CLASSICAL $N=2$ W-SUPERALGEBRAS AND SUPERSYMMETRIC GEL'FAND-DICKEY BRACKETS

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ABSTRACT

We construct an infinite series of classical $N=2$ W-superalgebras as a reduction of a recently constructed supersymmetric version of the Gel'fand-Dickey brackets for the generalized super KdV hierarchies. This is achieved via a supersymmetric Miura transformation which yields, as a by-product, classical free-field realizations.

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§1 Introduction

W-algebras[1],[2] play an increasing rôle in two-dimensional conformal field theory and in string theory. In the former context it has been shown that they appear as the symmetry[3] algebras of statistical mechanics systems at criticality; and in the latter case, their minimal models can be used for the construction of classical vacua. Furthermore, unexpected relationships have been recently unveiled between W-algebras and noncritical strings[4] via their matrix model formulation, where the "string equation" corresponds to generalized KdV flows with specific boundary conditions[5]. This observation allows one to translate computations of correlation functions in non-critical string theory to more manageable algebraic computations in terms of the associated Lax system, without having to appeal directly to the matrix model formulation. This has a variety of advantages, not the least of which is that it allows for generalizations—one very important generalization being the one to non-critical superstrings, where a direct treatment in terms of (super)matrix models is lacking[6].

W-superalgebras appear naturally in the context of supersymmetric Toda field theory[7] as the algebras of conserved currents. In this paper we present a systematic treatment of $N = 2$ W-superalgebras based on the formalism of supersymmetric KdV hierarchies, which seems especially well suited for the applications to superstring theory.

The formalism of supersymmetric Lax operators is, however, not as developed as its bosonic counterpart, where the relationship with W-algebras is well understood. Classical W-algebras, in fact, first appeared in the context of scalar Lax operators for the generalized KdV hierarchies[8], where they arise naturally as their "second hamiltonian structure." Indeed, the n^{th} order KdV hierarchy (for $n > 2$) is hamiltonian with respect to a classical version of W_n, generalizing the well-known fact[9] that the KdV equation is hamiltonian with respect to a classical version of the Virasoro algebra. Recently we[10] have defined a hamiltonian structure in the space of supersymmetric Lax operators yielding a supersymmetric version of the Gel'fand-Dickey brackets for the generalized KdV hierarchies. In this paper we construct reductions of this hamiltonian structure yielding $N = 2$ W-superalgebras.

In the non-supersymmetric case, several reductions of the second Gel'fand-Dickey bracket are known to exist. In fact, it follows from the work of Drinfel'd and Sokolov, that the second Gel'fand-Dickey bracket is the natural hamiltonian structure for an integrable system of evolution equations associated to the general linear algebra. Reductions of this bracket are related to some of its subalgebras; notably the A_n, B_n, and C_n series. The former series is obtained by demanding that a particular coefficient in the Lax operator vanish: whereas the latter series are obtained by imposing that the Lax operator have definite symmetry properties under a natural involution in the space of (pseudo)differential operators. All these reductions yield extensions of the Virasoro algebra which are generated by a finite number of primary fields of conformal weights equal to the exponents of the relevant Lie algebra. For the A_n series this is nothing but the W_n algebras of Fateev and Lykyanov.

In this paper we construct the reduction of the supersymmetric Gel'fand-Dickey brackets which is analogous to the one yielding W_n in the non-supersymmetric case. We find that this only makes sense for supersymmetric Lax operators of odd order, since for even order the constraint is not second-class and thus does not define a symplectic submanifold. Other reductions analogous to the ones connected to the B_n and C_n series in the non-supersymmetric case are also possible and yield $N = 1$ W-superalgebras. This requires different techniques in order to prove the Jacobi identity of the induced brackets and thus has been left for a separate publication[**11**].

The plan of this paper is the following. In section 2 we describe the basic formalism associated to the space of supersymmetric Lax operators and summarize the results of [**10**] . In section 3 we work out explicitly the simplest example of our reductions yielding the $N = 2$ supervirasoro algebra itself. In section 4 we proceed to define the general reduction. There we prove that the reduced brackets are indeed Poisson brackets and that they define W-superalgebras containing the $N = 2$ supervirasoro algebra as a subalgebra. Finally section 5 offers some concluding remarks.

§2 Supersymmetric Lax Operators

In this section we set up the formalism and the notation for the rest of the paper. It is our purpose to define Poisson brackets in certain spaces of (super) differential operators. These spaces are infinite-dimensional manifolds and their rigourous geometric treatment is beyond our scope. Fortunately these spaces can be endowed with a "formal" geometry which is sufficient for the rigourous treatment of Poisson brackets and their flows. As usual, this formal geometry consists in the algebraization of the necessary geometric notions. Therefore we proceed to define algebraically the ingredients needed to define Poisson structures: functions, vector fields, and 1-forms.

