THE SPIN 6 EXTENDED CONFORMAL ALGEBRA

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ABSTRACT

Using the conformal bootstrap method, we construct the unique extended conformal algebra generated by a single Virasoro primary field of spin 6. We write down the algebra explicitly by expressing the singular terms of the operator product expansion in terms of quasiprimaries.

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Introduction

Extended conformal algebras are an important tool in the classification of two dimensional conformal field theories – a major goal in string theory as well as in two dimensional critical phenomena. Modular invariant partition functions involving an infinite number of Virasoro characters can often be written using only a finite number of characters of a larger algebra, an extended algebra. Moreover, off-diagonal partition functions are often diagonal in an extended algebra. There exists by now a variety of extended algebras, the best known being affine and superconformal algebras. Another class of extended algebras consists of the so-called $\mathcal{W}_n$ algebras [**1**],[**2**].

Extended algebras are generated by an energy-momentum tensor $T(z) = L_{-2}\phi_0(z)$, and a set of primary fields $\phi_1(z), \ldots, \phi_\ell(z)$ with (half)integer conformal dimensions $\Delta_1, \ldots, \Delta_\ell$, so that the singular parts of the operator product expansion of any two of these fields can be expressed as a linear combination of them, their derivatives or normal ordered products, in such a way that the algebra is associative for all values of the central charge c of the Virasoro subalgebra [**3**]. Such an algebra is then denoted by[1] $\mathcal{V}_{(\Delta_1,\ldots,\Delta_\ell)}$. Affine and superconformal algebras are in a sense trivial examples of this.

Although extended algebras have recently received a lot of attention, only very few of them have been explicitly constructed. Following the conformal bootstrap method [**4**], A.B. Zamolodchikov [**1**] investigated the existence of extended algebras $\mathcal{V}_{(\Delta)}$ for (half)integer $\Delta \leq 3$. He found two previously unknown extended algebras: $\mathcal{V}_{(5/2)}$, which is associative if and only if $c = -13/14$, and $\mathcal{W}_3 \equiv \mathcal{V}_{(3)}$, which is associative for all values of c and thus an extended algebra in the sense of [**3**]. Unlike previously known extended algebras these ones are not Lie (super)algebras, since the commutators of the generators contain also normal ordered products of them. These results were later extended to $\mathcal{V}_{(3/2,5/2)}$, which is associative iff $c = -13/14$, and $\mathcal{V}_{(3/2,3)}$, which is associative for all c [**5**].

In [**6**], Bouwknegt generalised the approach of Zamolodchikov, investigating the existence of $\mathcal{V}_{(\Delta)}$ for (half)integer Δ. These algebras have, apart from the central charge, at most one free parameter: $C_{\Delta\Delta}^{\Delta}$ in the operator product $\phi_\Delta(z)\phi_\Delta(0) = (c/\Delta)[\phi_0] + C_{\Delta\Delta}^{\Delta}[\phi_\Delta]$. Using counting arguments, he concluded that for integer dimension, only $\Delta = 1, 2, 3, 4, 6$ gives an extended algebra for generic c. For $\mathcal{V}_{(4)}$, he found an expression for C_{44}^{4} in terms of c, but did not write down the algebra explicitly. Later, this was done by Hamada *et al.*, who also checked the Jacobi identities on the modes [**7**]. Bouwknegt also conjec-

[1] In [**3**] these algebras were denoted by $\mathcal{V}\mathrm{ir}_{(2,\Delta_1,\ldots,\Delta_\ell)}$. We find this notation slightly misleading since $T(z)$ is not a primary field.

tured the existence of $\mathcal{V}_{(6)}$. In this letter, we explicitly construct it using the conformal bootstrap method, thus checking associativity of the mode algebra for all values of c.

