THE CONFORMAL BOOTSTRAP AND SUPER W-ALGEBRAS

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ABSTRACT

We undertake a systematic study of the possible extensions of the $N = 1$ super Virasoro algebra by a superprimary field of spin $\frac{1}{2} \leq \Delta \leq \frac{7}{2}$. Besides new extensions which exist only for specific values of the central charge, we find a new non-linear algebra (super W_2) generated by a spin 2 superprimary which is associative for all values of the central charge. Furthermore, the spin 3 extension is argued to be the symmetry algebra of the $m = 6$ super Virasoro unitary minimal model, by exhibiting the (A_7, D_4)-type modular invariant as diagonal in terms of extended characters.

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§1 Introduction

Extended conformal and superconformal algebras have received a great deal of attention lately[1],[2],[3]. Their study is relevant for the classification problem of rational conformal field theories[4] (RCFTs) since every RCFT is by definition a minimal model of its chiral algebra (the operator subalgebra generated by its holomorphic fields) which, since it contains the Virasoro algebra as a subalgebra, extends it. As shown by Cardy[5], a conformal field theory (CFT) which is rational relative to the Virasoro algebra (*i.e.* , which contains a finite number of Virasoro primaries) must necessarily have $c < 1$. Similar arguments show that a superconformal field theory (SCFT) which is rational relative to the $N = 1$ super Virasoro algebra must have $c < \frac{3}{2}$. Therefore, in order to construct RCFTs for $c \geq 1$ (resp. rational SCFTs for $c \geq \frac{3}{2}$) one is lead to extended conformal (resp. superconformal) algebras.

There is no general agreement in the literature as to the definition of an extended conformal algebra; hence let us adopt the following definition for illustrative purposes. An extended conformal algebra is an associative operator product algebra which contains the (universal enveloping algebra of the) Virasoro algebra and is generated by a finite number of Virasoro primaries in the sense that in the singular part of the operator product expansion of these fields there appear only Virasoro descendents of the identity and of these fields as well as normal ordered products thereof. Similarly, we define an extended superconformal algebra by substituting 'Virasoro' by 'super Virasoro' in the above definition. Extended superconformal algebras, however, do not give rise to chiral algebras directly because the duality axiom of CFT forces the fields in the chiral algebra to have integer spin. Hence to obtain a chiral algebra from an extended superconformal algebra it is necessary to truncate the field content leaving only the fields with integer spins.

There is by now a wealth of examples of extended conformal algebras. Among the best known ones are the affine Lie algebras (extensions of Virasoro by weight 1 primaries), the super Virasoro algebras (extensions of the Virasoro algebra by one or more primaries of weight $\frac{3}{2}$), and Zamolodchikov's W_3 algebra (the unique extension by a field of weight 3).

It was Zamolodchikov[6] who initiated the systematic study of the existence of extended conformal algebras by analyzing the possible extensions of the Virasoro algebra by one or more fields of a given (integer or half-integer) spin $1/2 \leq \Delta \leq 3$ which were consistent with duality, *i.e.* , which yield crossing symmetric four point functions. He found a lot of already existing algebras: free fermions ($\Delta = \frac{1}{2}$), affine Lie algebras ($\Delta = 1$), super Virasoro algebras ($\Delta = \frac{3}{2}$), direct product of Virasoro algebras ($\Delta = 2$); as well as two new algebras ($\Delta = \frac{5}{2}$ and $\Delta = 3$) which, unlike the others, are not Lie (super)algebras, since they contain non-linear terms in the (anti)commutators of the modes. The case $\Delta = \frac{5}{2}$ satisfies duality for a specific value of the central charge ($c = -13/14$); whereas the case $\Delta = 3$ yields W_3 which is associative for all c. Zamolodchikov's work was extended in **[7]** and **[8]** by the introduction of primaries of different spins at once. In **[7]** the $\mathfrak{so}(N)$- and $\mathfrak{u}(N)$-extended superconformal algebras were obtained. For $N \leq 4$ the former algebras can be linearised by adding free spin $\frac{1}{2}$ fields. This is the inverse of the method by Goddard and Schwimmer[9] for decoupling free fermions in a CFT. In **[8]** two new algebras were obtained: the extension by primaries of spins $\frac{3}{2}$ and $\frac{5}{2}$, which is only associative for $c = -\frac{13}{14}$; and the one by primaries of spins $\frac{5}{2}$ and 3, which is associative for all c. Moreover we have investigated the existence of an extension of the Virasoro algebra by primaries of spin 3 and 4 only and have found[10] that duality is satisfied only for $c = 1$, -13, and $-\frac{116}{3}$.

The first results of a general nature were obtained by Bouwknegt in **[11]** , where he investigated the existence of extensions of the Virasoro algebra by a primary of integer or half-integer weight. Apart from finding new solutions for spins $\Delta > 3$, he argued based on group theoretic counting arguments that if one demands that the resulting mode algebra be associative for all values of the central charge, one can only have $\Delta = 1/2, 1, 3/2, 2, 3, 4, 6$. He also gave the value of the operator product coefficient C_{44}^4 for the spin 4 algebra. This algebra was later constructed in **[12]** , whereas in **[13]** we constructed the spin 6 algebra.

The body of knowledge concerning the extended $N = 1$ superconformal algebras is smaller in comparison; although using supsersymmetric Toda field theories one can, in principle, construct classical versions of these algebras[3]. The only hitherto known examples are the super Kac-Moody algebras[14] which are extensions by superprimaries (*i.e.* , primaries of the super Virasoro algebra) of spin $\frac{1}{2}$; the $N = 2$ and the small $N = 4$ super Virasoro algebras which are the unique extensions by one or three superprimary fields of spin 1; and the two algebras constructed in **[15]** : the extension by a spin 2 superprimary without self-coupling, which is associative for $c = -\frac{6}{5}$; and the one by a spin $\frac{5}{2}$ superprimary—called[1] super $W_{5/2}$—which is associative for $c = \frac{10}{7}$ and $c = -\frac{5}{2}$. The former value corresponds to the $m = 12$ minimal model[19] in the $N = 1$ unitary series, which possesses an exceptional modular invariant partition function of type[20] (E_6, D_8). It was argued in **[16]** that the superconformal field theory described by that partition function had super $W_{5/2}$ as a symmetry algebra and that the exceptional modular invariant partition function was in fact diagonal when written in terms of super $W_{5/2}$ characters, a fact which now appears to have been proven[21]. A coset construction for super $W_{5/2}$ analogous to the one of Bais *et al.*[22] for W_3 has been given in **[17]** .

In this paper we initiate a systematic investigation of extensions of the $N = 1$ super Virasoro algebra by additional superprimary fields of integer or half-integer spin $\frac{1}{2} \leq \Delta \leq \frac{7}{2}$. We restrict ourselves to fields of integer or half-integer spins since we are interested in algebras defined by the (anti)commutator of their modes. Our method is essentially a perturbative treatment of the conformal bootstrap[23],[24]: we write down the most general operator product expansion consistent with superconformal covariance, compute the superconformal blocks and the four point functions perturbatively, and impose duality order by order thus obtaining constraints

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[1] This algebra was originally termed super W_3 in **[15]** (*cf.* **[16]** and **[17]**); however, as noted in **[18]** , it is more natural to call the algebra generated by a spin Δ superprimary super W_Δ. We follow this terminology in this paper.

on the operator product coefficients and/or the central charge. Our results can be summarized as follows:

- $\Delta = \frac{1}{2}$: super Kac-Moody algebras, associative for all c.
- $\Delta = 1$: $N = 2$ and small $N = 4$ super Virasoro algebras, associative for all c.
- $\Delta = \frac{3}{2}$: direct product of super Virasoro algebras, associative for all c.
- $\Delta = 2$: a new (non-linear) algebra, associative for all c.
- $\Delta = \frac{5}{2}$: only associative for $c = \frac{10}{7}$ and $c = -\frac{5}{2}$.
- $\Delta = 3$: only associative for $c = \frac{5}{4}$, $c = -\frac{45}{2}$, and $c = -\frac{27}{7}$.
- $\Delta = \frac{7}{2}$: only associative for $c = \frac{7}{5}$ and $c = -\frac{17}{11}$.

In particular, notice that apart from the $\Delta = \frac{5}{2}$, $c = \frac{10}{7}$ case already discovered in **[15]** , we find two other values of the central charge corresponding to two $N = 1$ unitary minimal models: $\Delta = 3$, $c = \frac{5}{4}$, which corresponds to $m = 6$; and $\Delta = \frac{7}{2}$, $c = \frac{7}{5}$ which is $m = 10$. We prove that the $m = 6$ model indeed affords a unitary representation of the extended algebra constructed here. Moreover, we show that this extended algebra is the symmetry algebra of the unitary SCFT defined by the (A_7, D_4)-type modular invariant in the Cappelli classification[**20**],[**25**]. The analogous statement is not true for the $m = 10$ and $m = 12$ cases. Nevertheless, an intriguing structure reveals itself in the corresponding exceptional modular invariants (of types (D_6, E_6) and (E_6, D_8), respectively) which suggests that these extended algebras might play a rôle in these theories after all.

The rest of this paper is organized as follows. Section 2 sets the notation and explains the conventions we use for superconformal field theory. It explains in some detail the methods used to compute the coefficients of the superconformal families, the superconformal blocks, and the correlation functions. It also discusses the crossing symmetry constraints on the correlation functions arising from duality. The formulas we obtain differ from the ones usually found in the literature in some spin dependent phases. Since these formulas are crucial in applications of the conformal bootstrap we feel it is important to state them correctly at least once. This section also outlines the group theoretic method we follow to implement crossing symmetry perturbatively. Section 3 contains the results outlined above; although the details of the new spin 2 superconformal extension are relegated to an appendix. Section 4 discusses the relation between some of the extended algebras we find and some of the $N = 1$ unitary minimal models and their modular invariants. Finally section 5 contains some concluding remarks.

