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BIHAMILTONIAN STRUCTURE OF THE SUPERSYMMETRIC SKdV HIERARCHY

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ABSTRACT

We prove that the supersymmetric SKdV hierarchy is bihamiltonian. One of the hamiltonian structures (the "second") is a classical version of the supervirasoro algebra whereas the other structure is nonlocal. We exhibit these two structures as hamiltonian reductions of the recently constructed supersymmetric Gel'fand-Dickey brackets, derive Lenard relations, and prove the hamiltonian integrability of the hierarchy.

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Introduction

One of the simplest and best-known integrable system is the Korteweg-de Vries (KdV) equation. It is not surprising then that the first steps in the field of supersymmetric integrable systems consisted in constructing supersymmetric extensions of the KdV equation. In [**1**] Kupershmidt defined a fermionic extension of the KdV equation and proved the integrability of the resulting hierarchy. However his equations were not invariant under supersymmetry. Later in their seminal paper on the super KP hierarchy[2] Manin and Radul obtained the equation of Kupershmidt via a reduction of the super KP hierarchy and gave a Lax representation for it. Moreover they also constructed a different reduction that extended the KdV equation while at the same time preserving supersymmetry. This supersymmetric extension—hereafter denoted SKdV— was later studied by Mathieu in [**3**] . He constructed a one-parameter family of possible supersymmetric extensions of the KdV equation, but found that only for one value—which reproduced the SKdV equation of Manin and Radul—one obtained an integrable system. He showed that the SKdV equation is hamiltonian with respect to a classical version of the supervirasoro algebra: the supersymmetrization of the second hamiltonian structure of the KdV equation. He also demonstrated that a naive supersymmetrization of the first structure did not work and thus concluded that the SKdV equation was not bihamiltonian, as opposed to the equation of Kupershmidt which was shown, also by Mathieu, to be bihamiltonian. Moreover a Painlevé analysis[4] by Mathieu of possible supersymmetric extensions of the KdV equations seems to suggest that the only integrable extensions are the SKdV equation and the one of Kupershmidt. Thus one seemed forced to abandon either invariance under supersymmetry or the bihamiltonian structure of the equation.

Fortunately this is not true. We show in this letter that the SKdV equation is in fact bihamiltonian: the new hamiltonian structure being a nonlocal deformation of the naive supersymmetrization of the first hamiltonian structure of the KdV equation. Despite this fact, all the evolution equations in the hierarchy are local and, in fact, have a Lax representation. This bihamiltonian hierarchy is obtained as a reduction of the Lax hierarchy associated to the superdifferential operator $L = D^4 + \cdots$. Both hamiltonian structures—the nonlocal one and the one considered by Mathieu—arise as reductions of the recently constructed supersymmetric Gel'fand-Dickey brackets[5]. Moreover the Lenard relations for the unreduced hierarchy induce Lenard relations for the SKdV hierarchy. This allows us to prove the hamiltonian integrability of the SKdV hierarchy—a result which, to our knowledge, did not previously exist.

This letter is organized as follows. We first give the basic results on the bi-hamiltonian nature of the SKdV equation and only later do we exhibit this as a reduction from a more general system. We show by explicit computation of the Dirac brackets that the supersymmetric Gel'fand-Dickey brackets induce the two hamiltonian structures of the SKdV equation. We also show that the Lax flows are induced as well; that is, that they preserve the reduced submanifold. The reduction also induces Lenard relations and conserved charges and this allows the complete proof of hamiltonian integrability: the existence of an infinite number of nontrivial, independent, polynomial conserved charges in involution with respect to both hamiltonian structures.

Although we have tried to be self-contained, lack of space prevents us from reviewing the general formalism associated to the supersymmetric Lax hierarchies and to the differential calculus in the space of supersymmetric Lax operators. Details on the calculus can be found, for example, in **[6]** and **[7]** ; whereas the general results on the supersymmetric Lax hierarchies can be found in **[2]** for the odd order case and in **[8]** for the even order case.