Associativity and the Conformal Bootstrap

The conformal bootstrap starts by writing down the most general OPE between all the primary fields [**4**]

$$\phi_n(z)\phi_m(0) = \sum_p C_{nm}^p z^{\Delta_p-\Delta_n-\Delta_m} \sum_{\{k\}} z^{\sum k_i} \beta_{nm}^{p\,\{k\}} \phi_p^{\{k\}}(0), \tag{1}$$

where $\phi_p^{\{k\}}(0) = L_{-k_1}\cdots L_{-k_N}\phi_p(0)$ and L_m are the modes of the energy momentum tensor $L_m = \oint_0 \frac{dz}{2\pi i} z^{m+1}T(z)$. The coefficients $\beta_{nm}^{p\,\{k\}}$ are completely fixed by conformal covariance, the C_{nm}^p then follow from the requirement of associativity of the algebra. For convenience, we restrict ourselves to integer dimension bosonic fields.

To find the $\beta_{nm}^{p\,\{k\}}$'s, consider the matrix element

$$\langle k|\phi_l(1)L_{-k_1}\cdots L_{-k_N}|p\rangle, \tag{2}$$

where $\langle k| = \lim_{z\to\infty}\langle 0|\phi_k(z)z^{2L_0}$ and $|p\rangle = \lim_{z\to 0}\phi_p(z)|0\rangle$, $|0\rangle$ being the $SL(2,\mathbb{C})$ invariant vacuum. This matrix element can be evaluated in two different ways. Using the fact that ϕ_l is primary and that $\langle k|L_{-k_i} = 0$ for $k_i > 0$, we can rewrite this as differential operators acting on the three point function $\langle k|\phi_l(\zeta)|p\rangle$ evaluated at $\zeta = 1$. This yields $f_{kl}^{p\,\{k\}} C_{klp}$, where the $f_{kl}^{p\,\{k\}}$ are given by

$$f_{kl}^{p\,\{k_1,\ldots,k_N\}} = \prod_{i=1}^{N}\left(k_i\Delta_l + \Delta_p - \Delta_k + \sum_{j=i+1}^{N} k_i\right) . \tag{3}$$

On the other hand, one can use (1) to rewrite $\langle k|\phi_l(1)$. Plugging this into (2) gives

$$C_{klp}\sum_{\{k'\}} \beta_{lk}^{p\,\{k'\}} \mathcal{M}_p^{\{k'\}\{k\}}, \tag{4}$$

where the sum runs over all sets $\{k_i'\}$ such that $\sum k_i' = \sum k_i$ and where $\mathcal{M}_p^{\{k'\}\{k\}}$ is the Šapovalov form

$$\mathcal{M}_p^{\{k'\}\{k\}} = \langle p|L_{k_N'}\cdots L_{k_1'}L_{-k_1}\cdots L_{-k_N}|p\rangle. \tag{5}$$

Comparing (3) with (4) one finds

$$\beta_{nm}^{p\,\{k\}} = \sum_{\{k'\}} \left(\mathcal{M}_p^{-1}\right)^{\{k\}\{k'\}} f_{mn}^{p\,\{k'\}} . \tag{6}$$

Although these equations are equivalent to the ones derived in [**4**], this form makes manifest the observation of Al. B. Zamolodchikov that the poles of $\beta_{nm}^{p\,\{k\}}$ correspond to the zeros of the Kač determinant [**8**].

Associativity of the algebra up to null fields is then equivalent to crossing symmetry of the "four point function" [**4**]

$$G_{nm}^{lk}(x) = G_{nl}^{mk}(1-x) = x^{-2\Delta_n} G_{nk}^{lm}(\frac{1}{x}), \tag{7}$$

with $G_{nm}^{lk}(x)$ being defined as

$$\begin{aligned} G_{nm}^{lk}(x) &\equiv \langle k|\phi_l(1)\phi_n(x)|m\rangle \\ &= \sum_p C_{nm}^p C_{klp} \mathcal{F}_{nm}^{lk}(p|x) \ , \end{aligned} \tag{8}$$

where the conformal blocks $\mathcal{F}_{nm}^{lk}(p|x)$ are given by

$$\mathcal{F}_{nm}^{lk}(p|x) = x^{\Delta_p - \Delta_n - \Delta_m} \sum_{\{k\}} x^{\sum k_i} \beta_{nm}^{p\,\{k\}} f_{kl}^{p\,\{k\}} \ . \tag{9}$$

Notice that, using (6), the conformal block can be viewed as the sum of the entries in the inverse of the Šapovalov form weighted by some integers: the $f_{mn}^{p\,\{k\}}$'s [**9**].

