# A solution to the Einstein field equations in the presence of a plane-symmetric scalar Higgs field

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**Abstract**

Exact solutions to the Einstein field equations are obtained for a static configuration consisting of two regions of different energy density. These regions correspond to two different expectation values of a scalar Higgs field in the context of a GUT phase transition similar to that of the inflationary scenario. The two regions are separated by a "wall" of negligible thickness. These solutions constitute the necessary initial conditions to analyze the time evolution of the system, task which will be explored in further research.

A mi familia con orgullo y cariño


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# Contents

# 1 Introduction

*The effort to understand the universe*
*is one of the very few things that lifts*
*human life a little above the level of farce,*
*and gives it some of the grace of tragedy.*
***Steven Weinberg***

Cosmology is probably the oldest of the sciences. Ever since the beginning of civilized life on this planet, there have always existed men who have devoted their intellect to understanding the cosmos. A simple glance above during a clear starry night gives us a clue as to why this is so. Many models —scientific or otherwise— have been proposed in order to bring about order to this area, but it was not until the advent of Einstein's General Relativity, that we actually possessed a working theory to predict and examine the macroscopic phenomenology of gravity. Ever since many models have been put forward which attempt to represent the overall structure of the entire universe, concentrating only on the very large-scale features.

Today the most widely accepted cosmological model is the so-called hot big bang model, which we will hereafter denominate by the "standard model." In the following sections we will introduce the main features of this model and we will look at its main problems. We will then explore the so-called inflationary scenario as a possible solution to these problems, concentrating mainly in the "new" inflationary model, a slight variation of the original work. Finally, we will motivate the present work within that context.

## 1.1 The Standard Model

The hot big bang model treats the universe as a perfect fluid (gas) which is both homogeneous and isotropic. Hence it can be described (in comoving coordinates[1]) by the Robertson-Walker metric[2]

$$ds^2 = -dt^2 + \mathrm{R}^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right), \tag{1.1}$$

where $r$ and $\mathrm{R}(t)$ have been scaled in such a way that $k$ takes on the values $+1$,0 or $-1$, according to whether it describes a closed, flat or open universe, respectively.

[1]These are such that events, although in relative motion, have fixed spatial coordinates. This is analogous to lattice points in a permanently labeled but expanding (or contracting) Cartesian lattice.

[2]I have set $c = 1$ throughout.

The evolution of $R(t)$ is determined by the Einstein field equations

$$\frac{d^2R}{dt^2} = -\frac{4\pi}{3}G(\rho + 3p)R\,, \tag{1.2a}$$

and

$$H^2 + \frac{k}{R^2} = \frac{8\pi}{3}G\rho\,, \tag{1.2b}$$

where $p$ is the fluid's pressure, $\rho$ is its density and $H \equiv \frac{1}{R}\frac{dR}{dt}$ is the Hubble "constant." It is clear from equation (1.2b) that the condition $k = 0$ defines a critical density $\rho_c \equiv \frac{3H^2}{8\pi G}$ necessary for our space to be flat. If $k = +1$, then $\rho > \rho_c$ and consequently at some time $t$, $H = 0$. That is, there is enough gravitational self-attraction to stop the universe's expansion. Conversely, if $k = -1$, H will ever vanish. Let us, then, define the quantity $\Omega \equiv \frac{\rho}{\rho_c}$ and let us rewrite equation (1.2b) as follows

$$\frac{\Omega - 1}{\Omega} = \frac{3k}{8\pi G\rho R^2}\,. \tag{1.3}$$

In order to fully describe the dynamics of the system, we need an equation of state $F(p,\rho) = 0$ governing our perfect fluid. There are two different cases worth analyzing. These cases are thought to represent two different stages of our "post-early" universe. During the early stages of our "post-early" universe, the temperature (and, hence, the thermal energy) of the fluid was so high that the thermal excitation prevented the existence of bound states. The particles were, thus, free. Therefore, the mass density is dominated by the thermal radiation of effectively massless (since the thermal energy is much larger than the rest mass energy) particles at temperature T, whence

$$\rho = cT^4\,, \tag{1.4}$$

where $c$ is a constant depending on the number of species of effectively massless particles. Hence, we call this era radiation-dominated. Since the expansion is supposed to take place adiabatically (i.e., the system remains in thermal equilibrium), we obtain a further relation

$$\frac{d}{dt}\left(R(t)T\right) = 0\,. \tag{1.5}$$

Using these relations as the equations describing our state, we can substitute them into equations (1.2) yielding[3]

$$R(t) = b_r t^{\frac{1}{2}}\,, \tag{1.6}$$

[3] We have approximated these results by setting $k = 0$ in (1.2b) since we will see that for the present experimental values of $k$, it could not have been very far from 0 at that time.

where $b_r$ is a constant.

After some time (approximately $10^5$ years), the temperature was low enough for neutral atoms to form (the time of Hydrogen recombination) and the mass density was now beginning to be dominated by the matter in the universe. Assuming that there is no net mass flux through any hypothetical comoving sphere of radius $r\mathrm{R}(t)$, we are lead to the analogous equation to (1.4) for this matter-dominated era

$$\frac{d}{dt}\left(\rho \mathrm{R}^3\right) = 0\,. \tag{1.7}$$

Assuming that the universe is still expanding adiabatically, we can make use of equation (1.5) to obtain the following result

$$\mathrm{R}(t) = b_m t^{\frac{2}{3}}\,, \tag{1.8}$$

where $b_m$ is a constant.

Although in the next section we will examine the main problems of the standard model, it is also important to remark its successes. The standard model manages to explain the cosmic background radiation, the observed cosmological redshift and the primordial abundance of the light elements in good agreement with experimental data.

## 1.2 Problems of the Standard Model

With the relations obtained in the last section we are, in principle, equipped to extrapolate back in time the experimental results we obtain today. We do have a limit, however. The hot big bang model, and for that matter any (general) relativistic cosmological model, can only aspire to be valid from some time $t_{\mathrm{P}}$ onwards. The reason being, that for any earlier times quantum-gravitational effects are expected to dominate, and today a quantum theory of gravity is nowhere in sight. This time is denominated the Planck era, and it is defined as the time when the thermal energy of the universe is comparable to the rest mass energy of a particle having a mass $\mathrm{M_P} \equiv \mathrm{G}^{\frac{1}{2}}$, where G is the gravitational coupling constant.

This, however, is the least important of the problems facing the standard model. In this section we will examine the three major problems facing the hot big bang model: the horizon, flatness, and monopole production problems. It is precisely the existence of these three problems which motivates the inflationary scenario.

First of all we shall look into the monopole problem. The standard model predicts a tremendous overproduction of magnetic monopoles.[4] In order to understand how this happens, we must first mention that in the context of grand unified theories (GUTs), magnetic monopoles are in fact topologically stable knots in the expectation value of the Higgs field.[5] Assuming that the Higgs field has a correlation length $\xi$, the Kibble relation[6] gives a rough estimate of the density of monopoles

$$n_{\mathrm{M}} \approx \frac{1}{\xi^3}\,. \tag{1.9}$$

When the universe cools below the critical temperature[7] $T_c \approx 10^{14}$ GeV, it becomes (thermodynamically) favorable for the Higgs field expectation value to align over large distances. However it takes time for these correlations to be established. Causality alone implies[8] that

$$\xi \lessapprox \ell_{\mathrm{H}}\,, \tag{1.10}$$

where $\ell_{\mathrm{H}}$, the Horizon length, is defined to be the maximum distance a light pulse could have traveled since the beginning of time, and it is found to be $2t$ in the radiation-dominated era. This relation gives us an approximate lower bound on $n_{\mathrm{M}}$, and allows us to calculate the ratio $\frac{n_{\mathrm{M}}}{s}$, where $s$ is the entropy density

$$\frac{n_{\mathrm{M}}}{s} \gtrapprox 10^{-13}\,. \tag{1.11}$$

Assuming an adiabatic expansion and neglecting monopole-antimonopole annihilations,[9] we conclude that this ratio should still be approximately the same today. However this implies that $\Omega \gtrapprox 3\times 10^{11}$, and this is clearly wrong, since, among other things, it implies that the age of the universe today has an upper bound of $\approx 3\times 10^4$ years. This fact, reinforced by the lack of experimental evidence for the existence of magnetic monopoles makes it necessary to search for some scenario in which the production of magnetic monopoles is suppressed.