We will consider the space of differential operators on a $(1|1)$ superspace with coordinates (x, θ). These operators are polynomials in the supercovariant derivative $D = \partial_\theta + \theta\partial$ whose coefficients are superfields. The supercovariant derivative obeys $D^2 = \partial$. A supersymmetric Lax operator has the form $L = D^n + U_{n-1}D^{n-1} + \cdots + U_0$ and is homogeneous under the usual $\mathbb{Z}_2$ grading; that is, $|U_i| \equiv n+i \pmod 2$. We will define Poisson brackets on functions of the form:

$$F[L] = \int_B f(U) \,, \tag{2.1}$$

where $f(U)$ is a homogeneous (under the $\mathbb{Z}_2$ grading) differential polynomial of the U and $\int_B$ is defined as follows: if $U_i = u_i + \theta v_i$, and $f(U) = a(u,v) + \theta b(u,v)$, then $\int_B f(U) = \int b(u,v)$, where the precise meaning of integration will depend on the context. It denotes integration over the real line if we take the u_i and v_i to be rapidly decreasing functions; integration over one period if we take them to be periodic functions; or, more abstractly, a linear map annihilating derivatives so that we can "integrate by parts". It is worth remarking that whereas differential polynomials can be multiplied, this does not induce a multiplication on the functions we are considering. Thus the Poisson brackets will not enjoy the usual derivation property. This, however, does not affect the formalism.

Vector fields are parametrized by infinitesimal deformations $L \mapsto L + \epsilon A$ where $A = \sum A_l D^l$ is a homogeneous differential operator of order at most $n-1$. We denote the space of such operators by S_n. We don't demand that A have the same parity as L since we can have either odd or even flows. To such an operator $A \in S_n$ we associate a vector field D_A as follows. If $F = \int_B f$ is a function then

$$\begin{aligned} D_A\, F &\equiv \frac{d}{d\epsilon} F[L + \epsilon A]\bigg|_{\epsilon=0} \\ &= (-1)^{|A|+n} \int_B \sum_{k=0}^{n-1} \sum_{i=0}^{\infty} (-1)^{(|A|+n)i} A_k^{[i]} \frac{\partial f}{\partial U_k^{[i]}} \ , \end{aligned} \tag{2.2}$$

with $U_k^{[i]} = D^i U_k$ and the same for $A_k^{[i]}$. Integrating by parts we can write this as

$$D_A\, F = (-1)^{|A|+n} \int_B \sum_{k=0}^{n-1} A_k \frac{\delta f}{\delta U_k} \ , \tag{2.3}$$

where the Euler variational derivative is given by

$$\frac{\delta}{\delta U_k} = \sum_{i=0}^{\infty} (-1)^{|U_k| i + i(i+1)/2} D^i \frac{\partial}{\partial U_k^{[i]}} \ . \tag{2.4}$$

Since vector fields are parametrized by S_n, it is natural to think of 1-forms as parametrized by the dual space to S_n. This turns out to be given by super pseudo-differential operators[12] (SΨDO's) with the dual pairing given by the Adler supertrace to be defined below. We introduce a formal inverse D^{-1} to D and define SΨDO's as formal Laurent series in D^{-1} whose coefficients are differential polynomials in the U_i. The multiplication of SΨDO's is given by the following composition law (for any $k \in \mathbb{Z}$)

$$D^k \Phi = \sum_{i=0}^{\infty} \begin{bmatrix} k \\ k-i \end{bmatrix} (-1)^{|\Phi|(k-i)} \Phi^{[i]} D^{k-i} , \tag{2.5}$$

where the superbinomial coefficients are given by

$$\begin{bmatrix} k \\ k-i \end{bmatrix} = \begin{cases} 0 & \text{for } i < 0 \text{ or } (k,i) \equiv (0,1) \pmod 2; \\ \begin{pmatrix} \left[\frac{k}{2}\right] \\ \left[\frac{k-i}{2}\right] \end{pmatrix} & \text{for } i \geq 0 \text{ and } (k,i) \not\equiv (0,1) \pmod 2. \end{cases} \tag{2.6}$$

Given a SΨDO $P = \sum p_i D^i$ we define its super-residue as $\operatorname{sres} P = p_{-1}$ and its (Adler) supertrace as $\operatorname{Str} P = \int_B \operatorname{sres} P$. One can show[**12**] that the super-residue of a graded commutator is a perfect derivative so that its supertrace vanishes: $\operatorname{Str}[P\,,\,Q] = 0$, for $[P\,,\,Q] \equiv P\,Q - (-1)^{|P||Q|} Q\,P$. This then defines a supersymmetric bilinear form on SΨDO's: $\operatorname{Str}(PQ) = (-1)^{|P||Q|} \operatorname{Str}(QP)$. If P is any SΨDO we define its differential part P_+ as the part of P which is polynomial in D (including free terms) and its "integral" part P_- as simply $P - P_+$. It then follows that $\operatorname{Str} P_\pm Q_\pm = 0$ for any two SΨDO's.

Let us then define 1-forms as the space S_n^* of "integral" SΨDO's of the form $X = \sum_{k=0}^{n-1} D^{-k-1} X_k$, whose pairing with a vector field D_A, with $A = \sum A_k D^k$, is given by

$$(D_A, X) \equiv (-1)^{|A|+|X|+n+1} \operatorname{Str}(AX) = (-1)^{|A|+n} \int_B \sum_{k=0}^{n-1} (-1)^k A_k\, X_k\ , \tag{2.7}$$

which is nondegenerate. The choice of signs has been made to avoid undesirable signs later on. Given a function $F = \int_B f$ we define its gradient dF by $(D_A, dF) = D_A F$ whence, comparing with (2.3) , yields

$$dF = \sum_{k=0}^{n-1} (-1)^k D^{-k-1} \frac{\delta f}{\delta U_k}\ . \tag{2.8}$$

It is familiar from classical mechanics that to every function f one can associate a hamiltonian vector field ξ_f in such a way that $\xi_f\, g = \{f\,,\,g\}$. The hamiltonian vector field ξ_f is obtained from df by a tensor Ω mapping 1-forms to vector fields, so that the Poisson bracket of two functions is given by $\{f\,,\,g\} = (X_f, dg) = (\Omega(df), dg)$, with $(,)$ being the natural pairing between vector fields and 1-forms. In local coordinates, Ω coincides with the fundamental Poisson brackets. In other words, the map Ω carries the same information as the Poisson brackets.