The SKdV equation

The SKdV equation[2] is the nonlinear evolution equation for a superfield $U(x,\theta) = \xi(x) + \theta u(x)$ given by

$$\frac{\partial U}{\partial t} = \frac{1}{4} U^{[6]} + \frac{3}{4} (UU')'' \;, \tag{1}$$

where $'$ denotes the action of the supercovariant derivative $D = \partial_\theta + \theta\partial$ and $(D^i U) = U^{[i]}$. Breaking this equation in components we find

$$\frac{\partial\xi}{\partial t} = \frac{1}{4}\xi_{xxx} + \frac{3}{4}\left(\xi_x u + \xi u_x\right) \ , \tag{2}$$

$$\frac{\partial u}{\partial t} = \frac{1}{4}u_{xxx} + \frac{3}{2}u u_x - \frac{3}{4}\xi\xi_{xx} \ , \tag{3}$$

where the $_x$ subscripts denote the action of ∂. Setting $\xi = 0$ one recovers the KdV equation for u. As shown in **[2]** the SKdV equation has the Lax representation

$$\frac{\partial L}{\partial t} = [L_+^{\frac{3}{2}} \, , \, L] \ , \tag{4}$$

for $L = D^4 + UD$. This allows one to construct an infinite set of conserved charges $\mathrm{Str}\, L^{\frac{n}{2}}$, an infinite number of commuting flows, and hence to prove formal integrability[**3**].

As shown by Mathieu[**3**], the SKdV equation is hamiltonian with respect to the following Poisson bracket[1]

$$\{U(X)\, , \, U(Y)\}_2 = -\left[\frac{1}{2}D^5 + \frac{3}{2}U(X)D^2 + \frac{1}{2}U(X)'D + U(X)''\right]\cdot\delta(X-Y) \ , \tag{5}$$

which is a classical version of the supervirasoro algebra for $-U$; and with the following hamiltonian function

$$H^{(2)} = -\frac{1}{4}\int_B UU' \ , \tag{6}$$

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[1] The reason for the subscript $_2$ is historical. This is the supersymmetric analog of the second hamiltonian structure of the KdV equation. In keeping with tradition we shall call this the second hamiltonian structure, even though, it came first chronologically.

where $\int_B$ denotes the Berezin integral. In other words, the SKdV equation can be written as

$$\frac{\partial U}{\partial t} = -\left[\frac{1}{2}D^5 + \frac{3}{2}UD^2 + \frac{1}{2}U'D + U''\right] \cdot \frac{\delta H^{(2)}}{\delta U} \ . \tag{7}$$

However there is more. It turns out that the SKdV equation is also hamiltonian with respect to another hamiltonian function

$$H^{(1)} = \frac{1}{16}\int_B \left(U''U''' - 2U(U')^2\right) \ , \tag{8}$$

and relative to a *nonlocal* Poisson structure

$$\{U(X)\,,\,U(Y)\}_1 = -2\left[D^2(D^3+U)^{-1}D^2\right] \cdot \delta(X-Y) \ . \tag{9}$$

Indeed,

$$\frac{\delta H^{(1)}}{\delta U} = \frac{1}{4}UU'' - \frac{3}{8}(U')^2 - \frac{1}{8}U^{[5]} \ . \tag{10}$$

Applying D^2 we obtain

$$\left(\frac{\delta H^{(1)}}{\delta U}\right)'' = \frac{1}{4}UU^{[4]} - \frac{3}{4}U'U''' - \frac{1}{8}U^{[7]} \ , \tag{11}$$

which can be written as

$$\left(\frac{\delta H^{(1)}}{\delta U}\right)'' = (D^3+U) \cdot \left[-\frac{1}{8}U^{[4]} - \frac{3}{8}UU'\right] \ . \tag{12}$$