The second problem we will analyze is the so-called horizon problem, which was originally remarked by Wolfgang Rindler[10] as early as 1956. It is known experimentally[11] that there exists a cosmic background radiation of approxim-

[4] J. P. Preskill, Phys. Rev. Lett. **43**, 1365 (1979); Ya. B. Zeldovich and M. Y. Khlopov, Phys. Lett. **79B**, 239 (1978).

[5] The Higgs field, although treated like a scalar, is in fact a multicomponent field that can easily possess a non-trivial topology.

[6] T. W. B. Kibble, J. Phys. **A9**, 1387 (1976).

[7] This temperature corresponds to the energy below which the strong and electro-weak forces become distinguishable.

[8] A. H. Guth and S.-H. Tye, Phys. Rev. Lett. **44**, 631, 963 (E); M. B. Einhorn, D. L. Stein, and D. Toussaint, Phys. Rev. D**21**, 3295 (1980).

[9] See Ref. 4; T. Goldman, E. W. Kolb, and D. Toussaint, Phys. Rev. D**23**, 867 (1981).

[10] W. Rindler, Mon. Not. R. Astron. Soc. **116**, 663 (1956).

[11] This was discovered by Arno Penzias and Robert Wilson in 1964.

ately 2.7°K. Moreover, it was observed that this background radiation is isotropic to one part in $10^3$. This fact implies that at the time of the emission (or decoupling) of this radiation (i.e., the time of hydrogen recombination) the universe was in thermal equilibrium. However, if we extrapolate back in time what today constitutes the observable universe, we notice that, at the time of the radiation decoupling, it was approximately 90 horizon lengths.[12] Furthermore, at the time of the SU(5) phase transition, approximately $10^{-35}$ seconds after the big bang, the size of the region that must be assumed to be at thermal equilibrium is on the order of $10^{83}$ horizon lengths.[13] Therefore the standard model assumes as its initial conditions a thermal equilibrium over many causally disconnected regions. One must, thus, assume that whatever forces caused this equilibrium were capable of violating causality.

Last, let us look at the flatness problem, which was first pointed out by Robert Dicke and James Peebles in 1979.[14] The experimental determination of $\Omega$ gives us range of possible values. Today we feel we have enough evidence to support the fact that

$$0.1 \leq \Omega \leq 2\,. \tag{1.12}$$

Therefore we cannot conclude experimentally the ultimate fate of the universe. It is interesting to notice that these values closely sandwich the value $\Omega = 1$. This is especially worrisome when we realize that this precise value is a point of **unstable** equilibrium within the context of the standard model. Furthermore, the only time scale which appears in the equations of a radiation-dominated universe is the Planck time, $M_P^{-1} \approx 5 \times 10^{-44}$ seconds. A typical closed universe will reach its maximum size on the order of this time scale, whereas a typical open universe will evolve to a value of $\rho \ll \rho_c$. Therefore for a universe to have evolved for $\approx 10^{10}$ years to a value of $\Omega \approx 1$, it had to be very close to 1 in the past. Particularly, at T = 1 MeV, $\Omega$ had to be equal to 1 to one part in $10^{15}$; and at the time of the GUT phase transition, when $T \approx 10^{14}$ GeV, $\Omega$ had to equal 1 to one part in $10^{49}$! This very fine-tuning of parameters must be input as an initial condition to the model.

Given that the horizon and flatness problems depend purely on the initial conditions, it could be thought that, in principle, a working quantum theory of gravity could solve them. However we will see in the next section that the inflationary scenario manages to solve these two problems and also the monopole production problem using only well established physical processes, while at the

[12] A. H. Guth, in *Asymptotic Realms of Physics: Essays in Honor of Francis E. Low*, edited by A. H. Guth, R. L. Jaffe, and K. Huang (MIT Press, Cambridge, 1983).

[13] See Ref. 15

[14] R. H. Dicke and P. J. E. Peebles, in *General Relativity: An Einstein Centenary Survey*, edited by S. W. Hawking and W. Israel, (Cambridge University Press, Cambridge, England, 1979).

same time providing the necessary initial conditions for the standard model to retain its initial successes.

## 1.3 The Inflationary Scenario

Under a modest title, the inflationary universe made its *debut* in January 1981.[15] Created by Alan Guth, a particle theorist, the inflationary model of the universe managed to blend cosmology with recent advances in particle physics, especially in the area of unified gauge theories. Given that these theories are of paramount importance to the inflationary scenario I will briefly review them below.

### 1.3.1 Grand Unified Theories: an Overview

A grand unified theory is symmetric under some simple (i.e., containing no invariant subgroups) gauge group $\mathscr{G}$ which is a valid symmetry at the highest energies. As the energy is lowered the theory undergoes a series of "spontaneous" symmetry breakings into successive subgroups, the last two being the group describing the Weinberg-Salam unification, and the group responsible for the unification of QCD and electromagnetism. In order to understand this process better we shall describe what is meant by spontaneous symmetry breaking.

Consider a theory invariant under some gauge group[16] $\mathscr{G}$. The vacuum (lowest energy) state of the system described by this theory might not be manifestly invariant under $\mathscr{G}$. We then say that the symmetry of the theory is spontaneously broken. A particularly illustrative example is the field theory consisting of a self-interacting ($\phi^4$) massive complex scalar field coupled to a vector meson field. The theory is invariant under U(1). When we consider the vacuum state, several cases arise. In one of these cases, the vacuum state does not correspond to $\phi = 0$, but lies on a ring $|\phi| = \phi_o$. Expanding the Lagrangian density around the new minimum destroys the gauge invariance of the theory, and, moreover, the vector field gains a mass term. This is known as the Higgs mechanism and these scalar fields are known as the Higgs fields. An analogous, but slightly more complicated, Higgs field[17] breaks the GUT symmetry.

[15] A. H. Guth, Phys. Rev. D**23**, 347 (1981); For a general review about the inflationary scenario much along the same lines as this overview, see A. H. Guth, *The New Inflationary Universe*, in Proceedings of the XI Texas Symposium on Relativistic Astrophysics (Austin, Texas, December 13-17, 1982), published by the New York Academy of Sciences.

[16] For a review see, for example, E. S. Abers and B. W. Lee, Phys. Rep. **9C**, 1 (1973).

[17] Hereafter I will denote by Higgs field the linear combination of all the Higgs fields that break the underlying symmetry.

The simplest working GUT (and the one upon which the current inflationary scenario is based[18]) was proposed by Howard Georgi and Sheldon Glashow in 1974 and has SU(5) as the unifying gauge group.[19] At energies > $10^{14}$ GeV, the coupling constants for the strong, electromagnetic and weak interactions are all equal.[20] Hence we say that the forces are unified. For lower energies, the SU(5) symmetry breaks spontaneously into $SU(3) \times SU(2) \times U(1)$ through the workings of the Higgs field. Further down in the energy scale, the latter symmetry breaks into $SU(3) \times U^{em}(1)$ at $\approx 10^2$ GeV. In the inflationary context we will only be dealing with the first symmetry breaking.

There are two important predictions of GUTs worth mentioning. First is the equality of the magnitude of the charges of the electron and the proton, which can only be achieved in the Weinberg-Salam model by a very fine tuning of the parameters. The second important prediction is the value of $\sin^2\theta_W$, where $\theta_W$, the Weinberg angle, is a measure of the relative strengths of the weak and electromagnetic interactions.

There are three consequences of GUTs that are crucial to cosmology. The first is the existence of a phase transition at a critical temperature corresponding to the unification energy, which exists because the non-zero expectation value of the Higgs field is destroyed by thermal fluctuations.[21]

The second consequence concerns magnetic monopoles. Any GUT based on a simple gauge group which eventually breaks to $SU(3) \times U^{em}(1)$ will necessarily[22] contain magnetic monopoles of the t'Hooft-Polyakov type,[23] which are, as mentioned above, topologically stable knots in the expectation value of the Higgs field. These monopoles have masses on the order of $10^{16}$ GeV.