In analogy, we define Poisson brackets on the space of supersymmetric Lax operators by defining a map $J : S_n^* \to S_n$ in such a way that the Poisson bracket of two functions F and G is given by

$$\{F\,,\,G\} = D_{J(dF)}G = (D_{J(dF)}, dG) = (-1)^{|J|+|F|+|G|+n+1}\,\mathrm{Str}\,(J(dF)dG)\ . \quad (2.9)$$

The map Ω in this case is $dF \mapsto D_{J(dF)}$; although, because of the rather formal nature of our geometrical setting, it is the map J that plays the more relevant rôle. Demanding that the Poisson brackets defined by J obey the correct (anti)symmetry properties and the Jacobi identity imposes strong restrictions on the allowed maps J. Maps obeying these conditions are often called "hamiltonian". A hamiltonian map J was constructed in **[10]** via a supersymmetric Miura transformation. We briefly review this construction.

Let us factorize $L = (D - \Phi_n)(D - \Phi_{n-1}) \cdots (D - \Phi_1)$. This defines the U_i as differential polynomials in the Φ_j. We define the fundamental Poisson brackets of the Φ_j as follows. If we let $X = (x, \theta)$ and $Y = (y, \omega)$, then

$$\{\Phi_i(X)\,,\,\Phi_j(Y)\} = (-1)^i \delta_{ij} D\delta(X - Y)\ , \quad (2.10)$$

where $\delta(X - Y) = \delta(x - y)(\theta - \omega)$.

Let $F = \int_B f$ and $G = \int_B g$ be two functions with f and g differential polynomials in the U_k. Via the Miura transformation we can think of them as differential polynomials in the Φ_j. Their Poisson brackets can then be read off from (2.10) :

$$\{F\,,\,G\} = \int_B \sum_{i=1}^{n} (-1)^i \left(D \frac{\delta f}{\delta \Phi_i} \right) \frac{\delta g}{\delta \Phi_i}\ . \quad (2.11)$$

We must first calculate $\frac{\delta f}{\delta \Phi_i}$. The variation of F can be computed in two ways:

$$\delta F = \int_B \sum_{i=1}^{n} \delta\Phi_i \frac{\delta f}{\delta \Phi_i} = \int_B \sum_{k=0}^{n-1} \delta U_k \frac{\delta f}{\delta U_k} = (-1)^{|F|+n+1} \mathrm{Str}\,(\delta L dF) \ . \tag{2.12}$$

Defining $\nabla_i \equiv D - \Phi_i$, one computes

$$\delta L = -\sum_{i=1}^{n} \nabla_n \cdots \nabla_{i+1} \delta\Phi_i \nabla_{i-1} \cdots \nabla_1 \ . \tag{2.13}$$

Inserting this into (2.12) one finds after some reordering inside the integrals

$$\frac{\delta f}{\delta \Phi_i} = (-1)^{|F|(n+i+1)+i} \,\mathrm{sres}\, \left(\nabla_{i-1} \cdots \nabla_1 dF \nabla_n \cdots \nabla_{i+1}\right) \ . \tag{2.14}$$

Plugging this into (2.11) and after some standard manipulations[13],[10] with SΨDO's, one finds

$$\{F\, ,\, G\} = (-1)^{|F|+|G|+n}\, \mathrm{Str}\, \left[L(dF\, L)_+ dG - (LdF)_+ L dG\right] \ , \tag{2.15}$$

which is the supersymmetric analog of the second Gel'fand-Dickey bracket[8]. From this and (2.9) , the map $J : S_n^* \to S_n$ can be read off

$$J(X) = (LX)_+ L - L(XL)_+ = L(XL)_- - (LX)_- L \ . \tag{2.16}$$

The fundamental Poisson brackets of the U_k can, in turn, be read off from the map J as follows.

Suppose we write the fundamental Poisson brackets in the following form

$$\{U_i(X)\, ,\, U_j(Y)\} = \Omega_{ij} \cdot \delta(X - Y) \ , \tag{2.17}$$

where the Ω_{ij} are differential operators at the point X. If $F = \int_B \sum_i U_i A_i$, $G = \int_B \sum_j U_j B_j$ are homogeneous linear functions, then their Poisson bracket can be computed from (2.17) yielding

$$\{F\,,\,G\} = \sum_{ij} (-1)^{(|F|+1)(n+i)+i+j} \int_B A_i \Omega_{ij} B_j \ . \tag{2.18}$$