Therefore,

$$\frac{\partial U}{\partial t} = -2D^2(D^3+U)^{-1}D^2 \cdot \frac{\delta H^{(1)}}{\delta U} \tag{13}$$

recovers the SKdV equation. The identity

$$-2D^2(D^3+U)^{-1}D^2 \cdot \frac{\delta H^{(1)}}{\delta U} = -\left[\frac{1}{2}D^5 + \frac{3}{2}UD^2 + \frac{1}{2}U'D + U''\right] \cdot \frac{\delta H^{(2)}}{\delta U} \tag{14}$$

is an example of a Lenard relation. We will see in the next section that these Lenard relations connect an infinite number of polynomial conserved densities for the SKdV hierarchy. It is noteworthy that despite the nonlocal nature of one of the Poisson structures, the evolution equations for this hierarchy are local. It is also interesting to remark that the nonlocal Poisson operator $-2D^2(D^3+U)^{-1}D^2 = -2D+O(D^{-2})$ is a deformation of the naive supersymmetrization of the first hamiltonian structure of the KdV equation.

Bihamiltonian Structure by Hamiltonian Reduction

We now show how the bihamiltonian structure of the SKdV equation arises by reduction of the supersymmetric Gel'fand-Dickey brackets constructed in **[5]** . As pointed out in the previous section the SKdV equation is of Lax type with respect to the Lax operator D^4+UD. We can view this Lax operator as a reduction of the operator $L = D^4+U_3D^3+U_2D^2+U_1D+U_0$ where we have set $U_3 = U_2 = U_0 = 0$ and $U_1 = U$. Operators L of this form have been studied in **[8]** where they are analyzed as reductions of the even order SKP hierarchy. We first review the results of **[8]** for the general operator and then study the reduction.

It follows from the results of **[8]** that the Lax equations for the superdifferential operator $L = D^4 + U_3D^3 + U_2D^2 + U_1D + U_0$ are bihamiltonian with respect to the supersymmetric analogs of the Gel'fand-Dickey brackets. If $F = \int_B f$ and $G = \int_B g$ are functions of L, with f and g homogeneous differential polynomials in the U_i, we can define the following one-parameter family of Poisson brackets.

$$\{F\,,\,G\}_z = -(-1)^{|F|+|G|}\,\mathrm{Str}\,\left(J_z(dF)\,dG\right)\ , \tag{15}$$

where

$$dF = \sum_{i=0}^{3}(-1)^i D^{-i-1}\frac{\delta F}{\delta U_i}\ , \tag{16}$$

and similarly for dG, and J_z is defined by

$$J_z(X) = (LX)_+L - L(XL)_+ - z^2[L\,,\,X]_+\ , \tag{17}$$

for X any integral pseudodifferential operator. It was proven in [**8**] that $\{\,,\,\}_z$ is a Poisson bracket for all values of z. In particular the two Poisson brackets

$$\{F\,,\,G\}_0 = -(-1)^{|F|+|G|}\,\mathrm{Str}\,\left(J_0(dF)\,dG\right)\ , \tag{18}$$

$$\{F\,,\,G\}_\infty = -(-1)^{|F|+|G|}\,\mathrm{Str}\,\left(J_\infty(dF)\,dG\right)\ , \tag{19}$$

where J_0 and J_∞ are defined by $J_z = J_0 - z^2 J_\infty$, are the supersymmetric analogs of the Gel'fand-Dickey brackets. Let us define the following functions $H_n \equiv -\frac{2}{n}\mathrm{Str}\,L^{\frac{n}{2}}$. They are trivial for n even and it can be shown[**8**] that they are non-trivial for n odd. Therefore from now on whenever we write H_n we will assume that n is odd. The first three nonzero functions are

$$H_1 = -\int_B U\ , \tag{20}$$

$$H_3 = -\frac{1}{4}\int_B UU'\ , \tag{21}$$

and

$$H_5 = \frac{1}{16}\int_B \left(U''U''' - 2U(U')^2\right)\ . \tag{22}$$

The H_n are clearly conserved charges of the Lax equations since the supertrace annihilates (graded) commutators. Their gradients are given by[**8**] $dH_n = L_-^{\frac{n}{2}-1}$. Therefore the Lax flow