§2 Superconformal machinery

$N = 1$ Superconformal Field Theory

In this subsection we briefly discuss our conventions for $N = 1$ superconformal field theory (SCFT)[**26**],[**19**]. We focus on only one chiral sector. The $N = 1$ superconformal algebra is the unique associative extension of the Virasoro algebra by a holomorphic primary field $G(z)$ of weight $\frac{3}{2}$. It is defined by the following operator product expansions (OPEs):

$$
\begin{aligned}
T(z)T(w) =& \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.} \\
T(z)G(w) =& \frac{\frac{3}{2}G(w)}{(z-w)^2} + \frac{\partial G(w)}{z-w} + \text{reg.} \\
G(z)G(w) =& \frac{2c/3}{(z-w)^3} + \frac{2T(w)}{z-w} + \text{reg.}\ .
\end{aligned}
\tag{2.1}
$$

This superalgebra can be interpreted geometrically as the algebra of conformal superdiffeomorphisms of a superspace with points $Z = (z, \theta)$. Exploiting this fact, we can express operator products in a manifestly supercovariant fashion in terms of superfields $\Phi(Z) = \phi(z) + \theta\psi(z)$. The OPEs in (2.1) can indeed be succinctly written in terms of only one OPE involving the superfield $\mathbb{T}(Z) = \frac{1}{2}G(z) + \theta T(z)$:

$$
\mathbb{T}(Z_1)\mathbb{T}(Z_2) = \frac{c/6}{Z_{12}^3} + \frac{\frac{3}{2}\theta_{12}\mathbb{T}(Z_2)}{Z_{12}^2} + \frac{\frac{1}{2}D\mathbb{T}(Z_2)}{Z_{12}} + \frac{\theta_{12}\partial\mathbb{T}(Z_2)}{Z_{12}} + \text{reg.}\ , \tag{2.2}
$$

where $Z_{12} \equiv z_1 - z_2 - \theta_1\theta_2$ and $\theta_{12} \equiv \theta_1 - \theta_2$ are the superintervals and D is the covariant derivative defined by $D\Phi(Z) \equiv \psi(z) + \theta\partial\phi(z)$ and obeying $D^2 = \partial$. In the sequel we shall refer to OPEs in terms of superfields as SOPEs. A primary superfield $\Phi_\Delta(Z) = \phi_\Delta(z) + \theta\psi_{\Delta+1/2}(z)$ of weight Δ obeys the following SOPE with $\mathbb{T}(Z)$:

$$\mathbb{T}(Z_1)\Phi_\Delta(Z_2) = \frac{\Delta\theta_{12}\Phi_\Delta(Z_2)}{Z_{12}^2} + \frac{\frac{1}{2}D\Phi_\Delta(Z_2)}{Z_{12}} + \frac{\theta_{12}\partial\Phi_\Delta(Z_2)}{Z_{12}} + \text{reg.}\ , \tag{2.3}$$

which decodes into the following component OPEs:

$$T(z)\phi_\Delta(w) = \frac{\Delta\phi_\Delta(w)}{(z-w)^2} + \frac{\partial\phi_\Delta(w)}{z-w} + \text{reg.} \tag{2.4}$$

$$T(z)\psi_{\Delta+1/2}(w) = \frac{(\Delta+1/2)\psi_{\Delta+1/2}(w)}{(z-w)^2} + \frac{\partial\psi_{\Delta+1/2}(w)}{z-w} + \text{reg.} \tag{2.5}$$

$$G(z)\phi_\Delta(w) = \frac{\psi_{\Delta+1/2}(w)}{z-w} + \text{reg.} \tag{2.6}$$

$$G(z)\psi_{\Delta+1/2}(w) = \frac{2\Delta\phi_\Delta(w)}{(z-w)^2} + \frac{\partial\phi_\Delta(w)}{z-w} + \text{reg.}\ . \tag{2.7}$$

In other words, $\phi_\Delta(z)$ and $\psi_{\Delta+1/2}(z)$ are Virasoro primaries of weights Δ and $\Delta + 1/2$ respectively. Note that whereas ϕ_Δ is a superconformal primary, $\psi_{\Delta+1/2}$ is its descendent.

The $N = 1$ superconformal algebra has two kinds of representations depending on the monodromy around zero of the supercurrent $G(z)$. Being a spin $\frac{3}{2}$ field it can change by at most a sign. Accordingly, if $G(e^{2\pi i}z) = G(z)$ we say we are in the Neveu-Schwarz (NS) sector and if $G(e^{2\pi i}z) = -G(z)$ in the Ramond (R). The mode expansions for $T(z)$ and $G(z)$ are given by $T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n$ and $G(z) = \sum_r z^{-r-3/2}G_r$, where, in this last sum, r runs over the half-integers or integers in the NS or R sectors respectively. The mode algebra can be read off from (2.1) :

$$\begin{aligned} [L_m\,,\,L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n} \\ [L_m\,,\,G_r] &= (\frac{1}{2}m - r)G_{m+r} \end{aligned} \tag{2.8}$$

$$\{G_r\,,\,G_s\} =2L_{r+s} + \frac{c}{3}(r^2 - \frac{1}{4})\delta_{r,-s}\ .$$

In particular in the NS sector, the subalgebra generated by $\{L_0, L_{\pm 1}, G_{\pm 1/2}\}$, which is isomorphic to $\mathfrak{osp}(1|2)$, is represented without central extension. This subalgebra plays a rôle in SCFT analogous to the one played by the projective subalgebra of the Virasoro algebra in CFT. Geometrically it can be understood as superprojective transformations on the super Riemann sphere[**27**],[**28**]. In particular in a SCFT there is a unique state (the vacuum) which is annihilated by the superprojective subalgebra. Acting on the vacuum with a superprimary field at the origin sets up a bijective correspondence between superprimary fields and highest weight states in the NS sector. On the other hand, to each highest weight state in the R sector there corresponds a spin field[**19**] which creates it when acting on the vacuum. Spin fields are, of course, Virasoro primaries but not superprimaries; in fact, their OPE with $G(z)$ is non local. For the purposes of this paper we will deal mostly with the NS sector of the superconformal algebra since we are mainly interested in investigating the existence of extensions of this algebra by superprimary fields.

For future use we record here the action of the modes of $G(z)$ and $T(z)$ on primary superfields:

$$[L_n\,,\,\Phi_\Delta(Z)] =z^n \left[(n+1)(\Delta + \frac{1}{2}\theta D) + z\partial\right] \Phi_\Delta(Z) \tag{2.9}$$

$$[G_r\,,\,\Phi_\Delta(Z)]_\pm =z^{r-1/2} \left[-\Delta(2r+1)\theta + zQ\right] \Phi_\Delta(Z)\ , \tag{2.10}$$

where $Q\Phi(Z) = \psi(z) - \theta\partial\phi(z)$ is the generator of translations in superspace. It anticommutes with the covariant derivative and obeys $Q^2 = -\partial$.

Superprojective invariance of the vacuum implies that the correlation functions involving (quasi)primary superfields[2] obey two differential equations:

$$\sum_{i=1}^{N} Q_i \langle \Phi_{\Delta_1}(Z_1) \cdots \Phi_{\Delta_N}(Z_N) \rangle = 0 \ , \tag{2.11}$$

$$\sum_{i=1}^{N} (z_i Q_i - 2\Delta_i \theta_i) \langle \Phi_{\Delta_1}(Z_1) \cdots \Phi_{\Delta_N}(Z_N) \rangle = 0 \ . \tag{2.12}$$

In particular this fixes the two point functions of quasiprimaries up to a proportionality constant:

$$\langle \Phi_{\Delta_1}(Z_1) \Phi_{\Delta_2}(Z_2) \rangle \propto \frac{\delta_{\Delta_1,\Delta_2}}{Z_{12}^{2\Delta_2}} \ . \tag{2.13}$$

<u>Superconformal Covariance of the Operator Product Algebra</u>

Just as in ordinary (Virasoro) CFT the local fields assemble themselves into Virasoro families, in SCFT they assemble themselves into superconformal families constructed from the superprimary via the action of the operators $\widehat{L}_{-k}$ and $\widehat{G}_{-r}$ defined by their action on any local field $\phi(z)$:

$$\widehat{L}_{-k} \phi(z) = \oint_{C_z} \frac{d\zeta}{2\pi i} \frac{1}{(\zeta - z)^{k-1}} T(\zeta) \phi(z) \ , \tag{2.14}$$

$$\widehat{G}_{-r} \phi(z) = \oint_{C_z} \frac{d\zeta}{2\pi i} \frac{1}{(\zeta - z)^{r-1/2}} G(\zeta) \phi(z) \ . \tag{2.15}$$

The superconformal family of a superprimary $\phi(z)$ is isomorphic to the NS Verma module constructed from the highest weight vector obtained by acting with $\phi(0)$ on the vacuum.

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[2] By a quasiprimary superfield we mean a superfield which satisfies equation (2.10) for $r = \pm 1/2$.

The SOPE between two primary superfields can be decomposed into superconformal families:

$$\Phi_\Delta(Z)\Phi_{\Delta'}(W) = \sum_{\Delta''} \mathcal{C}_{\Delta\Delta'}^{\Delta''} \left[\Phi_{\Delta''}\right](Z|W) \ , \tag{2.16}$$

where $\left[\Phi_{\Delta''}\right](Z|W)$ is shorthand for the contribution to the above SOPE of the superconformal family of $\Phi_{\Delta''}$ and where $\mathcal{C}_{\Delta\Delta'}^{\Delta''}$ are some constants. For the purposes of this paper we restrict ourselves to integer or half-integer weights Δ, Δ', and Δ''. For convenience we put $W = 0$ in the above expansion. Then we can write $\left[\Phi_{\Delta''}\right](Z|0)$ more explicitly as follows:

$$\left[\Phi_{\Delta''}\right](Z|0) = \sum_{2N\in\mathbb{N}_0} Z^{\Delta''-\Delta'-\Delta+N} \sum_{\{r\},\{k\}} \beta_{\Delta\Delta'}^{\Delta''\,\{r,k\}} \widehat{G}_{-\{r\}} \widehat{L}_{-\{k\}} \Phi_{\Delta''}(0) \, , \tag{2.17}$$

where $\mathbb{N}_0$ denotes the non-negative integers, and the last sum is over all admissible pairs of tuples $\{r\} = \{r_1, \ldots, r_R\}$ and $\{k\} = \{k_1, \ldots, k_K\}$ at level N, that is, subject to the conditions $r_1 > \cdots > r_R > 0$, $k_1 \geq \cdots \geq k_K > 0$, and $\sum_{i=1}^R r_i + \sum_{j=1}^K k_j = N$; and where by $\widehat{G}_{-\{r\}}$ we mean the product $\widehat{G}_{-r_1} \cdots \widehat{G}_{-r_R}$ and similarly for $\widehat{L}_{-\{k\}}$. Finally note that we have introduced the convenient notation $Z^N = z^{N-1/2}\theta$ for N half integral.