On the other hand we can compute this from (2.9) . The gradient of F is given simply by $dF = \sum_k (-1)^k D^{-k-1} A_k$, and similarly for G with B_k replacing A_k. Since $J(dF) \in S_n$ is linear in dF, it defines differential operators J_{ij} by $J(dF) = \sum_{ij} (-1)^{|F|(n+i)+n+1} (J_{ij}(-1)^j A_j) D^i$. Plugging this into (2.9) one readily obtains

$$\{F\,,\,G\} = (-1)^n \sum_{ij} \int_B (-1)^{(|F|+n)(i+j)+ij+j+ni} A_i J^*_{ji} B_j \ , \tag{2.19}$$

where the adjoint K^* of a homogeneous differential operator K is defined by $\int_B (KA)B = (-1)^{|K||A|} \int_B A(K^*B)$ for A and B any two homogeneous superfields. Comparing (2.19) with (2.18) we get

$$\Omega_{ij} = (-1)^{ij+n} J^*_{ji} = J_{ij} \ . \tag{2.20}$$

From (2.15) it follows immediately that the Ω_{ij} are at most quadratic in the U_i. Thus, in terms of the U_i, we obtain associative superalgebras with quadratic relations. These are the supersymmetric analogues of the W algebras associated to $GL(n)$ in the Drinfel'd-Sokolov scheme[14]; and thus we tentatively call them $SWGL(n)$.

Drinfel'd and Sokolov constructed many other classical W algebras as reductions of $WGL(n)$. In particular, W_n was obtained by imposing that the next to highest order term in the n^{th} order Lax operator vanish. One can easily compute that this is a second class constraint so that the space of such Lax operators inherits a well defined Poisson bracket. In our case we see a sharp distinction between the even and odd n cases. It is easy to see that if n is even, the constraint $U_{n-1} = 0$ is actually first class; in fact, if and only if n is even, $\Omega_{n-1,n-1}$ vanishes. This means that the induced Poisson bracket in the constrained submanifold is not well defined unless we impose an additional constraint that has a nondegenerate Poisson bracket with U_{n-1}. On the other hand, if n is odd, then the constraint $U_{n-1} = 0$ is second class and the induced Poisson bracket is well defined and yields, as we will see, classical W-superalgebras extending the $N = 2$ superconformal algebra.

§3 An Explicit Example: from $SWGL(3)$ to $N = 2$ Supervirasoro.

In this section we compute explicitly the simplest of the series of algebras we construct: the reduction of $SWGL(3)$ by the constraint $U_2 = 0$. It is easily seen that $\{U_2(X)\,,\, U_2(Y)\} = -\delta'(X-Y)$ whence it defines a second-class constraint. It is customary when faced with second-class constraints to write down immediately the Dirac bracket. However, since our geometric setting is rather formal, this procedure is not guaranteed to be consistent. We will, aided by analogy with the symplectic setting, derive the induced Poisson bracket from first principles. For this we digress momentarily to review the finite dimensional symplectic case.

Suppose M is a finite-dimensional symplectic manifold and M_o is a submanifold given by some second-class constraints. Because the constraints are second-class, the symplectic form Ω on M restricts to a symplectic form on M_o: to evaluate it on two vector fields on M_o we simply view the vector fields as vector fields on M tangent to M_o and apply the symplectic form on M. This is possible since on $M_o \subset M$ there is a canonical map embedding the tangent bundle TM_o of M_o into the one of M. On the other hand, in order to write down the Poisson bracket of two functions on M_o we need to first define their associated hamiltonian vector fields and this requires knowledge of their gradients. The gradient of a function is a 1-form and, unlike vector fields, there is no canonical way to view 1-forms on M_o as 1-forms on M. In fact, such a choice is equivalent to a choice of complement to TM_o in TM on M_o; in other words, to a choice of normal bundle. Once a complement is chosen, $T^*M_o \subset T^*M$ can be identified with those 1-forms annihilating the normal vectors. In general there is no natural choice for the normal bundle, but in the symplectic case there is. We define the normal bundle as the symplectic complement of TM_o. This is to be compared with the riemannian setting. If we had a metric on M then we could always define the normal bundle as those tangent vectors on M perpendicular to M_o. In the symplectic case, this only works if the symplectic form is non-degenerate when restricted to M_o so that there are no vectors that are both tangent to M_o and symplectically perpendicular to it. Luckily when M_o is defined by second-class constraints this is the case. Then to define the Poisson brackets of two functions on M_o, we first extend the functions to M, we compute their gradients in M and from them their hamiltonian vector fields. We finally apply the symplectic form to their perpendicular projections. The resulting object is a function on M_o which can be shown to be independent of the extension. In symbols, if we write $TM = TM_o \oplus TM_o^\perp$ on M_o, and $\xi = \xi_o + \xi^\perp$ as the associated decomposition of a vector, then the induced bracket of two functions f and g on M_o is given by

$$\{f\,,\,g\}_o = \Omega(\xi_f - \xi_f^\perp, \xi_g - \xi_g^\perp)\ , \tag{3.1}$$

where ξ_f and ξ_g are, respectively, the hamiltonian vector fields associated to the extensions of f and g to functions on M. A natural basis for the normal bundle to M_o is given by the hamiltonian veotor fields associated to the constraints defining M_o. Computing the above bracket in that basis yields the familiar formula for the Dirac bracket.

Another equivalent way of defining the 1-forms on M_o is to say that they are the 1-forms on M which get mapped to TM_o under the map $T^*M \to TM$ induced by the symplectic form. This interpretation is better suited for our formal geometric setting, since we don't quite have a symplectic form, but rather a map J taking (formal) 1-forms to (formal) vector fields.