$$\begin{aligned}\frac{\partial L}{\partial t_n} &= [L_+^{\frac{n}{2}}\,,\,L] \\ &= [L\,,\,L_-^{\frac{n}{2}}] = J_\infty(dH_{n+2})\end{aligned} \tag{23}$$

is generated by H_{n+2} relative to the "first" hamiltonian structure J_∞. Furthermore

$$\begin{aligned}J_0(L_-^{\frac{n}{2}-1}) &= (LL_-^{\frac{n}{2}-1})_+L - L(L_-^{\frac{n}{2}-1}L)_+ \\ &= L(L_-^{\frac{n}{2}-1}L)_- - (LL_-^{\frac{n}{2}-1})_-L \\ &= L(L^{\frac{n}{2}-1}L)_- - (LL^{\frac{n}{2}-1})_-L \\ &= J_\infty(L_-^{\frac{n}{2}})\ ,\end{aligned} \tag{24}$$

whence the Lax flow (23) is also hamiltonian with respect to H_n if we use the "second" hamiltonian structure J_0. Thus we have the following Lenard relations connecting the conserved charges

$$J_\infty(dH_{n+2}) = J_0(dH_n) \ . \tag{25}$$

Let us now impose the reduction $U_3 = U_2 = U_0 = 0$ and $U_1 = U$. As we will see, both Gel'fand-Dickey brackets $\{\,,\,\}_0$ and $\{\,,\,\}_\infty$ induce Poisson brackets in the reduced submanifold. Let us write the fundamental Poisson brackets of the U_i as follows

$$\{U_i(X)\,,\,U_j(Y)\}_z = -J^{(z)}_{ij} \cdot \delta(X-Y) \ , \tag{26}$$

where the $J^{(z)}_{ij}$ are differential operators at the point X. They can be read off from the definition of J_z as follows. If $X = \sum_{i=0}^{3} D^{-i-1}X_i$ is a homogeneous integral operator, then

$$J_z(X) = \sum_{i,j=0}^{3} (-1)^{i|X|}(J^{(z)}_{ij} X_j)D^i \ . \tag{27}$$

Let us first work with the first hamiltonian structure J_∞. On the constrained submanifold, the nonzero entries are

$$J^{(\infty)}_{10} = J^{(\infty)*}_{01} = -2D^2 \ ,$$

and

$$J^{(\infty)}_{00} = 2(D^3+U) \ .$$

The Dirac bracket is then given by

$$\{U(X)\,,\,U(Y)\}_1 = -J^{(\infty)}_D \cdot \delta(X-Y) \ , \tag{28}$$

where

$$\begin{aligned} J^{(\infty)}_D &= -J_{10}\,J_{00}^{-1}\,J_{01} \\ &= 2D^2(D^3+U)^{-1}D^2 \ . \end{aligned} \tag{29}$$

The second hamiltonian structure is more involved. After a somewhat lengthy calculation, we find that on the constrained submanifold the nonzero components of $J^{(0)}$ are given by

$$
\begin{aligned}
J^{(0)}_{32} &= J^{(0)*}_{23} = 2D^2 \ , \\
J^{(0)}_{30} &= J^{(0)*}_{03} = -D(D^3+U) \ , \\
J^{(0)}_{22} &= -2(D^3+U) \ , \\
J^{(0)}_{21} &= J^{(0)*}_{12} = (D^3-U)D \ , \\
J^{(0)}_{20} &= J^{(0)*}_{02} = D^2(D^3+U) \ , \\
J^{(0)}_{11} &= 2UD^2+U'' \ , \\
J^{(0)}_{10} &= J^{(0)*}_{01} = (D^3+U)D^3 \ ,
\end{aligned}
$$

and

$$J^{(0)}_{00} = D^4(D^3+U) + UD^4 + UU' \ .$$

One then computes the Dirac bracket to be

$$\{U(X)\, ,\, U(Y)\}_2 = -J^{(0)}_D \cdot \delta(X-Y) \ , \tag{30}$$

where

$$J^{(0)}_D = \frac{1}{2}D^5 + \frac{3}{2}UD^2 + \frac{1}{2}U'D + U'' \ . \tag{31}$$

One can show that these two structures are coordinated.