To obtain the SOPE $\Phi_\Delta(Z)\Phi_{\Delta'}(W)$ for arbitrary W we merely perform a supertranslation. If $\Phi(Z)$ is a holomorphic superfield then it has a convergent expansion around 0 which we can write as follows:

$$\Phi(Z) = \sum_{2n\in\mathbb{N}_0} \frac{1}{\lfloor n \rfloor !} Z^n D^{2n} \Phi(0) \ , \tag{2.18}$$

where $\lfloor n \rfloor$ means the integer part of n. Let $W = (w, \varphi)$ and let $U(W) = \exp\left(\varphi G_{-1/2} + wL_{-1}\right)$. Then, Φ obeys

$$U(W)^{-1}\Phi(Z)U(W) = \Phi(Z - W) \ , \tag{2.19}$$

where $\Phi(Z - W)$ is defined by the expansion in (2.18) where we put $(Z - W)^n = (Z - W)^{n-1/2}(\theta - \varphi)$ for n half-integral. Then the SOPE $\Phi_\Delta(Z)\Phi_{\Delta'}(W)$ is given in terms of the one for $W=0$ simply by $U(W)\Phi_\Delta(Z-W)\Phi_{\Delta'}(0)U(W)^{-1}$ which, when plugged into (2.17) , becomes (2.16) with

$$
\begin{aligned}
\left[\Phi_{\Delta''}\right](Z|W) = & \sum_{2N\in\mathbb{N}_0} (Z-W)^{\Delta''-\Delta'-\Delta+N} \sum_{\{r\},\{k\}} \beta_{\Delta\Delta'}^{\Delta''\,\{r,k\}} \\
& \times \left(\widehat{G}_{-\{r\}}\widehat{L}_{-\{k\}} + [\varphi\widehat{G}_{-1/2}\,,\,\widehat{G}_{-\{r\}}\widehat{L}_{-\{k\}}]\right)\Phi_{\Delta''}(W)\,.
\end{aligned} \tag{2.20}
$$

The coefficients $\beta_{\Delta\Delta'}^{\Delta''\,\{r,k\}}$ are determined by superconformal covariance of the operator algebra; in other words, by the associativity of the operator product $\mathbb{T}(Z_1)\Phi_\Delta(Z_2)\Phi_{\Delta'}(Z_3)$. We shall need to compute these coefficients for the first few levels, so we digress momentarily to explain their computation. Applying both sides of (2.16) to the vacuum one obtains

$$
\Phi_\Delta(Z)\big|\Delta'\big\rangle = \sum_{\Delta''}\mathcal{C}_{\Delta\Delta'}^{\Delta''}\sum_{2N\in\mathbb{N}_0} Z^{\Delta''-\Delta'-\Delta+N}\left|\,{}^{\Delta''}_{\Delta\Delta'};N\right\rangle\,, \tag{2.21}
$$

where we have defined

$$
\left|\,{}^{\Delta''}_{\Delta\Delta'};N\right\rangle \equiv \sum_{\{r\},\{k\}} \beta_{\Delta\Delta'}^{\Delta''\,\{r,k\}} G_{-\{r\}}L_{-\{k\}}\big|\Delta''\big\rangle\,, \tag{2.22}
$$

where for any Δ, $|\Delta\rangle$ is the highest weight vector obtained by acting with $\Phi_\Delta(0)$ on the vacuum.

Acting on both sides of (2.21) with G_r for $r>0$ and using the fact that $G_r|\Delta'\rangle = 0$ together with (2.10) we obtain the following formulas for the action of $G_{r>0}$ on the states $\left|\,{}^{\Delta''}_{\Delta\Delta'};N\right\rangle$:

$$
G_r\left|\,{}^{\Delta''}_{\Delta\Delta'};N\right\rangle = \begin{cases} \left|\,{}^{\Delta''}_{\Delta\Delta'};N-r\right\rangle & \text{for } \sum\Delta+N\in\mathbb{Z} \\ \left[(2\Delta-1)r+\Delta''-\Delta'+N\right]\left|\,{}^{\Delta''}_{\Delta\Delta'};N-r\right\rangle & \text{for } \sum\Delta+N\in\mathbb{Z}+\frac{1}{2} \end{cases} \tag{2.23}
$$

where by $\sum\Delta$ we mean, here and in the sequel, $\Delta+\Delta'+\Delta''$. From these relations we can read off the effect of the action of $L_{k>0}$:

$$
\begin{aligned}
&L_k\Big|{}^{\Delta''}_{\Delta\Delta'};N\Big\rangle = \\
&\begin{cases}
[(\Delta-1)k+\Delta''-\Delta'+N]\Big|{}^{\Delta''}_{\Delta\Delta'};N-k\Big\rangle & \text{for } \sum\Delta+N\in\mathbb{Z}\\
\left[(\Delta-\frac{1}{2})k+\Delta''-\Delta'+N\right]\Big|{}^{\Delta''}_{\Delta\Delta'};N-k\Big\rangle & \text{for } \sum\Delta+N\in\mathbb{Z}+\frac{1}{2}
\end{cases}
\end{aligned}
\tag{2.24}
$$

In principle these equations are sufficient to compute the $\beta^{\Delta''\,\{r,k\}}_{\Delta\Delta'}$ levelwise; however we find it more convenient to derive a more explicit formula for them involving the inverse of the Šapovalov form of a Verma module of the NS algebra.

Let $\{r'\}=\{r'_1,\ldots,r'_{R'}\}$ and $\{k'\}=\{k'_1,\ldots,k'_{K'}\}$ be a pair of admissible tuples at level N and let $\{\bar{m}\}=\{m_M,\ldots,m_1\}$ for any tuple $\{m\}=\{m_1,\ldots,m_M\}$. Then, by definition,

$$
\begin{aligned}
L_{\{\bar{k}'\}}G_{\{\bar{r}'\}}\Big|{}^{\Delta''}_{\Delta\Delta'};N\Big\rangle &= \sum_{\{r\},\{k\}}\beta^{\Delta''\,\{r,k\}}_{\Delta\Delta'}L_{\{\bar{k}'\}}G_{\{\bar{r}'\}}G_{-\{r\}}L_{-\{k\}}|\Delta''\rangle\\
&= \sum_{\{r\},\{k\}}\beta^{\Delta''\,\{r,k\}}_{\Delta\Delta'}\mathcal{M}^{\{r',k'\}\,\{r,k\}}_{\Delta''}|\Delta''\rangle\ ,
\end{aligned}
\tag{2.25}
$$

where $\mathcal{M}_{\Delta''}$ is the Šapovalov form on the NS Verma module $V(\Delta'',c)$. On the other hand, we can use (2.23) and (2.24) iteratively to obtain an explicit expression for the LHS of (2.22) :

$$
L_{\{\bar{k}\}}G_{\{\bar{r}\}}\Big|{}^{\Delta''}_{\Delta\Delta'};N\Big\rangle = f^{\Delta''\,\{r,k\}}_{\Delta\Delta'}|\Delta''\rangle\ ,
\tag{2.26}
$$

where, for $\{r\}$ and $\{k\}$ a pair of admissible tuples at level N, $f^{\Delta''\,\{r,k\}}_{\Delta\Delta'}$ are given by

$$
\prod_i\left[2\Delta r_i+\Delta''-\Delta'+N-\sum_{j=1}^{i}r_j\right]\prod_{i=1}^{K}\left[\widetilde{\Delta}k_i+\Delta''-\Delta'+\sum_{j=i+1}^{K}k_j\right]\ ,
\tag{2.27}
$$

where the first product is taken over all $1\le i\le R$ obeying $i=2(\sum\Delta+N)$

(mod 2), and where

$$\widetilde{\Delta} = \begin{cases} \Delta & \text{for } \sum \Delta \in \mathbb{Z} \\ \Delta + \frac{1}{2} & \text{for } \sum \Delta \in \mathbb{Z} + \frac{1}{2} \end{cases} . \tag{2.28}$$

Comparing (2.25) with (2.26) we find that

$$\beta_{\Delta\Delta'}^{\Delta''\,\{r,k\}} = \sum_{\{r'\},\{k'\}} \left(\mathcal{M}_{\Delta''}^{-1}\right)^{\{r,k\}\,\{r',k'\}} f_{\Delta\Delta'}^{\Delta''\,\{r',k'\}} ; \tag{2.29}$$

which makes manifest the observation of Al. B. Zamolodchikov **[29]** for the Virasoro case that the analytic behaviour of the $\beta_{\Delta\Delta'}^{\Delta''\,\{r,k\}}$ as a function of c is such that its poles correspond to the zeros of the Šapovalov form.