The third consequence is the non-conservation of baryon number. Minimal

[18]At this point it is convenient to remark that although the inflationary scenario is dependent for most calculations on the underlying unified theory, it contains an essence all of its own. Thus, even when SU(5) will be referred as the particle theory upon which the inflationary scenario was originally based, it is crucial to recognize that this scenario is not committed to any one GUT in particular, but that it comprises a play-ground for particle theories in general. In this sense the inflationary scenario can be thought of as a test or experiment for particle theories, having at its disposition a most powerful accelerator: our very own early universe.

[19]H. Georgi and S. L. Glashow, Phys. Rev. Lett. **32**, 438 (1874).

[20]H. Georgi, H. R. Quinn, and S. Weinberg, Phys. Rev. Lett. **33**, 451 (1974); T. J. Goldman and D. A. Ross, Phys. Lett. **84B**, 208 (1979); Nuc. Phys. **B171**, 273 (1980).

[21]D. A. Kirzhnits and A. D. Linde, Phys. Lett. **42B**, 471 (1972); S. Weinberg, Phys. Rev. D**9**, 3357 (1974); L. Dolan and R. Jackiw, Phys. Rev. D**9**, 3320 (1974); for a review, see A. D. Linde, Rep. Prog. Phys. **42**, 389 (1979).

[22]See for example the following reviews: P. Goddard and D. I. Olive, Rep. Prog. Phys. **41**, 1357 (1978); S. Coleman, *The Magnetic Monopole Fifty Year Later* , in the Proc. of the International School of Subnuclear Physics, Ettore Majorana, Erice 1981, ed.

[23]G. t'Hooft, Nucl. Phys. **B79**, 276 (1974); A. M. Polyakov, Pis'ma Zh. Eksp. Teor. Fiz. **20**, 430 (1974) [JETP Lett. **20**, 194 (1974)].

SU(5) predicts a proton lifetime of the order of $10^{30}$ to $10^{33}$ years. This implies that the initial baryon number of the universe may be taken to be zero, and that the baryon number of the observable universe today may have been generated dynamically.[24]

### 1.3.2 The Inflationary Universe

The inflationary scenario makes no assumptions as to the homogeneity and isotropy of the early universe. The only starting assumption of the inflationary scenario is that it is hot ($T > 10^{14}$ GeV) in at least some places, and that at least some of the regions are expanding rapidly enough so that they will cool to the critical temperature $T_c$ before gravitational effects reverse the expansion. In these hot regions, thermal equilibrium would imply $\langle\phi\rangle = 0$, where $\langle\phi\rangle$ denotes the expectation value of $\phi$. However this is not the case, since the universe has not had time to thermalize.[25] Consequently, a further assumption is needed, namely that there exist regions of high energy density with $\langle\phi\rangle \approx 0$, and that some of these regions lose energy with $\phi$ being trapped in the false vacuum. By false vacuum we will refer to the field configuration $\phi = 0$. The term false vacuum is traditionally reserved for configurations that are classically stable. Historically, however, the original inflationary universe consisted of such a region, where the Higgs field was trapped and from where it escaped through quantum tunneling, thus forming bubbles of the new phase. In the "new" inflationary universe,[26] the potential is now different, so that what we call the false vacuum is nothing but a barely unstable configuration. Figure 1 compares the two potentials. The new potential obeys the Coleman-E. Weinberg condition[27]

$$\left.\frac{\partial^2 V}{\partial\phi^2}\right|_{\phi=0} = \left.\frac{\partial^3 V}{\partial\phi^3}\right|_{\phi=0} = 0\,. \tag{1.13}$$

These regions will cool to $T_c$ and, moreover, nucleation rate calculations[28] indicate that they will supercool below $T_c$. The energy density $\rho$ will approach $\rho_o \equiv V(\phi = 0)$. Since this false vacuum is Lorentz-invariant, the energy-momentum tensor must then have the form

$$T_{\mu\nu} = \rho_o g_{\mu\nu}\,. \tag{1.14}$$

[24]For a review, see M. Yoshimura, in *Grand Unified Theories and Related Topics: Proceedings of the 4th Kyoto Summer Institute*, eds. M. Konuma and T. Maskawa (World Scientific, Singapore, 1981).

[25]G. Steigman, Proceedings of the Europhysics Study Conference: Unification of the Fundamental Interactions II, Erice 6-14 October, 1981.

[26]A. D. Linde, Phys. Lett. **108B**, 389 (1982); A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982).

[27]S. Coleman and E. J. Weinberg, Phys. Rev. D**7**, 1888 (1973).

[28]M. Sher, Phys. Rev. D**24**, 1699 (1981).

Using our Robertson-Walker metric with $k = 0$, we obtain

$$\mathrm{R}(t) = \mathrm{e}^{\chi t}\,, \tag{1.15}$$

where

$$\chi = \sqrt{\tfrac{8}{3}\pi \mathrm{G}\rho_o}\,. \tag{1.16}$$

(For our parameters, $\chi \approx 10^{10}$ GeV.) This is nothing but de Sitter space. However the Robertson-Walker metric assumes homogeneity and isotropy, which are not assumptions in the inflationary scenario. If we, however, consider perturbations about this metric, we find that their behavior is governed by a "cosmological no-hair theorem," which states that whenever the energy-momentum tensor is given by (1.14), then any locally measurable perturbation about the de Sitter space will be damped exponentially on the time scale of $\chi^{-1}$. The theorem seems to hold in the context of linearized perturbation theory,[29] and it is conjectured to hold even for large perturbations.[30] Thus, a smooth de Sitter metric arises naturally, without any need to fine-tune the initial conditions.

As the space continues to supercool and exponentially expand, the energy density is fixed at $\rho_o$. Thus, the total matter energy is increasing. The inflationary model indicates that the false vacuum is the source of essentially all the matter, energy, and entropy in the observed universe.

This, at a first glance, seems to violate our notions of conservation of energy. However we can explain this by looking into the conservation law for our energy-momentum tensor, which is imposed by the Einstein field equations. In the case of matter (a perfect fluid) and a Robertson-Walker metric, the conservation law reduces to

$$\frac{d}{dt}\left(\mathrm{R}^3\rho\right) = -p\frac{d}{dt}\left(\mathrm{R}^3\right)\,. \tag{1.17}$$

Consequently, if the false vacuum has a large negative pressure, $p = -\rho_o$; equation (1.18) is satisfied identically, with the energy of the expanding gas increasing due to the negative pressure. If the space were asymptotically Minkowskian it would be possible to define a conserved total energy (matter plus gravitational). However, in a Robertson-Walker metric, this is only possible, perhaps, in the case that the total energy vanishes.

This is not as bad as it seems. The conserved quantities of the observed universe are either 0, or very large, the latter being the case of the baryon number

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[29] J. A. Frieman and C. M. Will, Ap. J. **259**, 437 (1982); J. D. Barrow, in Proceedings of the Nuffield Workshop on *The Very Early Universe*, eds. G. W. Gibbons, S. W. Hawking, and S. T. C. Siklos (Cambridge University Press, Cambridge, England); W. Boucher and G. W. Gibbons *ibid.*; P. Ginsparg and M. J. Perry, Semiclassical Perdurance of De Sitter Space, Harvard preprint HUTP-82/1035 (1982).

[30] G. W. Gibbons and S. W. Hawking. Phys. Rev. D**15**, 2738 (1977); S. W. Hawking and I. G. Moss, Phys. Lett. **110B**, 35 (1982).

and the matter energy. However, as we saw earlier, GUTs predict the nonconservation of baryon number. This seems to imply that the observed universe is entirely devoid of conserved quantities. Therefore it is very tempting to believe that the universe began from nothing,[31] or almost nothing. In fact, using the inflationary mechanism, it is possible to "create" the entire observed universe starting with a total matter energy of about ten kilograms.