We now show that the flow generated by the conserved charges H_n preserve the constrained submanifold. In other words, that the hamiltonian vector field associated to the function H_n is tangent to the constrained submanifold. As explained, for example, in **[6]** , we can think of $J(dH)$ as the infinitesimal displacement δL of L under the hamiltonian vector field associated to a function H relative to the Poisson structure J. Since on the constrained submanifold, L is of the form D^4+UD, its infinitesimal variation δL has the form δUD. Therefore to show that the flows generated by H_n preserve the constrained submanifold it is sufficient to show that $J_0(dH_n) = J_\infty(dH_{n+2})$ is a differential operator of the form $\delta U D$, for some differential polynomial δU. By the Lenard relations we can work with either J_∞ or J_0. We choose to work with J_∞. Notice that on the constrained submanifold $J_\infty(dH_{n+2})$ is a differential operator of order one. We have to show that the free term vanishes. It is easy to see that the free term of $J_\infty(dH_{n+2}) = [L\,,\, L_-^{\frac{n}{2}}]_+ = [L_+^{\frac{n}{2}}\,,\, L]$ vanishes if and only if the free term of $L^{\frac{n}{2}}$ vanishes, since L has no free term. Now, $L = (D^3+U)D$ obeys $L^* = DLD^{-1}$ and $\left(L^{\frac{1}{2}}\right)^* = -\left(L^*\right)^{\frac{1}{2}}$, whence $\left(L^{\frac{n}{2}}\right)^* = (-1)^n DL^{\frac{n}{2}}D^{-1}$. Using that the free term of any pseudodifferential operator P is given by $\operatorname{sres} PD^{-1}$ and that the super-residue obeys[**6**] $\operatorname{sres} P^* = \operatorname{sres} P$, we find that $\operatorname{sres} L^{\frac{n}{2}}D^{-1} = \operatorname{sres}\left(L^{\frac{n}{2}}D^{-1}\right)^* = \operatorname{sres} D^{-1}\left(L^{\frac{n}{2}}\right)^* = (-1)^n \operatorname{sres} L^{\frac{n}{2}}D^{-1}$. Hence for n odd, the free term of $L^{\frac{n}{2}}$ vanishes. Therefore the restriction of the functions H_n to the constrained submanifold generate hamiltonian flows on it. These flows are, of course, the specialization to this submanifold of the Lax flows for the unreduced operator.

The Lenard relations (25) induce Lenard relations in the constrained submanifold. To see this we must merely notice that the flow induced by H_n on the constrained submanifold can be computed in two ways. One can either compute it intrinsically using the induced Poisson brackets, or extrinsically using the original Poisson brackets but choosing an appropriate definition for the gradient of H_n. One can show[**8**] that the definition of the gradient that does the trick is $dH_n = \sum_{i=0}^{3} D^{-i-1}X_i$ with $X_3 = -\frac{1}{2}\left(\frac{\delta H_n}{\delta U}\right)''$, $X_2 = \frac{1}{2}\left(\frac{\delta H_n}{\delta U}\right)'$, $X_1 = -\frac{\delta H_n}{\delta U}$, and X_0 is defined by $X_0''' + UX_0 = \left(\frac{\delta H_n}{\delta U}\right)''$. Computing both sides of (25) with these definitions of the gradients we find the following relation for all n

$$-2D^2(D^3+U)^{-1}D^2\cdot\frac{\delta H_{n+2}}{\delta U} = -\left[\frac{1}{2}D^5 + \frac{3}{2}UD^2 + \frac{1}{2}U'D + U''\right]\cdot\frac{\delta H_n}{\delta U}\ , \qquad (32)$$

which coincides with (14) for the special case of $n=3$ (*cf.* (21) and (22)).