Duality and the Conformal Bootstrap

The operator algebra of a SCFT is fixed by superconformal covariance up to a few parameters: the dimensions of the superprimary fields and the operator product coefficients $\mathcal{C}_{\Delta\Delta'}^{\Delta''}$. The conformal bootstrap consists of fixing these parameters by demanding duality of the correlators. In CFT, Ward identities relate correlators containing secondary fields to the ones with the primaries from which they descend. Therefore[**24**] it is sufficient to impose duality only on the correlators of primary fields. In SCFT, however, things aren't quite so nice. In fact, superconformal Ward identities relate the correlators involving secondaries to the ones with only superprimaries and their superpartners, *i.e.* , the fields obtained from the superprimaries by acting with $G_{-1/2}$. These latter correlators are not all related by Ward identities, leading to what amounts to a superselection rule. Hence these correlators must be computed separately and duality must be imposed on them independently. For this reason, and since the superpartners are Virasoro primaries, we prefer to discuss duality of correlators involving Virasoro primaries.

It is easy to see that duality of any correlator follows from duality of the general four point functions and that duality of these in turn follow from that of the four point functions involving primaries only. Hence let $\phi_k, \ldots$ denote Virasoro primary fields. The four point functions $\langle\phi_k(z_1)\phi_l(z_2)\phi_n(z_3)\phi_m(z_4)\rangle$ can in principle be computed from the operator product expansions; but this requires the fields to be radially ordered: $|z_1| > |z_2| > |z_3| > |z_4|$. The value of the correlation function for any other ordering is obtained from this one via analytic continuation. Suppose that a different ordering is given for the $\{z_i\}$. Then there are two alternative ways to compute the correlator: we can either analytically continue the correlator computed from the former order, or we can compute it anew performing the operator product expansions in the latter one. Duality simply states that these two methods should yield the same result.

The requirement of duality (or crossing symmetry) of the four point functions translates into a generically infinite set of equations for the parameters in a CFT, which is in general hopeless to study except perturbatively. In order to set up the perturbative expansion it is convenient to work not with the four point function itself but with the related object:

$$\begin{aligned} G^{lk}_{nm}(x) &= \lim_{z\to\infty} z^{2\Delta_k} \langle\phi_k(z)\phi_l(1)\phi_n(x)\phi_m(0)\rangle \\ &= \langle k|\phi_l(1)\phi_n(x)|m\rangle \ , \end{aligned} \tag{2.30}$$

where as usual we define the out state $\langle k| \equiv \lim_{z\to\infty} z^{2\Delta_k}\langle 0|\phi_k(z)$. We can arrive at $G^{lk}_{nm}(x)$ from the general correlation function by making a projective transformation which sends (z_1, z_2, z_3, z_4) to $(\infty, 1, x, 0)$ where $x = (z_1 - z_2)(z_3 - z_4)/(z_1 - z_3)(z_2 - z_4)$ is the anharmonic ratio. The $G^{lk}_{nm}(x)$ can be computed perturbatively in the anharmonic ratio around $x = 0$. The series is guaranteed to have a finite radius of convergence if the OPE does, this being a basic axiom of CFT. But before describing the perturbative evaluation of $G^{lk}_{nm}(x)$ in terms of the (super)conformal blocks, we digress to derive the crossing relations for the $G^{lk}_{nm}(x)$.

Under a projective transformation $z \mapsto z' = (az+b)/(cz+d)$, with $ad-bc = 1$, a (quasi)primary field transforms as $\phi_k(z) \mapsto (cz+d)^{-2\Delta_k}\phi_k(z')$. When Δ_k is not an integer there is a phase ambiguity in the transformation law since (a, b, c, d)

and $(-a,-b,-c,-d)$ define the same transformation on the points. Fortunately this ambiguity is absent from the transformation laws of the correlation functions since we restrict ourselves to integer and half-integer weights only and in any non-vanishing correlator the sum of the weights is always an integer. Let us then perform the projective transformation $z \mapsto 1-z$ on $G^{lk}_{nm}(x)$. This transformation sends $(\infty,1,x,0)$ to $(\infty,0,1-x,1)$ and hence the radial order is upset. Reordering and taking into account the possible signs in which we may incur by commuting fermionic fields, we find that duality implies

$$G^{lk}_{nm}(x) = (-1)^{4\Delta_l(\Delta_n+\Delta_m)+4\Delta_n\Delta_m+3\Delta_k+\Delta_l+\Delta_m+\Delta_n} G^{mk}_{nl}(1-x) \ . \tag{2.31}$$

Similarly for the transformation $z \mapsto 1/z$, we find

$$G^{lk}_{nm}(x) = (-1)^{4(\Delta_k+\Delta_m)(\Delta_l+\Delta_n)+4\Delta_k\Delta_m+\Delta_k+\Delta_l+\Delta_m+\Delta_n} x^{-2\Delta_n} G^{lm}_{nk}(1/x) \ . \tag{2.32}$$

These signs are different from the ones usually found in the literature. We have verified them explicitly in a variety of examples. At the end of this section we will describe how to implement the crossing relations from a perturbative knowledge of the $G^{lk}_{nm}(x)$ around $x=0$; but first we discuss their perturbative evaluation.

Superconformal Blocks

The operator expansion $\phi_n(x)\phi_m(0)$ between two superprimary fields can be read off from the θ independent part of the SOPE $\Phi_n(X)\Phi_m(0)$ for $X=(x,\theta)$:[3]

[3] Sometimes it is convenient to rescale the primaries as follows $\Phi_m(Z) = \xi_m\phi_m(z) + \theta\psi_m(z)$ in such a way that the two point functions of both ϕ_m and ψ_m are normalized according to the standard convention that a Virasoro primary field ϕ_Δ (not equal to the identity) obeys $\langle\phi_\Delta(z)\phi_\Delta(0)\rangle = (c/\Delta)z^{-2\Delta}$. In that case, the operator product coefficients C^p_{nm} are related to those appearing in the SOPE by the formula $C^p_{nm} = \xi_p\xi_m^{-1}\xi_n^{-1}\mathcal{C}^p_{nm}$.

$$\phi_n(x)\phi_m(0) = \sum_p C^p_{nm} \sum_{N\in\mathbb{N}_0} x^{\Delta_p-\Delta_n-\Delta_m+\widetilde{N}} \sum_{\{r\},\{k\}} \beta^{p\,\{r,k\}}_{nm} \widehat{G}_{-\{r\}} \widehat{L}_{-\{k\}} \phi_p(0) \ , \tag{2.33}$$

where

$$\widetilde{N} = \begin{cases} N & \text{if } \Delta_p+\Delta_n+\Delta_m \in \mathbb{Z} \\ N+\frac{1}{2} & \text{if } \Delta_p+\Delta_n+\Delta_m \in \mathbb{Z}+\frac{1}{2} \end{cases} \ , \tag{2.34}$$

and where the second sum is over admissible pairs of tuples $\{r\}$, $\{k\}$ at level $\widetilde{N}$. In particular notice that for $\Delta_p+\Delta_n+\Delta_m \in \mathbb{Z}+\frac{1}{2}$ the first contribution from the superconformal family $[\phi_p]$ does not come from the primary field itself but from its descendent $\widehat{G}_{-1/2}\phi_p(0)$. This is a phenomenon of extended algebras which does not occur in Virasoro CFT. Plugging (2.33) into (2.30) we find

$$G^{lk}_{nm}(x) = \sum_p C^p_{nm} C_{lpk} \mathcal{F}^{lk}_{nm}(p|x) \ , \tag{2.35}$$

where the superconformal blocks $\mathcal{F}^{lk}_{nm}(p|x)$ are defined via

$$C_{lpk}\mathcal{F}^{lk}_{nm}(p|x) = \sum_{N\in\mathbb{N}_0} x^{\Delta_p-\Delta_n-\Delta_m+\widetilde{N}} \sum_{\{r\},\{k\}} \beta^{p\,\{r,k\}}_{nm} \langle k|\phi_l(1) G_{-\{r\}} L_{-\{k\}}|p\rangle \ , \tag{2.36}$$

where $C_{lpk} = \langle k|\phi_l(1)|p\rangle = \sum_q C^q_{lp} C^0_{kq}$. The above matrix elements can be calculated by (anti)commuting the super Virasoro modes to the left and using the facts that ϕ_l is superprimary and that they annihilate the out state. The result is, in fact, given in terms of the $f^{p\,\{r,k\}}_{lk}$ (defined in (2.27))

$$\mathcal{F}^{lk}_{nm}(p|x) = \varepsilon \sum_{N\in\mathbb{N}_0} x^{\Delta_p-\Delta_n-\Delta_m+\widetilde{N}} \sum_{\substack{\{r\},\{k\}\\ \{r'\},\{k'\}}} f^{p\,\{r,k\}}_{lk} \left(\mathcal{M}^{-1}_p\right)^{\{r,k\}\,\{r',k'\}} f^{p\,\{r',k'\}}_{nm} \ , \tag{2.37}$$

where

$$\varepsilon = \begin{cases} 1 & \text{if } \Delta_p+\Delta_k+\Delta_l \in \mathbb{Z} \\ (-1)^{2\Delta_l+1} & \text{if } \Delta_p+\Delta_k+\Delta_l \in \mathbb{Z}+\frac{1}{2} \end{cases} \ ,$$

and where we have used (2.29) to express the $\beta^{p\,\{r,k\}}_{nm}$ in terms of the $f^{p\,\{r,k\}}_{nm}$. As expected, (2.37) resembles the result in the Virasoro case[30],[13].

As a consequence of the superselection rules described in the previous subsection the above superconformal blocks are not sufficient to compute all correlation functions. In principle one could repeat the above calculations for four point functions containing superpartners and in the process obtain expressions for the other "superconformal blocks" in terms of which, and also of the ones given by (2.37) , any correlation function could then be written. Unfortunately, there does not seem to be a closed form expression analogous to that of (2.37) for all of these other blocks. We therefore find it more convenient to compute these other correlators by first decomposing the relevant OPEs into Virasoro families and then using the conformal (Virasoro) blocks. Moreover this provides a nontrivial check on our calculations of the superconformal family coefficients. The analogous formulas to (2.37) and (2.27) for the conformal blocks have been derived in **[13]** .