Within the region of space which is supercooling onto a de Sitter space, the Higgs field will undergo fluctuations of thermal and/or quantum character. At some point these fluctuations will become large enough, as to allow us to describe their evolution in a classical fashion. The size and shape of these fluctuation regions are presumably irregular,[32], but presumably homogeneous on a length scale of order $\chi^{-1}$.[33]

The Higgs field then "rolls" down the potential of Figure 1b, obeying the classical equation of motion

$$\frac{\partial^2\phi}{\partial t^2} + \frac{3}{R}\frac{dR}{dt}\frac{\partial\phi}{\partial t} = -\frac{\partial V}{\partial\phi}\,. \tag{1.18}$$

If the initial fluctuation is small, the the Coleman-E. Weinberg condition will imply that the rolling begins very slowly. As long as $\phi \approx 0$ the energy density $\rho$ remains close to $\rho_o$, and the exponential expansion continues. The exponential expansion takes place on a time scale $\chi^{-1}$ which is short compared to the time it takes for the field to roll down the potential. For the inflationary scenario to work, we require that the length scale of homogeneity be stretched from $\chi^{-1}$ to about 10cm before the Higgs field rolls off the plateau. This corresponds to an expansion factor of $10^{25}$.

When the Higgs field reaches the steep part of the potential, it falls quickly to the bottom and oscillates about the minimum. This motion occurs much more rapidly than the initial expansion. The oscillations of the Higgs field are quickly damped by the couplings to the other fields, and the energy is rapidly thermalized.[34] The oscillations of the Higgs field correspond to coherent states of Higgs particles, whereas the damping corresponds to the decay into other species. The release of energy ( i.e., the latent heat of the phase transition) raises the temperature to $\approx 10^{14}$ GeV.

[31] A. Vilenkin, Phys. Lett. **117B**, 25 (1982); *The Birth of Inflationary Universes*, Tufts University preprint (1982).

[32] A. Albrecht and P. J. Steinhardt, Phys. Rev. Lett. **48**, 1220 (1982).

[33] This length scale is on the order of $10^{-11}$ proton diameters; the entire observed will evolve from a region at most this size.

[34] A. Albrecht, P. J. Steinhardt, M. S. Turner, and F. Wilczek, Phys. Rev. Lett. **48**, 1437 (1982); L. F. Abbott, E. Farhi, and M. B. Wise, Phys. Lett. **117B**, 29 (1982); A. D. Dolgov and A. D. Linde, Phys. Lett. **116B**, 329 (1982).

From here on the standard scenario ensues, including the production of a net baryon number.

### 1.3.3 Solutions to the Cosmological Problems

We shall now see how the inflationary scenario overcomes the problems facing the standard model. Let us first look at the monopole problem. Since the Higgs field is correlated throughout the initial fluctuation region, and the exponential expansion increases the Higgs correlation length $\xi$ so that it is greater than $10^{10}$ light- years, the Kibble relation is totally ineffective. This mechanism will produce at most one monopole.[35] The horizon problem is clearly avoided in this scenario, since the entire universe evolves from a fluctuation region that was causally connected. The exponential expansion causes this very small region to encompass the whole observed universe.

The flatness problem is avoided by the dynamics of the exponential expansion of the fluctuation region. As the Higgs field begins to roll very slowly down the potential, the evolution of the metric is governed by the energy density $\rho_o$. Assuming that the fluctuation region can be described locally by a Robertson-Walker metric, then the scale factor evolves according to (1.2b)

$$H^2 = \frac{8\pi}{3} G\rho_o - \frac{k}{R^2} \, . \tag{1.19}$$

As the field rolls down the plateau, the first term of the right hand side of (1.19) is almost constant. While the second term is decreased by at least a factor $10^{50}$, thus the initial conditions necessary for the standard model in order to predict today's observed values for $\Omega$. The inflationary model further predicts, that the value of $\Omega$ today is extraordinarily close to 1.

### 1.3.4 Problems of the Inflationary Scenario

Not everything is perfect with the inflationary scenario, however. There is still one main problem, namely that of the calculation of density fluctuations. It is evident that the universe is not entirely homogeneous, given that *we* exist. At small scales, the universe is not homogeneous, and these small-scale inhomogeneities should be explained by density fluctuations arising from quantum fluctuations in the Higgs field. In the current model of the inflationary universe the density fluctuations of the universe are mis-predicted by approximately five orders of magnitude. This is known as the smoothness problem, and it plagues the standard model also.

[35] Experimental data can account for at most one monopole also.

This is not an inherent problem of the inflationary scenario, since it depends on the underlying particle theory. As we had pointed out above, inflation depends on the underlying GUT for most calculations, but it is not committed to any one in particular. So there is still hope that a GUT[36] will be found that solves the smoothness problem, while retaining all the other successes of the inflationary scenario.

## 1.4 Motivation

The current work stems from this last problem. The research on the inflationary scenario with homogeneous initial conditions has been extensive in the last couple of years. However, still we do not have a good understanding of what happens when the initial conditions are not homogeneous. And this is what probably occurred in the early universe. Nevertheless, the problem is by far non-trivial, so we must start by analyzing the simpler cases, those of spherical and planar symmetry. In this thesis we analyze the plane-symmetric configuration.

### 1.4.1 Definition of problem

We propose to investigate a region of space which is locally endowed of planar symmetry. We are considering a region of false vacuum embedded in a much larger region of true vacuum. The system is, by construction, symmetric about the origin, so that will only speak of one half of the space, while it is implicit that the same behavior is expected in the other half. The two regions are separated by a wall whose thickness we will allow to vanish in order to simplify the initial calculations. In one region the energy density of the Higgs field is $\rho_o$, corresponding to the false vacuum, while in the other region, the energy density vanishes, corresponding to the true vacuum. We will solve for a physically sensical solution that will serve as an initial configuration for further research on the time evolution of the system.

# 2 The action principle

This problem can be treated classically, and as is the case with classical field theories we must, first of all, set up a suitable action functional, varying which we hope to obtain the corresponding "equations of motion." In our case we have two independent field variables which are, however, coupled. We have

[36]In fact, some supersymmetric theories do predict this correctly, but they have other problems.

a scalar Higgs field whose dynamics will affect and be affected by the gravitational field, i.e., the dynamics of the (metric) tensor field.

Although the system is coupled, we can separate our action functional into two separate actions: one of them being independent of the scalar field variable. Consequently

$$\mathscr{A} = \mathscr{A}_{\mathrm{H}} + \mathscr{A}_{\mathrm{G}}, \tag{2.1}$$

where the subscripts H , G refer to the Higgs and gravitational fields respectively.

The action functional for the Higgs field is written as usual in terms of the Lagrangian density[37]

$$\mathscr{A}_{\mathrm{H}} = \int \mathscr{L} \sqrt{g}\, d^4x\,, \tag{2.2}$$

where $g = -\mathrm{D}et(g_{\mu\nu})$ and it is included in order to make the integral (i.e., the action functional) a world invariant. The Lagrangian density for our scalar Higgs field is the usual for a real scalar field [38]

$$\mathscr{L} = -\tfrac{1}{2}\,\partial^{\mu}\phi\,\partial_{\mu}\phi - \mathrm{V}(\phi)\,. \tag{2.3}$$

In the case of the gravitational field we set the usual Hilbert action given by

$$\mathscr{A}_{\mathrm{G}} = -\frac{1}{16\pi\mathrm{G}} \int \mathrm{R}\sqrt{g}\, d^4x\,, \tag{2.4}$$

where R is the curvature scalar obtained by contracting the Ricci tensor and G is Newton's gravitational constant.

In order to obtain the field equations we must vary the action functional with respect to the corresponding independent field variables and impose the condition that the variation vanish. In the case of the scalar Higgs field, the only action functional which depends on $\phi$ is $\mathscr{A}_{\mathrm{H}}$. Therefore,

$$\delta\mathscr{A}_{\mathrm{H}} = \int d^4x\sqrt{g}\left(\frac{\delta\mathscr{L}}{\delta\phi}\delta\phi + \frac{\delta\mathscr{L}}{\delta\partial_{\mu}\phi}\delta(\partial_{\mu}\phi)\right) = 0\,, \tag{2.5}$$

where the $\frac{\delta\ldots}{\delta\ldots}$ denote functional derivatives. Using equation (2.3) the variation becomes

$$\delta\mathscr{A}_{\mathrm{H}} = -\int d^4x\sqrt{g}\left(\frac{\partial\mathrm{V}}{\partial\phi}\delta\phi + \partial^{\mu}\phi\,\delta(\partial_{\mu}\phi)\right) = 0\,. \tag{2.6}$$

[37]A note on notation: $\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}}$ where $x^{\mu} = (t, \boldsymbol{x})$. In our convention the Minkowski metric $\eta \equiv diag(-1,1,1,1)$ and hence $x_{\mu} = (-t, \boldsymbol{x})$.