The Lenard relations say that the Poisson flows generated by H_n relative to the second Poisson structure and the one generated by H_{n+2} relative to the first coincide. In other words, for all functions H,

$$\{H\,,\,H_n\}_2 = \{H\,,\,H_{n+2}\}_1\ . \tag{33}$$

In particular, when H is one of the conserved quantities, say H_q, one has that

$$\begin{aligned}\{H_q\,,\,H_n\}_2 &= \{H_q\,,\,H_{n+2}\}_1\\ &= \{H_{q-2}\,,\,H_{n+2}\}_2\\ &\ \vdots\\ &= \{H_{q-2j+2}\,,\,H_{n+2j}\}_1\\ &= \{H_{q-2j}\,,\,H_{n+2j}\}_2\\ &\ \vdots\end{aligned}$$

Assuming, for definiteness, that $q > n$ we find in the above chain of equations either $\{H_m\,,\,H_m\}_1$ or $\{H_m\,,\,H_m\}_2$ both of which vanish since the H_i are bosonic. Therefore we find that the conserved charges are in involution with respect to both Poisson structures.

It remains to show that these charges are nontrivial and independent. Independence follows from the usual homogeneity arguments. One can give a (half)integer grading to D and to the differential polynomials in U in such a way that the Lax operator is homogeneous and that multiplication of pseudodifferential operators respects the grading. This then implies that the H_n have different degrees of homogeneity and hence are independent. Nontriviality is a consequence of the hamiltonian structure of the Lax flows. If $H_{n+2} = 0$ then, using the first hamiltonian structure, $\frac{\partial L}{\partial t_n} = 0$. Using the Lax equations it means that $L_+^{\frac{n}{2}}$ commutes with L. Now, one can show[8] that any pseudodifferential operator commuting with L is a constant linear combination of its fractional powers. By degree of homogeneity arguments it follows that $L_+^{\frac{n}{2}} = L^{\frac{n}{2}}$, whence that $H_n = 0$, since it is defined as the supertrace of $L^{\frac{n}{2}}$. Continuing in this fashion we see that if $H_n = 0$ for some n odd, then $H_1 = 0$. But we saw that this was nonzero (*cf.* (20)). Therefore all conserved charges are nontrivial.

Conclusions

In this letter we have shown that the supersymmetric SKdV equation of Manin and Radul is bihamiltonian. This has allowed us, after deriving the Lenard relations, to prove the hamiltonian integrability of the SKdV hierarchy. The novel feature of the new Poisson structure, which explains why the bihamiltonian structure was thought not to exist, is that it is nonlocal; even though it gives rise to local evolutions when the hamiltonians are chosen to be the conserved charges of the Lax equations. This result would have appeared rather puzzling were it not for the fact that the hierarchy can be seen to be a reduction of a local bihamiltonian hierarchy. The Lax flows in the unreduced hierarchy preserve the reduced submanifold and it just happens that the intrinsic description of this hierarchy requires a nonlocal Poisson structure. It should be remarked that this is not the only possible reduction of the Lax hierarchy associated to $L = D^4 + \cdots$ that yields the SKdV hierarchy. One can show that the reduction where $U_0 = -U'$ instead of $U_0 = 0$ also works. The two Poisson structures are now related to the ones discussed in this letter by simply $U \mapsto -U$ and an overall sign. It is interesting to notice that the resulting Lax operator is $L = D(D^3 - U)$, which obeys the intriguing relation $L^* = D^{-1}LD$ to be compared with the analogous relation in the case treated here.

It should be possible to obtain a supersymmetric extension of the Boussinesq hierarchy by starting with the general operator $L = D^6 + \cdots$ and impose some reduction which is consistent with the dynamics. The resulting hierarchy would then be bihamiltonian and integrable just because it is obtained by reduction from one that is. One would expect the first hamiltonian structure to be nonlocal but the second should lead to a supersymmetric version of W_3. In fact, the whole issue of possible reductions of the even order SKdV hierarchies is a very interesting open problem that we hope to address in a future publication.

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