Crossing Symmetry Constraints

We now describe a method essentially due to Bouwknegt**[11]** to implement the crossing symmetry constraints perturbatively. Although this method was applied in **[11]** to the special case of four point functions involving only one primary field, we shall need its straightforward generalization to the case of two distinct primaries. Assume then that only two different conformal weights occur in $G^{lk}_{nm}(x)$. In this case the crossing symmetry conditions can be rewritten as

$$G^{lk}_{nm}(x) = G^{mk}_{nl}(1-x) = (-1)^{4\Delta_l\Delta_n} x^{-2\Delta_n} G^{lm}_{nk}(1/x) \ . \tag{2.38}$$

These conditions can be given a very natural group theoretic interpretation. Consider first the case of all fields being the same and of weight Δ. Denote by $\mathbb{S}_\Delta$ the $4\Delta+1$ dimensional vector space spanned by $x^{-2\Delta}, \dots, x^{-1}, 1, (1-x)^{-1}, \dots, (1-x)^{-2\Delta}$. The symmetric group[4] $\mathfrak{S}_3$ acts on $\mathbb{S}_\Delta$ via the following transformations:

---

[4] This is the finite group of order 6 generated by S and T subject to the relations $S^2 = T^2 = (ST)^3 = 1$. It is isomorphic to the dihedral group $\mathcal{D}_3$: the symmetry group of the equilateral triangle.

$$
\begin{aligned}
(S\cdot f)(x) =&(-x)^{-2\Delta}f(1/x)\\
(T\cdot f)(x) =&f(1-x)\ ,
\end{aligned}
$$

for any function $f\in\mathbb{S}_\Delta$. Crossing symmetry of the $G^{\Delta\Delta}_{\Delta\Delta}(x)$ is then precisely the statement that it be in $\mathbb{S}_\Delta^{\mathfrak{S}_3}$ – the $\mathfrak{S}_3$-invariants of $\mathbb{S}_\Delta$, which is spanned by the $\mathfrak{S}_3$ orbits of some of the basis functions. Comparing term by term the perturbative expansion of $G^{\Delta\Delta}_{\Delta\Delta}(x)$ around $x=0$ with that of a linear combination of the basis of the invariants of $\mathbb{S}_\Delta^{\mathfrak{S}_3}$ gives a set of linear equations involving the operator product coefficients, the central charge, and the parameters in the linear combination of the invariant functions; which, generically, is overdetermined. Nevertheless, we are only interested in the equations coming from the poles at $x=0$; the reason being that the equations coming from regular terms are to be solved by adding new superprimaries which, since they only appear in the regular terms of the OPE are not present in the mode algebra.

Consider now the case where two distinct fields of weights Δ and Δ' appear in $G^{lk}_{nm}(x)$. Crossing transformations now permute the $\{k,l,m\}$ indices of the correlator as well as their functional dependence on x. We shall only be interested in the case where two of the fields have weight Δ and the other two Δ'. Therefore the correlators $G^{\Delta'\Delta'}_{\Delta\Delta}(x)$, $G^{\Delta\Delta'}_{\Delta\Delta'}(x)$, and $G^{\Delta'\Delta}_{\Delta\Delta'}(x)$ will transform into each other. We find it convenient therefore to introduce an auxiliary 3-dimensional vector space $\mathbb{V}$ spanned by $\{e_1,e_2,e_3\}$ and a representation of $\mathfrak{S}_3$ on $\mathbb{V}$ given by the following matrices relative to this basis

$$
T=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\ ,\quad S=\begin{pmatrix}0&\sigma&0\\\sigma&0&0\\0&0&1\end{pmatrix}\ , \tag{2.39}
$$

where $\sigma=(-1)^{2\Delta+4\Delta\Delta'}$. Then the crossing symmetry conditions are equivalent to the statement that

$$
G(x)\equiv G^{\Delta'\Delta'}_{\Delta\Delta}(x)\otimes e_1+G^{\Delta'\Delta}_{\Delta\Delta'}(x)\otimes e_2+G^{\Delta\Delta'}_{\Delta\Delta'}(x)\otimes e_3\in\mathbb{S}_\Delta\otimes\mathbb{V} \tag{2.40}
$$

be an element of the invariant subspace $(\mathbb{S}_\Delta\otimes\mathbb{V})^{\mathfrak{S}_3}$. This $2\Delta+1$ dimensional subspace is spanned by the orbits of a subset of the basis elements. Comparing order by order (and e_i by e_i) the Laurent expansion of $G(x)$ about $x = 0$ with that of a generic element of $(\mathbb{S}_\Delta \otimes \mathbb{V})^{\mathfrak{S}_3}$ we once again obtain a set of linear equations which, after determining the free parameters in the linear combination of invariants, impose further constraints on the OPE coefficients and/or the central charge. It is precisely these conditions which we will analyze systematically in the next section to determine possible extensions of the $N = 1$ superconformal algebra. It should be remarked that the extension of the above method to the case of the general four point function $G^{lk}_{nm}(x)$ is completely straightforward. Crossing transformations leave fixed the index n and hence in the case of the fields ϕ_k, ϕ_l, ϕ_m being different we merely tensor $\mathbb{S}_{\Delta_n}$ with a 6 dimensional representation $\mathbb{W}$ of $\mathfrak{S}_3$ to take into account the permutation of the indices. Crossing symmetry is then equivalent to the analogue of (2.40) being an element of the $4\Delta_n + 1$ dimensional space $(\mathbb{S}_{\Delta_n} \otimes \mathbb{W})^{\mathfrak{S}_3}$.

§3 Extended Superconformal Algebras

In this section we turn to the explicit construction of extended $N = 1$ superconformal algebras. We will mainly focus on those algebras which can be obtained by extending the $N = 1$ super Virasoro algebra by one superprimary ϕ_Δ of spin $\frac{1}{2} \leq \Delta \leq \frac{7}{2}$ subject to the following OPE:

$$\phi_\Delta \times \phi_\Delta = C^0_{\Delta\Delta}[\phi_0] + C^\Delta_{\Delta\Delta}[\phi_\Delta] + \text{regular terms} , \tag{3.1}$$

where $[\phi]$ denotes the superconformal family of the superprimary ϕ. In general, the self-coupling $C^\Delta_{\Delta\Delta}$ can only be nonzero for $\Delta = 2n$ or $2n + \frac{3}{2}$, with $n \in \mathbb{N}_0$. We leave open the possibility that further superprimaries may appear in the regular terms of the above OPE so that their presence may not be detectable in the mode algebra. The requirement that no new superprimaries may appear at all is much stronger and, in fact, always constrains the central charge to a finite set of values. In the case of $C^\Delta_{\Delta\Delta} = 0$ this has been investigated in **[30]** and **[31]** for the case of the Virasoro algebra and in **[32]** for the case of the $N = 1$ super Virasoro algebra.

The method we follow is the following. We write down the most general OPE of the fields involved which is consistent with superconformal covariance. The only free parameters are the central charge and the possible coupling(s). Following the (super)conformal bootstrap approach, we now impose crossing symmetry of the four point functions. As explained in the previous section, it is sufficient to check the correlators involving the primary fields and their superpartners. There is also good empirical evidence to suggest that it is sufficient to consider correlators involving only primaries or only superpartners. We do, however, also check the mixed correlators. The relevant four point functions are computed perturbatively either from the superconformal blocks (in the case of primaries) or from the conformal blocks, after breaking up the superconformal families into Virasoro ones. As we saw in the previous section, one can effectively impose crossing symmetry just from a perturbative knowledge of the correlators as long as one knows all the poles of the correlators $G^{nm}_{lk}(x)$ at $x = 0$.

Intuitively one should expect the following results. For $\Delta < 2$ the algebras are Lie superalgebras and one should recover the standard results. For $\Delta = 2$ one should find an algebra that satisfies duality for all values of the central charge. Indeed the study of the case without self-coupling in **[15]** revealed that the central charge was fixed. Including the coupling one expects this parameter to be fixed instead of the central charge. This is further supported by the existence of a classical version of this algebra arising from super Toda field theory[**3**]. For $\Delta > 2$ counting arguments similar to those in **[11]** teaches us that these algebras should only exist for a finite set of values of the central charge.

We now discuss our results case by case. In most cases we omit the details concerning the explicit form of the algebra but give schematically the Virasoro decomposition of the operator algebra whose couplings are unambiguously determined in terms of the couplings in the SOPE by superconformal covariance. We also discuss the origin of the restriction on the coupling (if any) and/or on the central charge for each algebra.

$\underline{\Delta = \frac{1}{2}}$

We introduce a primary superfield $\Phi_{1/2}(Z) = \phi_{1/2}(z) + \theta\phi_1(z)$. In this case there is no self-coupling and the OPEs are those of a free fermion and a free boson whose modes commute. This corresponds to a $\mathfrak{u}(1)$ super Kač-Moody algebra. The generalization to more than one primary superfield of weight $\frac{1}{2}$ is straightforward and gives rise to more general super Kač-Moody algebras[14]. We omit the details since these algebras are well known.

$\underline{\Delta = 1}$

Let $\Phi_1(Z) = \phi_1(z) + \theta\phi_{3/2}(z)$. Again there is no self-coupling possible. The resulting mode algebra is a Lie superalgebra with the following Virasoro primaries: ϕ_0, G, ϕ_1, and $\phi_{3/2}$. It is a classical result[33] that the only such algebra with this field content is the $N = 2$ super Virasoro algebra whose OPEs are given (schematically) by

$$\begin{aligned} \phi_1 \times \phi_1 &\to [\phi_0] \\ \phi_{3/2} \times \phi_{3/2} &\to [\phi_0] \\ \phi_1 \times \phi_{3/2} &\to [G] \\ G \times \phi_1 &\to [\phi_{3/2}] \\ G \times \phi_{3/2} &\to [\phi_1] \\ G \times G &\to [\phi_0] \ . \end{aligned} \tag{3.2}$$

This algebra has been the center of a lot attention recently because of its connections to string theory (since $N = 1$ spacetime supersymmetry requires $N = 2$ superconformal symmetry on the worldsheet[34]), Landau-Ginzburg theories[35], and topological field theory[36].