With these words behind us, we now resume the case at hand. Let M denote the space of supersymmetric Lax operators of order 3 and M_o the submanifold defined by $U_2 = 0$. If $L = D^3 + U_1 D + U_0$ is a point in M_o, and $X = D^{-1}X_0 + D^{-2}X_1 + D^{-3}X_2$ is a 1-form on M, we will see how its coefficients are related so that $J(X) = (LX)_+L - L(XL)_+$ is a vector tangent to M_o at L. We will see that demanding that the coefficient of D^2 in $J(X)$ vanishes, fixes X_2 in terms of X_0, X_1, and the U_i. A slightly tedious computation yields

$$\begin{aligned} J(X) =&(-1)^{|X|}\Big[X_2' + (X_0U_1)' - X_1'' - X_0'''\Big]D^2 \\ &+ \Big[U_1'X_1 - 2U_0X_1 - X_2'' - X_1''' - U_0X_0'\Big]D \\ &+ (-1)^{|X|}\Big[X_2''' - X_0^{[5]} + (U_0X_1)' - (U_0X_0)'' + (U_1X_0)''' \\ &- U_1X_0''' + U_1X_2' + U_1(U_1X_0)' - U_0X_0''\Big] \ . \end{aligned} \tag{3.2}$$

Demanding that the coefficient of D^2 vanishes yields $X_2 = X_0''+X_1'-U_1X_0$. Notice that the equation actually involves X_2' but that we could solve for X_2 without having to integrate since the whole equation was a perfect derivative. This is no accident, and we will see that this will always be the case. Whether or not this is important is not clear since in the rest of the the expression for $J(X)$, X_2 only appears through its derivatives, but this may just be an accident of low n. Substituting for X_2 in (3.2) yields after some more algebra

$$\begin{aligned} J(X) = &\left\{\left[-D^4 - U_0D + D^2U_1\right]\cdot X_0 - \left[2D^3 + 2U_0 - U_1'\right]\cdot X_1\right\} D \\ &+ (-1)^{|X|}\left\{\left[-U_0D^2 - D^2U_0\right]\cdot X_0 + \left[D^4 + U_1D^2 + DU_0\right]\cdot X_1\right\} \ , \end{aligned} \tag{3.3}$$

which define the operators J_{ij} and, hence, by (2.20) , the operators Ω_{ij} appearing in the fundamental Poisson brackets of the U_i:

$$\begin{aligned} \Omega_{00} =& \ \ U_0D^2 + D^2U_0 \\ \Omega_{01} =& - D^4 - U_1D^2 - DU_0 \\ \Omega_{10} =& - D^4 + D^2U_1 - U_0D \\ \Omega_{11} =& - 2D^3 - 2U_0 + U_1' \ . \end{aligned} \tag{3.4}$$

Notice that $\Omega_{00}^* = -\Omega_{00}$, $\Omega_{11}^* = \Omega_{11}$, and $\Omega_{01}^* = \Omega_{10}$ as expected from the grading of the U_i.

If we now define $\mathbb{T} \equiv U_0 - \frac{1}{2}U_1'$, $\mathbb{J} \equiv U_1$ we find the classical version of the $N=2$ supervirasoro algebra:

$$
\begin{aligned}
\{\mathbb{T}(X)\,,\,\mathbb{T}(Y)\} &= \left[\frac{1}{2}D^5 + \frac{3}{2}\mathbb{T}D^2 + \frac{1}{2}\mathbb{T}'D + \mathbb{T}''\right]\delta(X-Y)\\
\{\mathbb{T}(X)\,,\,\mathbb{J}(Y)\} &= \left[-\mathbb{J}D^2 + \frac{1}{2}\mathbb{J}'D - \frac{1}{2}\mathbb{J}''\right]\delta(X-Y)\\
\{\mathbb{J}(X)\,,\,\mathbb{J}(Y)\} &= -\left[2D^3 + 2\mathbb{T}\right]\delta(X-Y)\ .
\end{aligned}
\tag{3.5}
$$

The first equation identifies $\mathbb{T}$ as the super energy-momentum tensor, whereas the second identifies $\mathbb{J}$ as an $N=1$ superconformal primary of weight 1.

§4 The General Reduction: from $SWGL(2k+1)$ to $N=2$ W-superalgebras.

In this section we describe the general reduction of $SWGL(2k+1)$ obtained by imposing the constraint $U_{2k}=0$ on the space M of supersymmetric Lax operators of the form $L = D^{2k+1} + \cdots$. A quick calculation shows that $\{U_{2k}(X)\,,\,U_{2k}(Y)\} = -D\delta(X-Y)$. Since the operator $-D$ is formally invertible, the constraint is second-class and we expect that the constrained submanifold M_o inherits a well-defined Poisson bracket from M. We will first show that this in indeed the case and that the resulting fundamental Poisson bracket defines a W-superalgebra extending the $N=2$ supervirasoro algebra.

