r+s}\\
&+\frac{27}{5(4c+21)}(\partial GG)_{r+s}+\frac{54i\kappa}{\sqrt{5}(5c+6)}(GU)_{r+s}\\
&+\kappa\left[\frac{3(c-6)}{5(5c+6)}(r+s+2)(r+s+3)-\frac{1}{10}(4rs+6r+6s+9)\right]W_{r+s}\\
&+\left[\frac{3(2c-3)}{5(4c+21)}(r+s+2)(r+s+3)-\frac{1}{4}(4rs+6r+6s+9)\right]L_{r+s}\\
&+\frac{c}{60}(r^2-\frac{9}{4})(r^2-\frac{1}{4})\delta_{r+s,0}\ , \qquad (5)
\end{aligned}
$$

where $\kappa={C_{22}}^2/2$ is given by

$$
\kappa=\pm\frac{6+5c}{\sqrt{(21+4c)(15-c)}}\ . \qquad (6)
$$

A tedious calculation shows that the Jacobi identities are indeed satisfied for and only for the above value of the self-coupling. The sign ambiguity in (6) is a manifestation of the algebra automorphism taking $\Phi_2 \mapsto -\Phi_2$.

Recently, a classical version of super W_2 was constructed from the supersymmetric Toda field theory corresponding to $osp(3|2)$ **[3]** . This algebra was later quantised in **[22]** although the expression therein differs from the one given here. In fact, due to some computational errors which now seem to have been clarified[**23**], the form of the algebra in **[22]** is wrong, as can be easily inferred from the fact that it does not decompose into superconformal families.

Notice that for $c = -\frac{6}{5}$ the self-coupling κ vanishes and hence the superconformal family of the spin 2 superprimary does not appear. Thus, our results imply the existence of an extended conformal algebra for this value of the central charge, with $\phi_2 \times \phi_2 = C^0_{22}\phi_0 +$ regular terms. This algebra is precisely the one discovered in **[12]** . Alternatively, one can put $c = -\frac{6}{5}$ in the explicit form of the algebra as given in (3) , (4) , and (5) . One then sees that some superdescendents of W do remain, since the zero in κ is cancelled by a pole in the superconformal family coefficients. It has been remarked[**22**],[**23**] that the resulting algebra is then essentially different from that found in **[12]** . This, however, is not the case. Indeed, it is easy to see that the descendent fields which remain are null for this particular value of the central charge, hence decoupling from any correlation function. Therefore, they can be consistently set to zero and thus the "physical" content of the two algebras is the same. This implies that in a free field realization of super W_2 these null fields should be identically zero for that value of the central charge.

We now consider the extension of the $N = 1$ super Virasoro algebra by a primary superfield of weight $\Delta = 5/2$, which was investigated in detail in **[12]** . Our results— that the algebra is only associative for $c = \frac{10}{7}$ and $c = -\frac{5}{2}$ —confirm theirs. Recently a proposed extension of this algebra (for all values of the central charge) has been given in **[24]** and consists of 8 generating fields.

For $\Delta = 3$, crossing symmetry fixes the central charge to take only three values: $\frac{5}{4}$, $-\frac{45}{2}$, and $-\frac{27}{7}$. As shown in **[17]** this algebra is the symmetry algebra of the $c = \frac{5}{4}$ SCFT defined by the (A_7, D_4)-type modular invariant in the Cappelli classification **[25]** .

Finally, for $\Delta = \frac{7}{2}$, associativity is satisfied for only two values of the central charge: $\frac{7}{5}$ and $-\frac{17}{11}$. The former value lies in the unitary discrete series of the super Virasoro algebra. Its possible relation with the SCFT defined by the (D_6, E_6)-type modular invariant is discussed in **[17]** . The values for the self-couplings of this algebra are given by:

$$\left(C_{7/2,7/2}^{7/2}\right)^2 = \begin{cases} \frac{4563}{25840} & \text{for } c = \frac{7}{5} \\ \\ -\frac{34460181}{4187144} & \text{for } c = -\frac{17}{11} \end{cases} . \tag{7}$$

This result was recently confirmed in **[26]** by checking the Jacobi identities.

For extensions of the super Virasoro algebra by superprimaries of spin $\Delta \geq 4$ our method becomes, unfortunately, computationally too involved. Therefore we have restricted ourselves to the above range of conformal dimensions. Nevertheless there are two results of a general nature we can comment on. Counting arguments imply that there cannot exist any extension of the super Virasoro algebra by just one superprimary of spin $\Delta \geq \frac{5}{2}$ which is associative for all values of the central charge. However, a glance at the Kač tables of $N = 1$ minimal models reveals that there are certainly many more such extensions which are associative for particular c values. In fact, the m^{th} $N = 1$ unitary minimal model (for m even) always contains a representation of super $W_{m(m-2)/8}$, as can easily be inferred from the fusion rules.

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