Adding $n > 1$ weight 1 superprimaries (and nothing else) yields an associative algebra for all values of the central charge only for $n = 3$, resulting in the small $N = 4$ super Virasoro algebra.

$\underline{\Delta = \frac{3}{2}}$

Analogously to the conformal case treated by Zamolodchikov[6] the addition of n primary spin $\frac{3}{2}$ superfields $\Phi^i_{3/2}(Z) = \phi^i_{3/2}(z) + \theta\phi^i_2(z)$, where $i = 1, \ldots, n$, yields an algebra which is isomorphic to the direct product of $n+1$ copies of the $N = 1$ super Virasoro algebra. Indeed, since the resulting algebra is a Lie superalgebra it is actually easier to impose associativity by satisfying the Jacobi identities on the modes. Letting $\Phi^0_{3/2}(Z) = \mathbb{T}(Z)$ and extending the range of the indices from 0 to n, the algebra takes the following form

$$\Phi^i_{3/2} \times \Phi^j_{3/2} \to \sum_{k=0}^{n} \mathcal{C}^{ij}{}_k [\Phi^k_{3/2}] \ . \tag{3.3}$$

The Jacobi identities of the modes translate into the requirement that the couplings $\mathcal{C}^{ij}{}_k$ be the structure constants defining an associative commutative algebra. As is well known such an algebra can be diagonalized yielding the aforementioned direct product structure.

$\underline{\Delta = 2}$

Decomposing the SOPE of a primary superfield $\Phi_2(Z) = \phi_2(z) + \theta\phi_{5/2}(z)$ into Virasoro families we find:

$$\begin{aligned} \phi_2 \times \phi_2 &\to [\phi_0] + [\phi_2] \\ \phi_{5/2} \times \phi_{5/2} &\to [\phi_0] + [\phi_2] + [\phi_4] + [\widetilde{\phi}_4] \\ \phi_2 \times \phi_{5/2} &\to [G] + [\phi_{5/2}] + [\phi_{7/2}] \ , \end{aligned} \tag{3.4}$$

where $\{\phi_0, G, \phi_4, \ldots\}$ is the decomposition into Virasoro primaries of the superconformal family of the identity, and $\{\phi_2, \phi_{5/2}, \phi_{7/2}, \widetilde{\phi}_4, \ldots\}$ is that of the superconformal family of ϕ_2. We want to stress the fact that the couplings in the above operator products are completely determined from the self-coupling C^2_{22} of the superprimary by superconformal covariance. It should also be remarked that this is the first of the algebras we present which is not a Lie superalgebra since it contains quadratic terms in the RHS of the (anti)commutator of the modes.

The four point function $\langle \phi_2\phi_2\phi_2\phi_2 \rangle$ is crossing symmetric for any value of the central charge and of the self-coupling, whereas $\left\langle \phi_{5/2}\phi_{5/2}\phi_{5/2}\phi_{5/2} \right\rangle$ fixes the square of the self-coupling to

$$\left(C_{22}^{2}\right)^{2} = \frac{4(6+5c)^{2}}{(21+4c)(15-c)} \ , \tag{3.5}$$

in a normalization where $C_{22}^{0} = c/2$. In the sequel we will state all numerical results for the self-couplings in the standard normalization $C_{\Delta\Delta}^{0} = c/\Delta$ for the superprimary ϕ_{Δ}. Notice that the coupling vanishes precisely for $c = -\frac{6}{5}$ in agreement with the result of **[15]** where the case without self-coupling was treated. Furthermore the mixed correlators $\left\langle \phi_{5/2}\phi_{5/2}\phi_2\phi_2 \right\rangle, \ldots$ are crossing symmetric without imposing any constraints on the central charge, as we had expected. Therefore the mode algebra is associative for all values of the central charge. In the appendix we write the algebra down explicitly.

Recently, a classical version of super W_2 was constructed from the supersymmetric Toda field theory corresponding to $osp(3|2)$ **[3]** . This algebra was later quantised in **[37]** although the expression therein differs from the one given here. In fact, due to some computational errors which now seem to have been clarified[**38**], the form of the algebra in **[37]** is wrong, as can be easily inferred from the fact that it does not decompose into superconformal families.

Notice that for $c = -\frac{6}{5}$ the self-coupling vanishes and hence the superconformal family of the spin 2 superprimary does not appear. Thus, our results imply the existence of an extended conformal algebra for this value of the central charge, with $\phi_2 \times \phi_2 = C_{22}^{0}\phi_0 +$ regular terms. This algebra is precisely the one discovered in **[15]** . Alternatively, one can put $c = -\frac{6}{5}$ in the explicit form of the algebra as given in the appendix. One then sees that some superdescendents of W do remain, since the zero in the self-coupling is cancelled by a pole in the superconformal family coefficients. It has been remarked[**37**],[**38**] that the resulting algebra is then essentially different from that found in **[15]** . This, however, is not the case. Indeed, it is easy to see that the descendent fields which remain are null for this particular value of the central charge, hence decoupling from any correlation function. Therefore, they can be consistently set to zero and thus the "physical" content of the two algebras is the same. This implies that in a free field realization of super W_2 these null fields should be identically zero for that value of the central charge.

$\underline{\Delta = \frac{5}{2}}$

The extension of the $N = 1$ super Virasoro algebra by a primary superfield $\Phi_{5/2}(Z) = \phi_{5/2}(z) + \theta\phi_3(z)$ was investigated in detail in **[15]** to where we refer the reader for details. The algebra is only associative for $c = \frac{10}{7}$ and $c = -\frac{5}{2}$. As discussed in the introduction this algebra—super $W_{5/2}$— plays a rôle in the $c = \frac{10}{7}$ SCFT defined by the (E_6, D_8) exceptional modular invariant. We will try to illustrate this in the next section. Recently a proposed extension of this algebra (for all values of the central charge) has been given in **[39]** and consists of 8 generating fields.

$\underline{\Delta = 3}$

In the case of a primary superfield $\Phi_3(Z) = \phi_3(z) + \theta\phi_{7/2}(z)$ there is no self-coupling and the decomposition of the algebra into Virasoro primaries is schematically given by:

$$\begin{aligned} \phi_3 \times \phi_3 &\to [\phi_0] + [\phi_4] \\ \phi_{7/2} \times \phi_{7/2} &\to [\phi_0] + [\phi_4] + [\phi_6] \\ \phi_3 \times \phi_{7/2} &\to [G] \ , \end{aligned} \tag{3.6}$$

where $\{\phi_0, G, \phi_4, \phi_6, \ldots\}$ is the decomposition of the superconformal family of the identity into Virasoro primaries. In particular, ϕ_4 is the same primary field which appeared in the $\Delta = 2$ case.

Crossing symmetry of $\langle\phi_3\phi_3\phi_3\phi_3\rangle$ is satisfied for all values of the central charge, whereas $\left\langle\phi_{7/2}\phi_{7/2}\phi_{7/2}\phi_{7/2}\right\rangle$ fixes the central charge to be $\frac{5}{4}$, $-\frac{27}{7}$, and $-\frac{45}{2}$. Moreover the mixed correlators do not impose any further constraints on the central charge.

In the next section we will exhibit this algebra as the symmetry algebra of the $c = \frac{5}{4}$ SCFT defined by the (A_7, D_4)-type modular invariant.

$\underline{\Delta = \frac{7}{2}}$

Finally we discuss the algebra obtained by adding a primary superfield $\Phi_{7/2}(Z) = \phi_{7/2}(z) + \theta\phi_4(z)$. In this case the field can couple to itself. The decomposition of the OPEs into Virasoro primaries is now a little more complicated:

$$\begin{aligned} \phi_{7/2} \times \phi_{7/2} &\to [\phi_0] + [\phi_4] + [\widetilde{\phi}_4] + [\phi_6] + [\widetilde{\phi}_6] \\ \phi_4 \times \phi_4 &\to [\phi_0] + [\phi_4] + [\widetilde{\phi}_4] + [\phi_6] + [\widetilde{\phi}_6] \\ \phi_{7/2} \times \phi_4 &\to [G] + [\phi_{7/2}] + [\phi_{11/2}] + [\phi_{13/2}] \ , \end{aligned} \tag{3.7}$$

where $\{\phi_0, G, \widetilde{\phi}_4, \widetilde{\phi}_6, \ldots\}$ and $\{\phi_{7/2}, \phi_4, \phi_{11/2}, \phi_6, \phi_{13/2}, \ldots\}$ are the Virasoro decompositions of the superconformal families of ϕ_0 and $\phi_{7/2}$ respectively.

Duality of the four point function $\left\langle \phi_{7/2}\phi_{7/2}\phi_{7/2}\phi_{7/2} \right\rangle$ fixes the square of the self-coupling to a rational function of the central charge and $\langle \phi_4\phi_4\phi_4\phi_4 \rangle$ restricts the central charge to take only the following values: $c = \frac{7}{5}$ and $c = -\frac{17}{11}$. Again, the mixed correlators do not restrict the central charge further.

In conclusion, the values for the self-couplings of this algebra are given by:

$$\left(C_{7/2,7/2}^{7/2}\right)^2 = \begin{cases} \frac{4563}{25840} & \text{for } c = \frac{7}{5} \\ \\ -\frac{34460181}{4187144} & \text{for } c = -\frac{17}{11} \end{cases} . \tag{3.8}$$

After a preliminary version of this paper, we received a preprint by Hornfeck[40] confirming this result via an explicit check of the Jacobi identities.

We shall comment on the possible rôle of this algebra in the $c = \frac{7}{5}$ SCFT defined by the (D_6, E_6)-type modular invariant in the next section.