[38]A repeated pair of covariant (down) and contravariant (up) indices implies summation over all possible values of that index. In this case we sum from 0 to 3.

Integrating the second term by parts and making use of the trivial identity

$$\delta(\partial_\mu \phi) \equiv \partial_\mu(\delta\phi) \tag{2.7}$$

we obtain

$$\delta\mathscr{A}_{\rm H} = \int d^4x\,\sqrt{g}\left(\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}\partial^\mu\phi\right) - \frac{\partial V}{\partial\phi}\right)\delta\phi = 0\,, \tag{2.8}$$

where we have used the fact that there exists a conserved current in order to make our "surface" term vanish.

Since $\delta\mathscr{A}_{\rm H} = 0$ for any variation $\delta\phi$ , the *fundamental lemma of the calculus of variations* implies that

$$\mathscr{D}_\mu\partial^\mu\phi - \frac{\partial V}{\partial\phi} = 0\,. \tag{2.9}$$

where $\mathscr{D}_\mu \equiv \partial_\mu + \frac{1}{2}\partial_\mu \ln g$, is the covariant derivative operator acting on a contravariant vector. Rewriting this in terms of the curved-space D'Alembertian, we obtain the equation of motion for our scalar Higgs field

$$\Box\phi = \frac{\partial V}{\partial\phi}\,. \tag{2.10}$$

In order to obtain the gravitational field equations we impose that the action be stationary with respect to the variation of the metric tensor. Now both action functionals will have to be varied. The variation of $\mathscr{A}_{\rm G}$ will produce after lengthy algebra

$$\delta\mathscr{A} = \frac{1}{16\pi G}\int d^4x\,\sqrt{g}\left(R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R\right)\delta g_{\mu\nu} + \delta\mathscr{A}_{\rm H} = 0\,. \tag{2.11}$$

Following Weinberg[39] we define the energy-momentum tensor $T_{\mu\nu}$ as the functional derivative of $\mathscr{A}_{\rm H}$ with respect to the (metric) tensor field. Consequently

$$\delta\mathscr{A}_{\rm H} = \tfrac{1}{2}\int d^4x\,\sqrt{g}\,T^{\mu\nu}\,\delta g_{\mu\nu}\,. \tag{2.12}$$

Substituting equation (2.12) into (2.11) and multiplying by $16\pi G$ yields

$$\delta\mathscr{A} = \int d^4x\,\sqrt{g}\left(R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R + 8\pi G T^{\mu\nu}\right)\delta g_{\mu\nu} = 0\,. \tag{2.13}$$

Again this must be true for any $\delta g_{\mu\nu}$ , and hence the fundamental lemma of the calculus of variations implies

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu}\,, \tag{2.14}$$

[39] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972) p.360.

which are the Einstein Field Equations written in their fully covariant form.

We still have to calculate the energy-momentum tensor for our system. Using equation (2.12) we arrive at the following expression for $T^{\mu\nu}$ :

$$T^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta}{\delta g_{\mu\nu}} \left( \sqrt{g} \mathscr{L} \right) . \tag{2.15}$$

Using the chain rule and the following identity

$$\frac{\delta \ln \sqrt{g}}{\delta g_{\mu\nu}} = \tfrac{1}{2} g^{\mu\nu} , \tag{2.16}$$

equation (2.15) becomes

$$T^{\mu\nu} = g^{\mu\nu} \mathscr{L} + 2 \frac{\delta \mathscr{L}}{\delta g_{\mu\nu}} . \tag{2.17}$$

Since the only term in the Lagrangian density which depends on the metric tensor is the "kinetic" term, the derivative can be easily evaluated. Using the following identity

$$\frac{\delta g^{\rho\sigma}}{\delta g_{\mu\nu}} = -g^{\rho\mu} g^{\sigma\nu} , \tag{2.18}$$

the energy-momentum tensor is found to be

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - g^{\mu\nu} \left( \tfrac{1}{2} \partial_\rho \phi \partial^\rho \phi + V(\phi) \right) . \tag{2.19}$$

For our purposes it will be more convenient to rewrite equation (2.14). Let us take the trace of both sides of the equation. This yields the following relation between the curvature scalar R and the trace of the energy-momentum tensor T:

$$R = 8\pi G T . \tag{2.20}$$

Substituting for R in equation (2.14) and rearranging terms, we obtain

$$R_{\mu\nu} = -8\pi G \left( T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T \right) . \tag{2.21}$$

Finally we define $\mathscr{T}_{\mu\nu} \equiv T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T$ and our field equations take their final form

$$R_{\mu\nu} = -8\pi G \mathscr{T}_{\mu\nu} . \tag{2.22}$$

# 3 Symmetry considerations

The Einstein field equations are generally ten independent non-linear partial differential equations. It is only by imposing certain symmetries that we can reduce the number of equations — although, generally, not their non-linear character. These symmetry requirements constitute the initial *Ansatz* and generally define the problem. In our case we define our symmetry as planar and by that we mean that our field variables should be invariant under the following set of transformations

1. Translations in the $y$-axis,
2. Translations in the $z$-axis,
3. Rotations in the $y$-$z$ plane.

This *Ansatz* will restrict the form and functional dependence of our independent field variables. This set of symmetry transformations form a 3-parameter Lie group of isometries —since they leave the metric invariant— known as the Euclidean group in two dimensions , $\mathscr{E}(2)$. The three parameters of this group correspond to the three quantities needed to specify the magnitude of the transformations. It can be shown that there is a direct and natural relationship between the group of isometries and the inherent (i.e., coordinate-independent) symmetries of the space.

Imagine we take a point in our space and we operate on it successively with any single one infinitesimal transformation. The locus of this point will describe a curve which is continuous since the group is. If we repeat this procedure with other points in the space we will obtain a set of paths congruent to each other —a *congruence* of curves. If the space is symmetric with respect to that particular transformation then any field defined in that space will be unchanged as we move it along any one of the congruences. The precise mathematical construct which tells us how a given field changes as it is dragged along such congruence is called the Lie derivative and it is taken with respect to the vector defining the congruence. In the case of a symmetry transformation the Lie derivative vanishes. In that case the vector defining the congruence is called a *Killing* vector, and the restriction that the Lie derivative vanishes is called Killing's equation. Naturally the vectors defining the congruence are nothing but the generators of the Lie group, i.e., the infinitesimal isometries.

These are found to be $\{\boldsymbol{e}_y, \boldsymbol{e}_z, y\boldsymbol{e}_z - z\boldsymbol{e}_y\}$. Using a coordinate (i.e., commuting) basis we can express these generators as $\{\partial_y, \partial_z, y\partial_z - z\partial_y\}$. These can be seen to satisfy the following commutation relations

$$[\xi_1, \xi_2] = 0\ , \ [\xi_1, \xi_3] = \xi_2\ , \ [\xi_3, \xi_2] = \xi_1\ , \tag{3.1}$$

where the $\{\xi_n\} \equiv \left\{\partial_y, \partial_z, y\partial_z - z\partial_y\right\}$.

Imposing the $\mathscr{E}(2)$ symmetry and recognizing the Killing vectors of our space as the above generators, we can obtain the most general form of our field variables by direct application of Killing's equation.

In the case of the scalar Higgs field, Killing's equation is very simple since the Lie derivative of a scalar field reduces to the directional derivative of the field along the congruence defined by our Killing vectors. Symbolically this reduces to

$$\partial_y \phi = \partial_z \phi = 0\,. \tag{3.2}$$

This implies that $\phi$ is independent of $y$ and $z$ , i.e., $\phi = \phi(x, t)$.