Let $L = D^{2k+1} + U_{2k-1}D^{2k-1} + \cdots U_0$ be a point in the constrained submanifold M_o. We want to define 1-forms on M_o. According to the discussion in the previous section, we define them as 1-forms on M which get mapped under J to tangent vectors to M_o at L, *i.e.* , to differential operators of order at most $2k-1$. Let $X = \sum D^{-l-1}X_l$ be a 1-form on M. Under J it gets mapped to the tangent vector $X \mapsto J(X) = (LX)_+L - L(XL)_+$, which is, in general, a differential operator of order at most $2k$. X will be a 1-form on M_o if we demand that the coefficient of $J(X)$ multiplying D^{2k} vanish. The coefficient of $J(X)$ of order $2k$ is given by the super-residue of $J(X)D^{-2k-1}$. Now, using that $J(X)$ is also given by

$L(XL)_- - (LX)_-L$, we find

$$\begin{aligned}\operatorname{sres} J(X)D^{-2k-1} &= \operatorname{sres}\left[L(XL)_-D^{-2k-1} - (LX)_-LD^{-2k-1}\right]\\ &= \operatorname{sres}\left[L(XL)_-D^{-2k-1} - LX(LD^{-2k-1})_+\right]\ ,\end{aligned}$$

where we have used that $\operatorname{sres} A_-B = \operatorname{sres} A_-B_+ = \operatorname{sres} AB_+$ for any two SΨDO's A and B. Using further that $(LD^{-2k-1})_+ = 1$, we find

$$\begin{aligned}0 =& \operatorname{sres} J(X)D^{-2k-1}\\ &= \operatorname{sres}\left[L(XL)_-D^{-2k-1} - LX\right]\\ &= \operatorname{sres}\left[(-1)^{|X|}XL - LX\right]\\ &= -\operatorname{sres}\,[L\,,\,X],\end{aligned} \tag{4.1}$$

where for the next to last step we have simply noticed that $\operatorname{sres} L(XL)_-D^{-2k-1} = \operatorname{sres} D^{2k+1}(XL)_-D^{-2k-1}$ and we have then computed it. Therefore 1-forms on M_o are 1-forms X on M which satisfy the additional condition that $\operatorname{sres}\,[L\,,\,X] = 0$. A closer look at this relation shows that this determines X_{2k} in terms of the other coefficients and of the U_i. Indeed,

$$\begin{aligned}0 &= \operatorname{sres}\,[L\,,\,X]\\ &= \operatorname{sres} LD^{-2k-1}X_{2k} + \operatorname{sres}\,[L\,,\,\sum_{l=0}^{2k-1} D^{-l-1}X_l]\\ &= -(-1)^{|X|}X'_{2k} + \operatorname{sres}\,[L\,,\,\sum_{l=0}^{2k-1} D^{-l-1}X_l]\ ,\end{aligned}$$

whence

$$X'_{2k} = (-1)^{|X|}\operatorname{sres}\,[L\,,\,\sum_{l=0}^{2k-1} D^{-l-1}X_l]\ . \tag{4.2}$$

Notice that although this is an equation for X'_{2k}, we can actually solve for X_{2k} since the RHS, being the super-residue of a graded commutator, is also a perfect derivative. Thus in solving for X_{2k} we simply drop the "constant of integration".

We now write down the fundamental Poisson brackets induced on M_o from those of M. Let us write $J(X) = \sum_{ij}(-1)^{(|X|+1)(i+1)} J_{ij} X_j D^i$ which defines the differential operators J_{ij}. Demanding that the top term vanishes becomes a linear relation $\sum_j J_{2k,j} X_j = 0$, from which it follows that

$$X_{2k} = -J_{2k,2k}^{-1} \sum_{j=0}^{2k-1} J_{2k,j} X_j \ . \tag{4.3}$$

Plugging this back into $J(X)$ we find

$$J(X) = \sum_{i,j=0}^{2k-1} (-1)^{(|X|+1)(i+1)} \widetilde{J}_{ij} X_j D^i \ , \tag{4.4}$$

with

$$\widetilde{J}_{ij} = J_{ij} - J_{i,2k}\, J_{2k,2k}^{-1}\, J_{2k,j} \ , \tag{4.5}$$

which implies that the new fundamental Poisson bracket is none other than the "naïve" Dirac bracket. In the usual geometric setting the Jacobi identity for the Dirac bracket is automatic since the Jacobi identity is equivalent to the symplectic form being closed on M_o and this follows trivially from the fact that the induced symplectic form on the constrained submanifold is the pull-back of the one on M via the embedding $M_o \hookrightarrow M$. In our formal setting, however, we cannot appeal too much to the geometry unless the necessary geometric facts have been suitably algebraized and this, although interesting and possible, is beyond the scope of this paper.

Therefore to prove that the induced bracket does indeed obey the Jacobi identity we will proceed as in **[13]** and make sure that the proof in **[10]** of the analogous fact for the unconstrained manifold goes through essentially unmodified. Since the proof invoked the Miura transformation, we need to understand the constraint $U_{2k} = 0$ in this context. A simple computation shows that $U_{2k} = \sum_i (-1)^i \Phi_i$. Thus, in terms of the fundamental fields Φ_i, the constrained submanifold is a "hyperplane". Our first task is to define the gradient of a function on this hyperplane.

As before, we define it in such a way that the associated hamiltonian vector field is tangent to the hyperplane. For this we have to find out what the hamiltonian vector field of a function looks like. Let $F = \int_B f$ be any function. We write its gradient as a $2k+1$-tuple $dF = \left(\frac{\delta f}{\delta \Phi_i}\right)$. We also write vector fields as $2k+1$-tuples $\xi = \left(\xi^i\right)$, with pairing given by

$$(\xi, dF) = \int_B \sum_i \xi^i \frac{\delta f}{\delta \Phi_i} = \xi \cdot F \ . \tag{4.6}$$