§4 $N = 1$ Superconformal Unitary Minimal Models with Extended Symmetry

In this section we explore the relationship between some of the extended algebras constructed in the previous section and some of the unitary minimal models of the $N = 1$ super Virasoro algebra[26],[19]. The central charge of these models are indexed by a positive integer $m \geq 3$

$$c(m) = \frac{3}{2}\left(1 - \frac{8}{m(m+2)}\right) , \tag{4.1}$$

and the spectrum is given by

$$\Delta_{r,s} = \frac{((m+2)r - ms)^2 - 4}{8m(m+2)} + \frac{\varepsilon}{16} , \tag{4.2}$$

where $1 \leq r \leq m-1$, $1 \leq s \leq m+1$, and $\varepsilon = 0$ for $r-s$ even (NS sector) and $\varepsilon = 1$ for $r-s$ odd (R sector).

Notice that for $m = 6$, $c = \frac{5}{4}$ and $\Delta_{5,1} = \Delta_{1,7} = 3$; for $m = 10$, $c = \frac{7}{5}$ and $\Delta_{9,5} = \Delta_{1,7} = \frac{7}{2}$; and for $m = 12$, $c = \frac{10}{7}$ and $\Delta_{5,1} = \Delta_{7,13} = \frac{5}{2}$; and that these are the positive c-values for which the relevant extended algebras were found to exist. It follows from the Cappelli classification of modular invariants for the unitary super Virasoro minimal models, that for these c-values there exist non-diagonal (in the super Virasoro characters) invariants defining SCFTs in which either the primary or its superpartner appear—for $m = 6$ we find the (A_7, D_4)-type invariant, for $m = 10$ the (D_6, E_6)-type invariant, and the (E_6, D_8)-type invariant for $m = 12$. It is therefore tempting to conjecture that these extended algebras are in fact realized in these theories. However, this is clearly not sufficient. Indeed, one must verify that the representations of the super Virasoro algebra appearing in that particular SCFT assemble themselves into representations of the extended algebra. To do this it is convenient to study the fusion rules of the relevant minimal model.

The fusion rules for the minimal models of the $N = 1$ super Virasoro algebra were derived in [**26**] for the NS sector and the R sector was incorporated in [**41**] . Using them we will see that for $m = 6$ the extended spin 3 algebra is indeed realized in the fusion rules, whereas for $m = 10$ and $m = 12$ we have not been able to show that this is the case. We therefore discuss them in separate subsections—the one for $m = 6$ being the more detailed, since our results in this case are more conclusive.

$\underline{m = 6}$

The Kač table of this minimal model is given by

5	3	$\frac{67}{32}$	$\frac{5}{4}$	$\frac{23}{32}$	$\frac{1}{4}$	$\frac{3}{32}$	0
4	$\frac{29}{16}$	$\frac{33}{32}$	$\frac{9}{16}$	$\frac{5}{32}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{5}{16}$
3	$\frac{5}{6}$	$\frac{41}{96}$	$\frac{1}{12}$	$\frac{5}{96}$	$\frac{1}{12}$	$\frac{41}{96}$	$\frac{5}{6}$
2	$\frac{5}{16}$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{5}{32}$	$\frac{9}{16}$	$\frac{33}{32}$	$\frac{29}{16}$
1	0	$\frac{3}{32}$	$\frac{1}{4}$	$\frac{23}{32}$	$\frac{5}{4}$	$\frac{67}{32}$	3
r/s	1	2	3	4	5	6	7

From the fusion rules[5] one sees that the spin 3 primary ϕ_3 obeys

$$\phi_3 \times \phi_3 = \phi_0 \ , \tag{4.3}$$

which already proves that the spin 3 extended algebra found in the previous section is realized in this minimal model. Equation (4.3) implies[**42**] that ϕ_3 is a simple current[**43**] of order 2. Therefore it breaks up the primaries and their superpartners into orbits of at most length 2. Explicitly the fusion products with ϕ_3 in the NS

[5] For notational convenience we denote the primary field corresponding to the Kač label (r, s) by $\phi_{\Delta_{r,s}}$.

sector are given by

$$
\begin{aligned}
\phi_3 \times \phi_{1/4} &= \phi_{5/4} \\
\phi_3 \times \phi_{1/32} &= \phi_{\overline{33/32}} \\
\phi_3 \times \phi_{5/32} &= \phi_{\overline{5/32}} \\
\phi_3 \times \phi_{5/6} &= \phi_{5/6} \\
\phi_3 \times \phi_{1/12} &= \phi_{1/12} \ ,
\end{aligned}
\tag{4.4}
$$

and in the R sector by

$$
\begin{aligned}
\phi_3 \times \phi_{3/32} &= \phi_{67/32} \\
\phi_3 \times \phi_{1/16} &= \phi_{9/16} \\
\phi_3 \times \phi_{5/16} &= \phi_{29/16} \\
\phi_3 \times \phi_{23/32} &= \phi_{23/32} \\
\phi_3 \times \phi_{5/96} &= \phi_{5/96} \\
\phi_3 \times \phi_{41/96} &= \phi_{41/96} \ .
\end{aligned}
\tag{4.5}
$$

In the NS sector we denote by $\phi_{\overline{\Delta}}$ the superpartner of ϕ_Δ. Although these superpartners are not primary with respect to the super Virasoro algebra, they are with respect to the chiral algebra of the SCFT (which consists of the maximal bosonic subalgebra of the universal enveloping algebra of the super Virasoro algebra) and hence they can appear in the fusion products. Their fusion products, however, follow from those of the superprimaries by acting with the supercurrent G (itself a simple current of order 2) and hence we omit them. In particular, this means that the superpartner $\phi_{\overline{3}}$ is also a simple current of order 2. Recall that the primaries in the R sector are doubly degenerate; although we shall not clutter the notation distinguishing between them.

The procedure outlined in **[43]** in order to obtain a non-diagonal modular invariant as a result of the existence of a simple current can be straightforwardly applied in this case yielding, starting from the diagonal modular invariant, the following partition function:

$$
\begin{aligned}
Z =&\frac{1}{2}|\chi_0+\chi_3|^2+\frac{1}{2}|\chi_{1/4}+\chi_{5/4}|^2+|\chi_{5/6}|^2+|\chi_{1/12}|^2+(\chi\leftrightarrow\widetilde{\chi}) \\
&+\frac{1}{2}|\widehat{\chi}_{3/32}+\widehat{\chi}_{67/32}|^2+|\widehat{\chi}_{23/32}|^2+|\widehat{\chi}_{41/96}|^2+\frac{1}{2}|\widehat{\chi}_{5/96}|^2 ,
\end{aligned}
\tag{4.6}
$$

where χ denotes the NS character; $\widetilde{\chi}$ the NS character with a $(-1)^F$ insertion; and $\widehat{\chi}$ denotes the R character. This modular invariant is precisely the (A_7, D_4) invariant in the Cappelli classification. Notice that the SCFT defined by this invariant does contain a holomorphic primary of spin 3 that generates the extended algebra found in the previous section. Hence, this invariant seems to provide us with an explicit example of a SCFT with super W_3 symmetry.

<u>$m = 10$ and $m = 12$</u>

We refrain from giving the Kač table for these models since they involve too many primaries. The fusion rules for the $m = 10$ model teach us that $\phi_{7/2}$ is not a simple current, but rather it satisfies the following fusion product:

$$
\phi_{7/2}\times\phi_{7/2}=\phi_0+\phi_{3/2}+\phi_{19/3}+\phi_{\overline{7/2}}+\phi_{\overline{1/3}} . \tag{4.7}
$$

Therefore, unlike in the $m = 6$ model, one cannot a priori conclude that this minimal model realizes super $W_{7/2}$ symmetry. Nevertheless, an intriguing algebraic structure surrounds the spin 7/2 field. First of all, $\phi_{7/2}$ generates a subalgebra of this minimal model containing (apart from itself) those primaries appearing in the above fusion product, their superpartners and a simple current J of order 2 (and its superpartner) of spin $\Delta_{9,1} = 10$ which pairs up $\phi_{7/2}$ with $\phi_{3/2}$, and $\phi_{1/3}$ with $\phi_{19/3}$. Moreover $\phi_{7/2}$ has unique fusion rules with a number of other primaries in this model (not belonging to this subalgebra) and, remarkably, these fields all contribute to the exceptional modular invariant of the (D_6, E_6)-type. Indeed this invariant has the following structure:

$$
Z =\frac{1}{4}\sum_{\Delta}|\chi_\Delta+\chi_{\phi\Delta}+\chi_{J\Delta}+\chi_{J\phi\Delta}|^2+(\chi\leftrightarrow\widetilde{\chi})
$$

$$+\frac{1}{2}\sum_{\Delta}|\widehat{\chi}_{\Delta}+\widehat{\chi}_{J\Delta}|^2 \ , \tag{4.8}$$

where $\chi_{\phi\Delta}$ denotes the character associated with the primary field obtained by acting with $\phi_{7/2}$ on ϕ_{Δ}, and similarly for $J\Delta$ and $J\phi\Delta$. Moreover, the first sum runs over those weights Δ such that ϕ_{Δ} has unique fusion rules with $\phi_{7/2}$.

The case $m = 12$ parallels the previous one closely, hence we omit most details. In this case, $\phi_{5/2}$ is not a simple current; but generates a subalgebra containing a simple current of order 2 of spin $\Delta_{1,11} = 15$. The (E_6, D_8)-type exceptional modular invariant for this model has the form (4.8) where ϕ is now $\phi_{5/2}$ and J is the spin 15 simple current.

We shall not attempt to prove here that these SCFTs do indeed afford such extended symmetries; although the structure of (4.8) clearly suggests that it can be obtained via the procedure in **[43]** applied to the simple current J, but starting from a non-diagonal invariant that presumably is a consequence of the algebraic structure related to $\phi_{7/2}$ (for $m = 10$) or $\phi_{5/2}$ (for $m = 12$).