Unlike for the $f_{kl}^{p\,\{k\}}$ coefficients, there is no closed form expression for the $\beta_{nm}^{p\,\{k\}}$. This makes the full perturbative calculation of the conformal blocks impossible. Nevertheless, crossing symmetry imposes very strong constraints [**6**] on the leading terms (the poles at $x = 0$) of (8). It is precisely these constraints that we will exploit in order to fix the C_{nm}^p.

Perturbative Evaluation of the Four Point Function

Having reviewed the general method, we now proceed to the construction of $\mathcal{V}_{(6)}$. Besides the energy-momentum tensor $T(z)$, satisfying a Virasoro algebra with central charge c, this algebra contains a spin 6 primary field $\phi_6(z)$. The only nontrivial OPE is

$$\phi_6(z)\phi_6(0) = \frac{c}{6}[\phi_0] + C_{66}^6[\phi_6], \tag{10}$$

where $[\phi_0]$ and $[\phi_6]$ stand for the entire conformal family of ϕ_0 and ϕ_6 respectively,

$$[\phi_0] = z^{-12}\left(1 + z^2\beta_{66}^{0\,\{2\}}L_{-2} + z^3\beta_{66}^{0\,\{3\}}L_{-3} + \cdots\right)\phi_0(0) \tag{11}$$

$$[\phi_6] = z^{-6}\left(1 + z\beta_{66}^{6\,\{1\}}L_{-1} + \cdots\right)\phi_6(0) \ . \tag{12}$$

Thus the computation of the singular terms of the four point function requires the knowledge of $\beta_{66}^{6\,\{k\}}$ up to level $\sum k_i = 5$ and $\beta_{66}^{0\,\{k\}}$ up to level 11. We refrain from giving the explicit values for these coefficients. Later on we will write down the algebra in terms of quasiprimary families so we will only need special linear combinations of the $\beta^{p\,\{k\}}_{mn}$'s.

The perturbative expansions of the conformal blocks are the following:

$$\mathcal{F}^{66}_{66}(6|x) = x^{-6}\sum_{n\geq 0} x^n \mathcal{F}^{(6)}_n \ , \qquad \mathcal{F}^{66}_{66}(0|x) = x^{-12}\sum_{n\geq 0} x^n \mathcal{F}^{(0)}_n \ , \tag{13}$$

where the first few terms are given by

$$\begin{aligned}
\mathcal{F}^{(6)}_0 &= 1 \qquad \mathcal{F}^{(6)}_1 = 3\\
\mathcal{F}^{(6)}_2 &= 3(49c+2832)/2\nu_0 \qquad \mathcal{F}^{(6)}_3 = 8(14c+1221)/\nu_0\\
\mathcal{F}^{(6)}_4 &= 9(168c^3+29019c^2+1115930c+2192728)/10\nu_1\\
\mathcal{F}^{(6)}_5 &= 9(42c^3+8973c^2+424358c+888760)/2\nu_1
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{F}^{(0)}_0 &= 1 \qquad \mathcal{F}^{(0)}_1 = 0 \qquad \mathcal{F}^{(0)}_2 = 72/c \qquad \mathcal{F}^{(0)}_3 = 72/c\\
\mathcal{F}^{(0)}_4 &= 36(9c+424)/c\mu_0 \qquad \mathcal{F}^{(0)}_5 = 144(2c+201)/c\mu_0\\
\mathcal{F}^{(0)}_6 &= 300(12c^3+1971c^2+34804c-6976)/c\mu_1\\
\mathcal{F}^{(0)}_7 &= 72(45c^3+9754c^2+307334c-19608)/c\mu_1\\
\mathcal{F}^{(0)}_8 &= 18(2450c^5+691282c^4+41443715c^3+598084042c^2\\
&\qquad +477826632c-3736896)/c\mu_2\\
\mathcal{F}^{(0)}_9 &= 24(1680c^5+551694c^4+42266925c^3+839141524c^2\\
&\qquad +909443564c+678288)/c\mu_2\\
\mathcal{F}^{(0)}_{10} &= 36(11340c^6+4452055c^5+482858192c^4+18776361980c^3\\
&\qquad +244485733072c^2+324323108352c+222461184)/c\mu_3\\
\mathcal{F}^{(0)}_{11} &= 1800(210c^6+90894c^5+11360119c^4+526575812c^3\\
&\qquad +8352275156c^2+13755620848c+2898432)/c\mu_3\ .
\end{aligned}$$