In the case of the metric tensor the Lie derivative is more complicated and it is defined by

$$\mathscr{L}_{\xi_n} g_{\mu\nu} \equiv \xi_n^{\sigma}\partial_\sigma g_{\mu\nu} + g_{\mu\sigma}\partial_\nu \xi_n^{\sigma} + g_{\sigma\nu}\partial_\mu \xi_n^{\sigma}\,. \tag{3.3}$$

Imposing that this vanish we find the following functional dependencies

$$\partial_y g_{\mu\nu} = \partial_z g_{\mu\nu} = 0\,, \tag{3.4}$$

and the following restrictions on the general form of the metric

$$g_{zz} = g_{yy}\,, \tag{3.5}$$

and

$$g_{z\mu} = 0 \;\; \forall\, \mu \neq z\,,\; g_{y\mu} = 0 \;\; \forall\, \mu \neq y\,. \tag{3.6}$$

Therefore, our metric is independent of $y$ and $z$ and our line element has the following general form

$$ds^2 = g_{tt} dt^2 + g_{xx} dx^2 + 2 g_{xt} dx\, dt + g_{yy}\left(dy^2 + dz^2\right)\,. \tag{3.7}$$

However since $g_{\mu\nu}$ is symmetric we can always diagonalize the $x$-$t$ submatrix by redefining $x$ and $t$ accordingly. Therefore, the final form of our most general plane-symmetric line element will be

$$ds^2 = -\mathrm{A}(x,t) dt^2 + \mathrm{B}(x,t) dx^2 + \mathrm{C}(x,t)\left(dy^2 + dz^2\right)\,. \tag{3.8}$$

where A, B, C are the unknown functions for our metric.

With these results we can proceed to solve the field equations. However, before doing so, two remarks are in order.

In the first place, since the Ricci tensor is constructed fully from the metric tensor, it obeys Killing's equation. Therefore we conclude that we will only have at most four independent components of the Ricci tensor $\mathrm{R}_{xx}, \mathrm{R}_{tt}, \mathrm{R}_{yy}$ and, possibly, $\mathrm{R}_{xt}$ since we must recall that the procedure by which we obtained a diagonal metric was a redefinition of the coordinates and not by direct application of Killing's equations.

Finally, according to the generalization of Birkhoff's theorem [40] spaces admitting a 3-parameter group of isometries have an additional Killing vector (i.e., admit a 4-parameter group), provided that the energy-momentum tensor satisfies certain conditions. Furthermore, in the case that $T_{\mu\nu}$ does not depend on time —which is true in both our boundary conditions— the space has a fourth Killing vector which is time-like. Therefore it is possible to find a coordinate system in which the metric is static, i.e., independent of the time coordinate.

[40] See for example, Hubert Goenner, Comm. math. Phys. **16**, 34-47 (1970).

# 4 The field equations

In the last section we found the functional dependence of our field variables. We shall now construct the field equations explicitly taking our symmetry *Ansatz* into consideration.

## 4.1 Scalar Field

The equation of motion of the scalar Higgs field was found in section 2 and is given by equation (2.10). Since $\phi = \phi(x,t)$, equation (2.10) reduces to

$$\left(\partial_x + \tfrac{1}{2}\partial_x \ln g\right)\left(g^{xx}\frac{\partial\phi}{\partial x}\right) + \left(\partial_t + \tfrac{1}{2}\partial_t \ln g\right)\left(g^{tt}\frac{\partial\phi}{\partial t}\right) = -\frac{\partial V}{\partial\phi} \,. \tag{4.1}$$

and since our metric is diagonal, i.e., $g^{\mu\mu} = \frac{1}{g^{\mu\mu}}$ we can use the general form of the metric given by (3.8) to rewrite the equation of motion in the following fashion

$$\frac{1}{B}\frac{\partial^2\phi}{\partial x^2} - \frac{1}{A}\frac{\partial^2\phi}{\partial t^2} + \left(\frac{A_t}{A} - \frac{1}{2}\frac{\partial}{\partial t}\ln\left(ABC^2\right)\right)\frac{1}{A}\frac{\partial\phi}{\partial t} - \left(\frac{B_x}{B} - \frac{1}{2}\frac{\partial}{\partial x}\ln\left(ABC^2\right)\right)\frac{1}{B}\frac{\partial\phi}{\partial x} = -\frac{\partial V}{\partial\phi} \,. \tag{4.2}$$

## 4.2 Gravitational Field

We shall find the explicit form of the Einstein field equations taking into account the symmetries we imposed. In order to calculate the Ricci tensor we must first compute a series of geometrical objects which comprise the machinery of General Relativity. Let us define the affine connections of our Riemannian space in the following way

$$\Gamma^{\lambda}_{\mu\nu} \equiv \tfrac{1}{2} g^{\lambda\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) \,. \tag{4.3}$$

These objects are symmetric under the interchange of its lower indices, but they are not tensors since they transform in a different manner. From the metric given by equation (3.8) we find the following linearly independent non-vanishing affine connections

$$\Gamma^{t}_{tt} = \frac{A_t}{2A} \,,\, \Gamma^{t}_{xx} = \frac{B_t}{2A} \,,\, \Gamma^{t}_{yy} = \Gamma^{t}_{zz} = \frac{C_t}{2A} \,,$$

$$\Gamma^{x}_{tt} = \frac{A_x}{2B} \,,\, \Gamma^{x}_{xx} = \frac{B_x}{2B} \,,\, \Gamma^{x}_{yy} = \Gamma^{x}_{zz} = -\frac{C_x}{2B} \,, \tag{4.4}$$

$$\Gamma^t_{tx} = \frac{A_x}{2A}\,,\ \Gamma^x_{tx} = \frac{B_t}{2B}\,,\ \Gamma^y_{ty} = \Gamma^z_{tz} = \frac{C_t}{2C}\,,\ \Gamma^y_{xy} = \Gamma^z_{xz} = \frac{C_x}{2C}\,.$$

Next we define the Riemann curvature tensor, $R^\kappa_{\mu\lambda\nu}$ by the following relation

$$R^\kappa_{\mu\lambda\nu} \equiv -\partial_\lambda \Gamma^\kappa_{\mu\nu} - \Gamma^\sigma_{\mu\nu}\Gamma^\kappa_{\sigma\lambda} + \partial_\nu \Gamma^\kappa_{\mu\lambda} + \Gamma^\sigma_{\mu\lambda}\Gamma^\kappa_{\sigma\nu}\,. \tag{4.5}$$

However, we are not interested in the Riemann curvature *per se* but in its contraction: the Ricci tensor, which we define as follows

$$R_{\mu\nu} \equiv R^\rho_{\mu\rho\nu} = -\partial_\rho \Gamma^\rho_{\mu\nu} - \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} + \partial_\nu \Gamma^\rho_{\mu\rho} + \Gamma^\sigma_{\mu\rho}\Gamma^\rho_{\sigma\nu}\,. \tag{4.6}$$

Since this tensor is symmetric we find only four independent components which we list below

$$R_{tt} = \frac{B_{tt}}{2B} + \frac{C_{tt}}{C} - \frac{B_t^2}{4B^2} - \frac{C_t^2}{2C^2} - \frac{A_{xx}}{2B} + \frac{A_x^2}{4AB} - \frac{A_t B_t}{4AB} - \frac{A_t C_t}{2AC} - \frac{A_x C_x}{2BC} + \frac{A_x B_x}{4B^2}\,, \tag{4.7$a$}$$

$$R_{xt} = \frac{C_{xt}}{C} - \frac{C_t C_x}{2C^2} - \frac{A_x C_t}{2AC} - \frac{B_t C_x}{2BC}\,, \tag{4.7$b$}$$

$$R_{xx} = \frac{A_t B_t}{4A^2} - \frac{B_{tt}}{2A} + \frac{B_t^2}{4AB} - \frac{B_t C_t}{2AC} + \frac{C_{xx}}{C} - \frac{C_x^2}{2C^2} + \frac{A_{xx}}{2A} - \frac{A_x^2}{4A^2} - \frac{A_x B_x}{4AB} - \frac{B_x C_x}{2BC}\,, \tag{4.7$c$}$$

$$R_{yy} = R_{zz} = -\frac{C_{tt}}{2A} + \frac{A_t C_t}{4A^2} - \frac{B_t C_t}{4AB} - \frac{B_x C_x}{4B^2} + \frac{C_{xx}}{2B} + \frac{A_x C_x}{4AB}\,. \tag{4.7$d$}$$

where $f_x \equiv \frac{\partial f}{\partial x}$, etc.