§5 Conclusions

In this paper we have initiated a systematic investigation of the existence of extensions of the $N = 1$ super Virasoro algebra by one or in some cases several primary superfields of (half-)integer dimension $\frac{1}{2} \leq \Delta \leq \frac{7}{2}$ using the conformal bootstrap method. Consequently we developed in detail the decomposition of operator products into superconformal families and the implementation of the requirement of crossing symmetry of the correlators. Since this method becomes computationally quite involved for high Δ, we have restricted ourselves to the above range of conformal dimensions. However, a glance at the Kač tables of $N = 1$ minimal models reveals that there are certainly many more such algebras. In fact, the m^{th} $N = 1$ unitary minimal model (for m even) always contains a representation of super $W_{m(m-2)/8}$, as can easily be inferred from the fusion rules.

The extended algebras we constructed in that way fall into two classes. For $\Delta \leq \frac{3}{2}$ the algebras are Lie (super)algebras and, hence, it is easier to verify the Jacobi identities directly in order to check associativity. Some of these results are already known; but we have included them for completeness. For $\Delta \geq 2$, however, the algebras are non-linear since in the (anti)commutator of the modes there also appear normal ordered products of them. For $\Delta = 2$ we found an algebra—super W_2—which is associative for all values of the central charge. For higher values of Δ the requirement of crossing symmetry constraints the central charge to lie in a finite set of values. It is remarkable that all such positive c-values lie in the $N = 1$ unitary minimal series. Therefore we investigated the relation between these minimal models and these extended algebras. We argued that the $m = 6$ unitary minimal model realizes the super W_3 algebra and that the (A_7, D_4)-type modular invariant in the Cappelli classification is diagonal in the super W_3 characters. For super $W_{5/2}$ and super $W_{7/2}$ we have not been able to reach similar conclusions. Nevertheless we pointed out an intriguing connection between the (D_6, E_6) (resp. (E_6, D_8)) exceptional modular invariants and super $W_{7/2}$ (resp. super $W_{5/2}$). It would be interesting to explore the existence of similar connections in, say, Virasoro minimal models.

Undoubtedly, the most tangible result of this paper is the construction of super W_2—the first non-linear extension of the $N = 1$ super Virasoro algebra which is associative for all values of the central charge. There are known examples of two-dimensional statistical mechanics models displaying superconformal invariance [**19**],[**44**] at criticality as well as models with a non-linear extension of the conformal algebra as a symmetry[**45**]. This prompts the question of whether there are models displaying both. Our investigations are a first step in the construction of such supersymmetric conformal field theories. In particular, the non-linear algebra super W_2 invites the search for its (unitary) minimal models. On the other hand, recent advances in W-gravity make super W_2 a prime candidate for the construction of a W-supergravity theory, which could lead to new insights into the nature of superstring theory.

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Appendix A The Spin 2 Extended Algebra: super W_2

In this appendix we describe in some detail the extension of the $N = 1$ super Virasoro algebra by a primary of spin 2. This algebra is the first known non-linear $N = 1$ superconformal extension which is associative for all values of the central charge. The SOPE is given by[6]

$$\Phi_2 \times \Phi_2 \to \mathcal{C}_{22}^0 \, [\Phi_0] + \mathcal{C}_{22}^2 \, [\Phi_2] \ , \tag{A.1}$$

where $\mathcal{C}_{22}^0 = -c/10$ and $\mathcal{C}_{22}^2 = i{C_{22}}^2/\sqrt{5}$; and can be easily written explicitly using equations (2.16) and (2.20) and the following coefficients for the superconformal families:

$\{r\}$	$\{k\}$	$\beta_{22}^{0\,\{r,k\}}$
$\{\frac{3}{2}\}$	$\{\}$	$\frac{6}{c}$
$\{\}$	$\{2\}$	$\frac{4}{c}$
$\{\frac{5}{2}\}$	$\{\}$	$\frac{4}{c}$
$\{\}$	$\{3\}$	$\frac{2}{c}$
$\{\frac{7}{2}\}$	$\{\}$	$\frac{12(c+3)}{c(4c+21)}$
$\{\frac{3}{2}\}$	$\{2\}$	$\frac{108}{c(4c+21)}$

$\{r\}$	$\{k\}$	$\beta_{22}^{2\,\{r,k\}}$
$\{\frac{1}{2}\}$	$\{\}$	$\frac{1}{2}$
$\{\}$	$\{1\}$	$\frac{1}{2}$
$\{\frac{3}{2}\}$	$\{\}$	$\frac{27}{5c+6}$
$\{\frac{1}{2}\}$	$\{1\}$	$\frac{3(c-6)}{2(5c+6)}$

(A.2)

[6] We use the normalization $\xi_2 = i/\sqrt{5}$; see footnote 3. We also write W and U for ϕ_2 and $\phi_{5/2}$ respectively.

However we find it more illuminating to break the algebra into its Virasoro primaries. The general form of the algebra is given by (3.4) where the primary fields are explicitly given by:

$$\begin{aligned}
\Theta(z) \equiv \phi_4(z) &= (\partial G G)(z) - \frac{17}{5c+22}\Lambda(z) - \frac{3}{10}\partial^2 T(z) , \\
\Delta(z) \equiv \widetilde{\phi}_4(z) &= -i\sqrt{5}\,(GU)(z) - \frac{76}{5c+44}\Omega(z) - \frac{1}{5}\partial^2 W(z) , \\
\Gamma(z) \equiv \phi_{7/2}(z) &= (GW)(z) + \frac{2i}{\sqrt{5}}\partial U(z) ,
\end{aligned} \tag{A.3}$$

where as usual the parenthesis denote normal ordering defined via point-splitting regularization, and the quasiprimary fields $\Lambda(z)$ and $\Omega(z)$ are defined by

$$\Lambda(z) = (TT)(z) - \frac{3}{10}\partial^2 T(z) , \tag{A.4}$$

and

$$\Omega(z) = (TW)(z) - \frac{3}{10}\partial^2 W(z) . \tag{A.5}$$

In terms of these fields the OPEs are given by (2.1) , (2.4) , (2.5) , (2.6) , (2.7) , and

$$\begin{aligned}
W(z)W(0) =& \frac{c/2}{z^4} + \frac{2T(0) + {C_{22}}^2 W(0)}{z^2} + \frac{\partial T(0) + \frac{1}{2}{C_{22}}^2 \partial W(0)}{z} + \text{reg.} , \\
U(z)W(0) =& \frac{\frac{3i}{\sqrt{5}}G(0)}{z^3} + \frac{\frac{2i}{\sqrt{5}}\partial G(0) + \frac{1}{2}{C_{22}}^2 U(0)}{z^2} \\
&+ \frac{a_\Xi \Xi(0) + \frac{3i}{4\sqrt{5}}\partial^2 G(0)}{z} \\
&+ {C_{22}}^2 \frac{\frac{3}{10}\partial U(0) + a_\Gamma \Gamma(0)}{z} + \text{reg.} , \\
U(z)U(0) =& \frac{2c/5}{z^5} + \frac{2T(0) + \frac{2}{5}{C_{22}}^2 W(0)}{z^3} + \frac{\partial T(0) + \frac{1}{5}{C_{22}}^2 \partial W(0)}{z^2} \\
&+ {C_{22}}^2 \frac{\frac{3}{50}\partial^2 W(0) + a_\Omega \Omega(0) + a_\Delta \Delta(0)}{z} \\
&+ \frac{a_\Lambda \Lambda(0) + \frac{3}{10}\partial^2 T(0) + a_\Theta \Theta(0)}{z} + \text{reg.} ,
\end{aligned} \tag{A.6}$$

where we have introduced another quasiprimary field:

$$\Xi(z) = (TG)(z) - \frac{3}{8}\partial^2 G(z) \ , \tag{A.7}$$

and the constants $a_\Xi = \frac{54i}{\sqrt{5}(4c+21)}$, $a_\Gamma = \frac{27i}{\sqrt{5}(5c+6)}$, $a_\Omega = \frac{54}{5(5c+44)}$, $a_\Delta = -\frac{27}{5(5c+6)}$, $a_\Lambda = \frac{27}{5c+22}$, and $a_\Theta = \frac{27}{5(4c+21)}$; and where the coupling ${C_{22}}^2$ is given (up to a sign reflecting the algebra automorphism $\phi_2 \mapsto -\phi_2$) by (3.5) . It is now straightforward to compute the mode algebra of which we give the most relevant (anti)commutators for completeness. We define the modes ϕ_n for a field $\phi(z)$ of weight Δ by $\phi(z) = \sum_n \phi_n z^{-n-\Delta}$, where n is half-integer for the NS sector of $\Delta \in \mathbb{Z} + \frac{1}{2}$ fields, and integer otherwise. One then obtains

$$\begin{aligned}
[W_m\,,\,W_n] =& \frac{c}{12}m(m^2-1)\delta_{m+n,0} + (m-n)\left(L_{m+n} + \frac{1}{2}{C_{22}}^2 W_{m+n}\right) \ , \\
[U_r\,,\,W_n] =& \frac{i}{4\sqrt{5}}(r^2 - 2nr + 3n^2 - \frac{9}{4})G_{n+r} + {C_{22}}^2(\frac{1}{5}r - \frac{3}{10}n)U_{n+r} \\
&+ a_\Xi \Xi_{n+r} + {C_{22}}^2 a_\Gamma \Gamma_{n+r} \ ,
\end{aligned} \tag{A.8}$$

and

$$\begin{aligned}
\{U_r\,,\,U_s\} =& \frac{c}{60}(r^2 - \frac{1}{4})(r^2 - \frac{9}{4})\delta_{r+s,0} \\
&+ \frac{1}{10}(3r^2 + 3s^2 - 4rs - \frac{9}{2})\left(L_{r+s} + \frac{1}{5}{C_{22}}^2 W_{r+s}\right) \\
&+ a_\Lambda \Lambda_{r+s} + a_\Theta \Theta_{r+s} + {C_{22}}^2\left(a_\Omega \Omega_{r+s} + a_\Delta \Delta_{r+s}\right) \ .
\end{aligned}$$

A tedious calculation shows that the Jacobi identities are indeed satisfied for and only for the given value of the self-coupling.

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