where for future reference we have defined the following denominators

$$\begin{aligned}
\nu_0 &= 13c+516 \qquad \nu_1 = \nu_0(c+47)(c+2)\\
\mu_0 &= 5c+22 \qquad \mu_1 = \mu_0(7c+68)(2c-1)\\
\mu_2 &= \mu_1(5c+3)(3c+46) \qquad \mu_3 = \mu_2(11c+232)\ .
\end{aligned}$$

The only nontrivial crossing symmetry check comes from the four point function $G(x) \equiv G^{66}_{66}(x)$, whose first few terms can easily be written down plugging (14) into (8).

Crossing symmetry of $G(x)$ can be given a very natural group theoretic interpretation [**6**]. Let $\mathbb{V}$ denote the 25-dimensional vector space spanned by the functions $x^{-12}, \ldots, x^{-1}, 1, (1-x)^{-1}, \ldots, (1-x)^{-12}$. The dihedral group $\mathcal{D}_3$ acts on $\mathbb{V}$ via the following transformations:

$$\begin{aligned}(S \cdot f)(x) =& x^{-12} f(\frac{1}{x}) \\ (T \cdot f)(x) =& f(1-x) \ ,\end{aligned}$$

for any function $f \in \mathbb{V}$. Crossing symmetry of $G(x)$ is then precisely the statement that it be in $\mathbb{V}^{\mathcal{D}_3}$ – the $\mathcal{D}_3$-invariants of $\mathbb{V}$. It is easy to show that this space is 5-dimensional and, furthermore, that it is spanned by the group orbits of the following elements: x^{-12}, x^{-10}, x^{-8}, x^{-7}, and x^{-6}. Trying to fit the pole structure of $G(x)$ at $x = 0$ with that of a linear combination of these orbits imposes a consistency condition which is only satisfied provided[2]

$$\left(C_{66}^{6}\right)^2 = \frac{400(c^2 - 388c + 4)(13c + 516)^2(c + 47)^2(c + 2)}{3(11c + 232)(7c + 68)(5c + 22)(5c + 3)(3c + 268)(3c + 46)(2c - 1)} \tag{15}$$

where c remains arbitrary. The regular terms in the perturbative expansion of $G(x)$ yield further consistency conditions. These can be solved by introducing extra primary fields of dimensions ≥ 12 in the operator product algebra (10) which, however, do not appear in the singular part of the OPE and, hence, in the mode algebra. We have checked this explicitly for the first few regular terms.

The Explicit Form of the Algebra

We now proceed to write down the algebra explicitly. Rather than expand the field in modes and write down their commutation relations, we prefer to write down the singular terms in the operator product expansion in terms of quasiprimary fields. This is a more economic way to write the same amount of information. The conformal family $[\phi_6]$ appearing in (10) can be decomposed into quasiprimary families as follows:

$$[\phi_6] = \{\phi_6^{(0)}\} + \beta_6^{(1)}\{\phi_6^{(1)}\} + \beta_6^{(2)}\{\phi_6^{(2)}\} + \beta_6^{(3)}\{\phi_6^{(3)}\} + \cdots \tag{16}$$

with

$$\{\phi_6^{(i)}\} = z^{-12+\Delta_i} \sum_{n \geq 0} z^n \alpha_n^i \partial^n \phi_6^{(i)} \ , \tag{17}$$