Having calculated the "geometrical" terms of the Einstein field equations all we need to do is calculate the explicit form of the energy momentum tensor $T_{\mu\nu}$. However, as we mentioned at the end of section 2, it will be more convenient to work with $\mathscr{T}_{\mu\nu}$ instead which was defined as $T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T$. Taking the trace of $T_{\mu\nu}$ as given by equation (2.19) we obtain

$$T = -\partial_\rho \phi\, \partial^\rho \phi - 4V(\phi)\,. \tag{4.8}$$

From this $\mathscr{T}_{\mu\nu}$ becomes

$$\mathscr{T}_{\mu\nu} = \partial_\mu \phi\, \partial_\nu \phi + g_{\mu\nu} V(\phi)\,. \tag{4.9}$$

This tensor has four independent components

$$\mathscr{T}_{tt} = \phi_t^2 - AV(\phi)\,, \tag{4.10$a$}$$

$$\mathscr{T}_{xt} = \phi_x \phi_t\,, \tag{4.10$b$}$$

$$\mathscr{T}_{xx} = \phi_x^2 + BV(\phi)\,, \tag{4.10$c$}$$

$$\mathscr{T}_{yy} = \mathscr{T}_{zz} = \mathrm{C}\,\mathrm{V}(\phi)\,, \tag{4.10d}$$

corresponding to the four independent components of the Ricci Tensor.

The field equations are given by equation (2.22) and there are four independent equations. Letting $\kappa \equiv 8\pi G$ , we can write them in the following way

$$\frac{B_{tt}}{2B}+\frac{C_{tt}}{C}-\frac{B_t^2}{4B^2}-\frac{C_t^2}{2C^2}-\frac{A_{xx}}{2B}+\frac{A_x^2}{4AB}-\frac{A_tB_t}{4AB}-\frac{A_tC_t}{2AC}-\frac{A_xC_x}{2BC}+\frac{A_xB_x}{4B^2} = -\kappa\left(\phi_t^2 - A V(\phi)\right)\,, \tag{4.11a}$$

$$\frac{C_{xt}}{C}-\frac{C_tC_x}{2C^2}-\frac{A_xC_t}{2AC}-\frac{B_tC_x}{2BC} = -\kappa\,\phi_x\,\phi_t\,, \tag{4.11b}$$

$$\frac{A_tB_t}{4A^2}-\frac{B_{tt}}{2A}+\frac{B_t^2}{4AB}-\frac{B_tC_t}{2AC}+\frac{C_{xx}}{C}-\frac{C_x^2}{2C^2}+\frac{A_{xx}}{2A}-\frac{A_x^2}{4A^2}-\frac{A_xB_x}{4AB}-\frac{B_xC_x}{2BC} = -\kappa\left(\phi_x^2 + B V(\phi)\right)\,, \tag{4.11c}$$

$$-\frac{C_{tt}}{2A}+\frac{A_tC_t}{4A^2}-\frac{B_tC_t}{4AB}-\frac{B_xC_x}{4B^2}+\frac{C_{xx}}{2B}+\frac{A_xC_x}{4AB} = -\kappa C V(\phi)\,. \tag{4.11d}$$

We can divide these equations into two kinds —those which contain second time derivatives of the functions (known as *dynamical* equations since they provide information about the time evolution of the functions) and those which do not (known as *constraint* equations). In our case we have three equations containing second time derivatives but of only two of the functions, thus we can form suitable linear combinations of our equations in order to obtain an extra equation of constraint. In particular we can solve for $B_{tt}$ and $A_{tt}$ from equations (4.11c) and (4.11d) respectively and substitute these into (4.11a). This way we obtain two dynamical equations and two equations of constraint. The dynamical equations are given by (4.11c) and (4.11d), whereas the equations of constraint are given by (4.11b) and the following linear combination of equations (4.11a,c,d)

$$\frac{A}{B}\left(-2\frac{C_{xx}}{C}+\frac{C_x^2}{2C^2}+\frac{C_xB_x}{CB}-\kappa\phi_x^2-2\kappa B V(\phi)\right)+\left(\frac{B_tC_t}{BC}+\frac{C_t^2}{2C^2}-\kappa\phi_t^2\right)=0\,. \tag{4.12}$$

# 5 Boundary conditions

In this section we will set up the boundary conditions for our problem. In General Relativity boundary conditions are themselves actually solutions to the Einstein field equations in which the energy-momentum tensor exhibits its boundary values. As we discussed in the introduction our boundaries are the two stationary values of the energy density of the Higgs field, the so called *true*, and *false*, vacua. In order to ease the solution to the dynamical equations we hope to set up an initial configuration which is static. This is analogous to the "start from rest" of a Newtonian problem. In general this is not always possible to achieve, however we mentioned at the end of the third section that in our case this was possible due to the time-like nature of the fourth Killing vector of our space, whose existence was guaranteed by the generalization of Birkhoff's theorem and the fact that our initial conditions involve a static $T_{\mu\nu}$. Consequently, this amounts to setting all the time derivatives to zero and solving the problem as if $\phi, A, B, C$ were functions of $x$ alone. We shall now solve these equations for the two vacua.

## 5.1 True Vacuum

In this phase the Higgs field is constant and has its minimum expectation value, i.e., $V(\phi)$ vanishes. Therefore the field equations remain

$$-\frac{A_{xx}}{2B}+\frac{A_x^2}{4AB}+\frac{A_xB_x}{4B^2}-\frac{A_xC_x}{2BC}=0\,, \tag{5.1a}$$

$$\frac{C_{xx}}{C}-\frac{C_x^2}{2C^2}+\frac{A_{xx}}{2A}-\frac{A_xB_x}{4AB}-\frac{A_x^2}{4A^2}-\frac{B_xC_x}{2BC}=0\,, \tag{5.1b}$$

$$-\frac{B_xC_x}{4B^2}+\frac{C_{xx}}{2B}+\frac{A_xC_x}{4AB}=0\,. \tag{5.1c}$$

In order to solve these equations we multiply (5.1a) and (5.1c) by $\frac{B}{A}$ and $\frac{B}{C}$ respectively and observe that we can express all terms as logarithmic derivatives of our functions. Equivalently, we let $A \equiv e^{2\alpha}$ , $B \equiv e^{2\beta}$ and $C \equiv e^{2\gamma}$ and rewrite equations (5.1) in terms of the new functions. This results in the following equations

$$\alpha_{xx}+\alpha_x^2+2\alpha_x\gamma_x-\alpha_x\beta_x=0\,, \tag{5.2a}$$

$$2\gamma_{xx}+2\gamma_x^2+\alpha_{xx}+\alpha_x^2-\left(2\gamma_x+\alpha_x\right)\beta_x=0\,, \tag{5.2b}$$

$$\gamma_{xx}+2\gamma_x^2-\gamma_x\beta_x+\gamma_x\alpha_x=0\,. \tag{5.2c}$$

We notice that the above equations do not contain a term involving $\beta_{xx}$. Thus the evolution of $\beta$ along the $x$-trajectory (the only trajectory in this case) is not fixed. This stems from the following fact. Consider the metric given by (3.8) without the time dependence. We can always redefine $x \to x' = \int^x e^{\beta(\tau)} d\tau$ such that the metric looks like

$$ds^2 = -e^{2\alpha'} dt^2 + dx'^2 + e^{2\gamma'} \left( dy^2 + dz^2 \right) . \tag{5.3}$$