Now if $F = \int_B f$ and $G = \int_B g$ are any two functions, their Poisson bracket is given by $\xi_F \cdot G$, where ξ_F is the hamiltonian vector field associated to F. Comparing (4.6) with (2.11) we read off the coefficients of the hamiltonian vector field $\xi_F^i = (-1)^i \left(\frac{\delta f}{\delta \Phi_i}\right)'$. We now demand that ξ_F be tangent to the constrained submanifold; in other words, that the hamiltonian vector field preserves the constraint. This translates into

$$\begin{aligned} \xi_F \cdot \sum_i (-1)^i \Phi_i = 0 &\Rightarrow \sum_i (-1)^i \xi_F^i = 0 \\ &\Rightarrow \sum_i \left(\frac{\delta f}{\delta \Phi_i}\right)' = 0 \ . \end{aligned} \tag{4.7}$$

Observing that the proof in **[10]** was essentially combinatorial starting from the expression (2.14) for the gradient of a function, we can appeal to it provided that (2.14) is consistent with the constraint; that is, provided that it obey (4.7) . This will provide a constraint on dF which will turn out to be precisely (4.1) . We proceed to prove this now.

Replacing the supercovariant derivative of $\frac{\delta f}{\delta \Phi_i}$ by the graded commutator $[\nabla_i \, , \, \frac{\delta f}{\delta \Phi_i}]$, and using that for any SΨDO P its super-residue can be written as

$$\operatorname{sres} P = (P_- \nabla_i)_+ = (-1)^{|P|+1} (\nabla_i P_-)_+ \ , \tag{4.8}$$

we find

$$\sum_i \left(\frac{\delta f}{\delta \Phi_i}\right)' = \sum_i (-1)^{(|F|+1)i} \Big[\nabla_i (\mathrm{sres}\, \nabla_{i-1} \cdots \nabla_1 dF \nabla_n \cdots \nabla_{i+1}) - (-1)^{|F|} \left(\mathrm{sres}\, \nabla_{i-1} \cdots \nabla_1 dF \nabla_n \cdots \nabla_{i+1}\right) \nabla_i\Big]$$
$$= \sum_i (-1)^{(|F|+1)i} \Big[\nabla_i ((\nabla_{i-1} \cdots \nabla_1 dF \nabla_n \cdots \nabla_{i+1})_- \nabla_i)_+ - (\nabla_i (\nabla_{i-1} \cdots \nabla_1 dF \nabla_n \cdots \nabla_{i+1})_-)_+ \nabla_i\Big] \ .$$

Notice that we can drop the $-$ subscripts because, if we replace them by $+$, we could drop the outer $+$ subscripts and the terms cancel pairwise. Dropping the $-$ subscripts, we find

$$\sum_i \left(\frac{\delta f}{\delta \Phi_i}\right)' = \sum_i (-1)^{(|F|+1)i} \Big[\nabla_i (\nabla_{i-1} \cdots \nabla_1 dF \nabla_n \cdots \nabla_i)_+ - (\nabla_i \cdots \nabla_1 dF \nabla_n \cdots \nabla_{i+1})_+ \nabla_i\Big]$$
$$= \sum_i (-1)^{(|F|+1)i} \Big[(\nabla_i \cdots \nabla_1 dF \nabla_n \cdots \nabla_{i+1})_- \nabla_i - \nabla_i (\nabla_{i-1} \cdots \nabla_1 dF \nabla_n \cdots \nabla_i)_-\Big] \ .$$

Since this is a zeroth order differential operator we can project to its $+$ part for free and using the expressions above for the super-residue we obtain

$$\sum_i \left(\frac{\delta f}{\delta \Phi_i}\right)' = \sum_i (-1)^{(|F|+1)i} \Big[\mathrm{sres}\, \nabla_i \cdots \nabla_1 dF \nabla_n \cdots \nabla_{i+1} - (-1)^{|F|+1} \mathrm{sres}\, \nabla_{i-1} \cdots \nabla_1 dF \nabla_n \cdots \nabla_i\Big]$$
$$= (-1)^{|F|+1} \mathrm{sres}\, L\, dF - \mathrm{sres}\, dF\, L$$
$$= - \mathrm{sres}\, [dF\, ,\, L] \ .$$

Therefore as long as the gradient of a function is defined so that it obeys $\mathrm{sres}\, [dF\, ,\, L] = 0$, the proof in **[10]** goes through and the reduced supersymmetric Gel'fand-Dickey brackets obey the Jacobi identity.

Finally we prove that the fundamental Poisson brackets (4.5) on the constrained submanifold contain the $N = 2$ supervirasoro algebra as a subalgebra. This is done by computation. After reducing the supersymmetric Lax operator, the first two fields appearing are U_{2k-1} and U_{2k-2} which have weights 1 and $\frac{3}{2}$, respectively. It can be shown quite generally that, before reduction, the fundamental Poisson brackets of U_{2k}, U_{2k-1}, and U_{2k-2} close among themselves. It then follows that after reduction the induced Poisson brackets of U_{2k-1} and U_{2k-2} will still close into them, so that these two fields generate a subalgebra. We will see that this is (after some field redefinition) the $N = 2$ supervirasoro subalgebra.