[2] The sign ambiguity in C_{66}^{6} is a reflection of the algebra automorphism $\phi_6 \mapsto -\phi_6$.

where Δ_i is the conformal weight of $\phi_6^{(i)}$ and the α_n^i are numerical coefficients given by the formula

$$\alpha_n^i = \prod_{j=1}^{n} \frac{j+\Delta_i-1}{j^2+j(2\Delta_i-1)} \ . \tag{18}$$

The quasiprimary fields $\phi_6^{(i)}$ are given in the following table:

i	Δ_i	$\phi_6^{(i)}$
0	6	ϕ_6
1	8	$\left(L_{-2} - \frac{3}{26}L_{-1}^2\right)\phi_6$
2	10	$\left(L_{-2}^2 - \frac{3}{2}L_{-3}L_{-1} + 3L_{-4}\right)\phi_6$
3	10	$\left(\frac{78}{5}L_{-4} - \frac{13}{2}L_{-3}L_{-1} + L_{-2}L_{-1}^2 - \frac{1}{20}L_{-1}^4\right)\phi_6$

and the $\beta_6^{(i)}$ coefficients are given by

$$\begin{aligned} \beta_6^{(1)} &= 186/\nu_0 \qquad \beta_6^{(2)} = 12(572c+2089)/5\nu_1 \\ \beta_6^{(3)} &= (13c^2-9682c-62256)/34\nu_1 \ . \end{aligned}$$

Similarly, but a bit more complicated, the conformal family $[\phi_0]$ appearing in (10) can be decomposed into quasiprimary families in the following manner:

$$[\phi_0] = \phi_0 + \sum_{i=1}^{11} \beta_0^{(i)}\{\phi_0^{(i)}\} + \cdots \ , \tag{19}$$

where the $\{\phi_0^{(i)}\}$ are defined by

$$\{\phi_0^{(i)}\} = z^{-12+\Delta_i} \sum_{n\geq 0} z^n \alpha_n^i \partial^n \phi_0^{(i)} \ , \tag{20}$$

with the α_n^i being given once again by (18).

The table of quasiprimaries is now the following:

i	Δ_i	$\phi_0^{(i)}$
1	2	$L_{-2}\phi_0$
2	4	$\left(L_{-2}^2 - \frac{3}{5}L_{-4}\right)\phi_0$
3	6	$\left(L_{-3}^2 - \frac{8}{5}L_{-4}L_{-2} - \frac{4}{7}L_{-6}\right)\phi_0$
4	6	$\left(L_{-2}^3 - \frac{9}{5}L_{-4}L_{-2} - \frac{6}{7}L_{-6}\right)\phi_0$
5	8	$\left(L_{-3}^2L_{-2} - \frac{8}{5}L_{-4}L_{-2}^2 + \frac{4}{5}L_{-5}L_{-3} - \frac{36}{35}L_{-6}L_{-2}\right)\phi_0$
6	8	$\left(L_{-4}^2 - \frac{5}{3}L_{-5}L_{-3} + \frac{20}{21}L_{-6}L_{-2} - \frac{5}{9}L_{-8}\right)\phi_0$
7	8	$\left(L_{-2}^4 - \frac{18}{5}L_{-4}L_{-2}^2 + \frac{9}{5}L_{-5}L_{-3} - \frac{156}{35}L_{-6}L_{-2} - 3L_{-8}\right)\phi_0$
8	10	$\left(L_{-5}^2 - \frac{12}{7}L_{-6}L_{-4} + \frac{15}{14}L_{-7}L_{-3} - \frac{10}{21}L_{-8}L_{-2} - \frac{6}{11}L_{-10}\right)\phi_0$
9	10	$\left(L_{-4}^2L_{-2} - \frac{5}{3}L_{-5}L_{-3}L_{-2} + \frac{20}{21}L_{-6}L_{-2}^2 - \frac{5}{14}L_{-7}L_{-3} - \frac{25}{63}L_{-8}L_{-2}\right)\phi_0$
10	10	$\left(L_{-3}^2L_{-2}^2 - \frac{8}{5}L_{-4}L_{-2}^3 - \frac{3}{5}L_{-4}L_{-3}^2 - \frac{12}{15}L_{-6}L_{-2}^2 + \frac{12}{7}L_{-6}L_{-4} + \frac{16}{5}L_{-5}L_{-3}L_{-2} - \frac{6}{35}L_{-7}L_{-3} + \frac{8}{105}L_{-8}L_{-2} + \frac{12}{11}L_{-10}\right)\phi_0$
11	10	$\left(L_{-2}^5 - 6L_{-4}L_{-2}^3 + 9L_{-5}L_{-3}L_{-2} - \frac{96}{7}L_{-6}L_{-2}^2 + \frac{36}{7}L_{-6}L_{-4} + \frac{27}{14}L_{-7}L_{-3} - \frac{111}{7}L_{-8}L_{-2} - \frac{144}{11}L_{-10}\right)\phi_0$