There are two (at least) different ways we could compute the induced fuundamental Poisson brackets of U_{2k-1} and U_{2k-2}. We could compute the fundamental Poisson brackets Ω_{ij} for $i, j = 2k, 2k-1, 2k-2$ and then use formula (4.5) or, equivalently, compute the mapping $J(X)$ for X a 1-form on the constrained submanifold, *i.e.* , after imposing $\mathrm{sres}\,[L\,,\,X] = 0$; or we could compute the brackets directly via the Miura transformation from the fundamental brackets of the basic fields Φ_i after taking into account the reduction to the hyperplane $\sum_i (-1)^i \Phi_i = 0$. Both ways are, of course, equivalent and equally computationally involved. We choose to present the former computation since at least it has the advantage of keeping things manifestly in terms of the U_j; whereas the Miura calculation would, at the end of the day, involve recombining the basic fields Φ_i back into the U_j: a slightly tedious task.

Thus let $L = D^{2k+1} + U_{2k-1} D^{2k-1} + U_{2k-2} D^{2k-2} + \cdots$ and let $X = \sum_j D^{-j-1} X_j$ be a 1-form. Because we are interested in fundamental Poisson brackets involving only U_{2k-1} and U_{2k-2} it suffices to extract the terms in $J(X)$ which depend on X_{2k-1} and X_{2k-2}. Part of this dependence will come from imposing $\mathrm{sres}\,[L\,,\,X] = 0$. Computing this we find

$$\mathrm{sres}\,[L\,,\,X] = 0 \Rightarrow \; X_{2k} = k X'_{2k-1} + k X''_{2k-2} - X_{2k-2} U_{2k-1} + \cdots \tag{4.9}$$

where the $\cdots$ stand for terms involving $X_{j<2k-2}$. We now compute $J(X)$ and keep only those terms involving X_{2k-1} and X_{2k-2}. After a little algebra one finds

$$\begin{aligned}
\widetilde{J}_{2k-1,2k-1} &= -k(k+1)D^3 + U'_{2k-1} - 2U_{2k-2} \\
\widetilde{J}_{2k-2,2k-1} &= \frac{1}{2}k(k+1)D^4 + U_{2k-1}D^2 + DU_{2k-2} \\
\widetilde{J}_{2k-1,2k-2} &= -\frac{1}{2}k(k+1)D^4 + D^2U_{2k-1} - U_{2k-2}D \\
\widetilde{J}_{2k-2,2k-2} &= -U_{2k-2}D^2 - D^2U_{2k-2}
\end{aligned}$$

from where we can deduce the fundamental Poisson brackets $\widetilde{\Omega}$ via equation (2.20) . Defining $\mathbb{T} \equiv U_{2k-2} - \frac{1}{2}U'_{2k-1}$ and $\mathbb{J} = U_{2k-1}$, we find that they obey

$$\begin{aligned}
\{\mathbb{T}(X)\,,\,\mathbb{T}(Y)\} &= \left[\frac{1}{4}k(k+1)D^5 + \frac{3}{2}\mathbb{T}D^2 + \frac{1}{2}\mathbb{T}'D + \mathbb{T}''\right]\delta(X-Y) \\
\{\mathbb{T}(X)\,,\,\mathbb{J}(Y)\} &= \left[-\mathbb{J}D^2 + \frac{1}{2}\mathbb{J}'D - \frac{1}{2}\mathbb{J}''\right]\delta(X-Y) \\
\{\mathbb{J}(X)\,,\,\mathbb{J}(Y)\} &= -\left[k(k+1)D^3 + 2\mathbb{T}\right]\delta(X-Y)\ ,
\end{aligned} \tag{4.10}$$

which, once again, is the classical $N = 2$ supervirasoro algebra. One can check that for the special case $k = 1$ we do, in fact, recover equation (3.5) .

§5 Conclusions

In this paper we have obtained an infinite series of $N = 2$ exteded superalgebras by reduction of a recently constructed Poisson structure on the space of supersymmetric Lax operators of odd order. This reduction is the supersymmetric analogue of the Drinfel'd-Sokolov reduction of the second Gel'fand-Dickey bracket which is associated to the A_n series of classical Lie algebras and which yields the W_n algebras of Fateev and Lykyanov.

For the Lax operator of order $2k+1$, the spectrum of the resulting algebra consists of ($N = 1$) superfields U_j for $j = 0, 1, \ldots, 2k-1$, of naïve weights $k-(j-1)/2$. In particular, U_{2k-1} and U_{2k-2} have been shown to generate a $N = 2$ supervirasoro algebra. It is then a natural conjecture to expect that each remaining field U_j for j odd gives rise to an $N = 2$ superconformal primary field $\widetilde{U}_j$ obtained by deforming U_j via the addition of differential polynomials in the $U_{i>j}$. Similarly the remaining U_j with j even give rise to their partners. We have checked this explicitly for the simplest example, but we have not proven it in general. Nevertheless, we have no doubt of the validity of this statement and we hope to return to this point in a future paper. It should be remarked that the bosonic spectrum of these algebras does not in general agree with the spectrum of the nonsupersymmetric W_n algebras. Therefore one does not recover these algebras upon truncation, like the simple case of $SWGL(2)$ might suggest. It seems plausible—like the reader can check explicitly for $SWGL(3)$—that further reductions might yield conformal algebras with the spectrum of W_n, but these reductions do not seem natural in this context, since they involve constraining part of the bosonic spectrum as well.

As mentioned in the introduction, this series of algebras has been also obtained in [**7**] from the $Sl(n+1|n)$ Toda field theory, where it appears as the algebra of conserved quantities. It is a very interesting open question to obtain this algebra as hamiltonian reduction of the corresponding affine algebra, thus recovering the connection between Lie superalgebras and W-superalgebras existing in the bosonic case. Work on this is in progress.

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