and the $\beta_0^{(i)}$ coefficients are given by

$$
\begin{aligned}
\beta_0^{(1)} =& 12/c \qquad \beta_0^{(2)} = 372/c\mu_0 \\
\beta_0^{(3)} =& -5(85c^2 + 1122c - 15160)/3c\mu_1 \qquad \beta_0^{(4)} = 160(139c + 35)/c\mu_1 \\
\beta_0^{(5)} =& -6(40465c^3 + 892878c^2 - 9324952c + 1870944)/13c\mu_2 \\
\beta_0^{(6)} =& 9(2282c^4 - 217685c^3 + 13103548c^2 + 81974228c - 53857248)/143c\mu_2 \\
\beta_0^{(7)} =& 48(22992c^2 + 57652c - 2179)/c\mu_2 \\
\beta_0^{(8)} =& -6(5565c^5 - 175231c^4 - 3584360c^3 + 188510972c^2 \\
& - 397046576c - 17045662464)/221c\mu_3 \\
\beta_0^{(9)} =& 36(8785c^4 - 1006740c^3 + 44073460c^2 \\
& + 612878464c + 131856048)/17c\mu_3 \\
\beta_0^{(10)} =& -120(145940c^3 + 4588515c^2 - 36867098c - 4696416)/17c\mu_3 \\
\beta_0^{(11)} =& 192(172800c^2 + 874936c + 15707)/c\mu_3 \ .
\end{aligned}
$$

It is now straightforward to write down the singular terms of the operator product expansion and hence the mode algebra in terms of the modes of the quasiprimary fields.

Conclusions

In this letter we have constructed a new extended conformal algebra obtained by adding a dimension 6 primary field to the Virasoro algebra. The algebra was written down explicitly by expressing the singular part of the operator expansion in terms of quasiprimary fields. We have shown that the mode algebra is indeed associative for all values of the central charge.

Exploiting the relation between classical $\mathcal{W}$-algebras and Toda field theory, the Poisson brackets of Casimir algebras [**10**] can in principle be calculated; in particular, this has been done for G_2 in [**11**]. Given that the casimirs of G_2 have orders 2 and 6, it is clear that the spin 6 algebra constructed here must correspond to the quantum Casimir algebra of G_2 should the algebra constructed in [**11**] survive quantization. This being the case would allow one, in principle, to construct unitary highest weight representations of the spin 6 algebra. A quick glance at the Virasoro minimal models reveals at least two such models containing a spin 6 primary: $c = 129/154$ and $c = 592/595$. The latter one is in fact in the unitary series as well, although not in the unitary series of the Casimir algebra of G_2. It would be interesting to find statistical models displaying a spin 6 symmetry.

Up to miraculous cancellations between different conformal blocks which would allow the existence of $\mathcal{V}_{(\Delta)}$ for higher Δ, our construction exhausts the list of all possible extended conformal algebras which are associative for arbitrary values of the central charge and which are generated by only one extra integer dimension primary in addition to the Virasoro generator [**6**].

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