Thus dropping the primes, we get the same equations (5.2) but with $\beta = 0$

$$\alpha_{xx} + \alpha_x^2 + 2\alpha_x \gamma_x = 0 , \tag{5.4a}$$

$$2\gamma_{xx} + 2\gamma_x^2 + \alpha_{xx} + \alpha_x^2 = 0 , \tag{5.4b}$$

$$\gamma_{xx} + 2\gamma_x^2 + \gamma_x \alpha_x = 0 . \tag{5.4c}$$

Substituting (5.4c) into (5.4a) we obtain

$$\alpha_{xx} + \alpha_x^2 = 2\gamma_{xx} + 4\gamma_x^2 = 0 . \tag{5.5}$$

Substituting this into (5.4b) we obtain a differential equation for $\gamma$

$$\gamma_{xx} + \tfrac{3}{2}\gamma_x^2 = 0 , \tag{5.6}$$

which can be easily solved to yield

$$\gamma = \tfrac{1}{2} \ln k_3 + \tfrac{2}{3} \ln(x - k_2) . \tag{5.7}$$

Substituting (5.6) into (5.4c) we obtain the following relation between $\alpha$ and $\gamma$

$$\alpha_x + \tfrac{1}{2}\gamma_x = 0 . \tag{5.8}$$

This can be easily solved for $\alpha$

$$\alpha = \tfrac{1}{2} \ln k_1 - \tfrac{1}{3} \ln(x - k_2) . \tag{5.9}$$

The metric functions $A(x)$ and $B(x)$ are found to be

$$A(x) = \frac{k_1}{(x - k_2)^{\frac{2}{3}}} , \tag{5.10a}$$

$$C(x) = k_3 (x - k_2)^{\frac{4}{3}} , \tag{5.10b}$$

where $k_1$ , $k_2$ and $k_3$ are constants of integration.

However we can get rid of all the constants by suitably redefining the coordinates, consequently the metric takes the final form

$$ds^2 = -\frac{1}{x^{\frac{2}{3}}} dt^2 + dx^2 + x^{\frac{4}{3}} \left( dy^2 + dz^2 \right) . \tag{5.11}$$

It is a trivial exercise to show that the equation of constraint (4.12) is satisfied by our solutions (5.10a,b).

It is worthwhile to remark several properties of the metric just obtained. First, it is singular at the origin. This is not just an artificiality induced by our choice of coordinates, but a physical singularity, given that the Kretschmann invariant —a world scalar—, defined as $S \equiv R_{\mu\nu\kappa\lambda} R^{\mu\nu\kappa\lambda}$, is singular at $x = 0$. Last, in the asymptotic limit $x \to \infty$, our metric reduces to that of (flat) Minkowski space.

## 5.2 False Vacuum

In this phase the expectation value of the Higgs field is at a local stationary point and therefore $\phi_x$ vanishes. However because the expectation value is not a minimum the potential does not vanish. In fact, and due to the stationarity of $\phi$ , $V(\phi) = \rho_o$, where $\rho_o$ is a constant. It is the energy density of this phase and hence it is positive. The field equations remain

$$-\frac{A_{xx}}{2B} + \frac{A_x^2}{4AB} + \frac{A_x B_x}{4B^2} - \frac{A_x C_x}{2BC} - A\kappa\rho_o = 0 , \tag{5.12$a$}$$

$$\frac{C_{xx}}{C} - \frac{C_x^2}{2C^2} + \frac{A_{xx}}{2A} - \frac{A_x B_x}{4AB} - \frac{A_x^2}{4A^2} - \frac{B_x C_x}{2BC} + B\kappa\rho_o = 0 , \tag{5.12$b$}$$

$$-\frac{B_x C_x}{4B^2} + \frac{C_{xx}}{2B} + \frac{A_x C_x}{4AB} + C\kappa\rho_o = 0 . \tag{5.12$c$}$$

We now follow the same process as for the previous case and rewriting the above equations in terms of the logarithmic derivatives, we arrive at the following equations

$$\alpha_{xx} + \alpha_x^2 + 2\alpha_x \gamma_x - \alpha_x \beta_x + \kappa e^{2\beta} \rho_o = 0 , \tag{5.13$a$}$$

$$2\gamma_{xx} + 2\gamma_x^2 + \alpha_{xx} + \alpha_x^2 - \left( 2\gamma_x + \alpha_x \right) \beta_x + \kappa e^{2\beta} \rho_o = 0 , \tag{5.13$b$}$$

$$\gamma_{xx} + 2\gamma_x^2 - \gamma_x \beta_x + \gamma_x \alpha_x + \kappa e^{2\beta} \rho_o = 0 . \tag{5.13$c$}$$

Again we can redefine $x$ in order to get rid of $\beta$ , thus obtaining the following equations

$$\alpha_{xx} + \alpha_x^2 + 2\alpha_x \gamma_x + \kappa\rho_o = 0 , \tag{5.14$a$}$$

$$2\gamma_{xx} + 2\gamma_x^2 + \alpha_{xx} + \alpha_x^2 + \kappa\rho_o = 0\,, \tag{5.14b}$$

$$\gamma_{xx} + 2\gamma_x^2 + \gamma_x\alpha_x + \kappa\rho_o = 0\,. \tag{5.14c}$$

Substituting (5.14a) and (5.14c) into (5.14b) we obtain the following differential equation involving $\gamma$

$$\gamma_{xx} + \tfrac{3}{2}\gamma_x^2 + \tfrac{1}{2}\kappa\rho_o = 0\,. \tag{5.15}$$

Let $\xi \equiv \frac{3}{2}\gamma$ then we can rewrite (5.15) as

$$\xi_{xx} + \xi_x^2 + \tfrac{3}{4}\kappa\rho_o = 0\,. \tag{5.16}$$

Now we notice that $\xi_{xx} + \xi_x^2 = e^{-\xi}\frac{d^2}{dx^2}e^{\xi}$ and letting $\lambda^2 \equiv \frac{3}{4}\kappa\rho_o$ we arrive at

$$\frac{d^2}{dx^2}e^{\xi} + \lambda^2 e^{\xi} = 0\,. \tag{5.17}$$

Letting $\Xi \equiv e^{\xi}$ we find that

$$\Xi_{xx} = -\lambda^2\Xi\,. \tag{5.18}$$

This is solved trivially, yielding

$$\Xi = k_3 \sin\lambda\,(x + k_1)\ . \tag{5.19}$$

From this we obtain $\xi$ and consequently $\gamma$

$$\gamma = \tfrac{2}{3}\ln\left(k_3 \sin\lambda(x + k_1)\right)\ . \tag{5.20}$$

Substituting (5.16) into (5.14c) we obtain the following equation for $\alpha$

$$\alpha_x = \frac{\gamma_{xx}}{\gamma_x} + \gamma_x\,. \tag{5.21}$$

Substituting (5.20) into (5.21) we found $\alpha$ to be

$$\alpha = \tfrac{1}{2}\ln k_2 + \ln\left(\cos\lambda(x + k_1)\right) - \tfrac{1}{3}\ln\left(\sin\lambda(x + k_1)\right)\ . \tag{5.22}$$

The metric functions remain

$$A(x) = k_4\frac{\cos^2\lambda(x + k_1)}{\sin^{\frac{2}{3}}\lambda(x + k_1)}\,, \tag{5.23a}$$

$$C(x) = k_5 \sin^{\frac{4}{3}} \lambda(x + k_1) \,, \tag{5.23b}$$

where $\lambda \equiv \sqrt{\frac{3}{4}\kappa\rho_o}$ and the $\{k_i\}$ are constants of integration. We can get rid of these constants, however, by suitably redefining our coordinates. Hence we can rewrite our line element as follows P

$$ds^2 = -\cos^2 \lambda x \left(\frac{\lambda}{\sin \lambda x}\right)^{\frac{2}{3}} dt^2 + dx^2 + \left(\frac{\sin \lambda x}{\lambda}\right)^{\frac{4}{3}} \left(dy^2 + dz^2\right) . \tag{5.24}$$

It can be easily verified that the equation of constraint (4.12) is satisfied by our functions.

It is interesting to notice that in the limit as $\rho_o \to 0$ (hence $\lambda \to 0$ also) the metric can be expanded around that point and taking terms up to $\mathcal{O}(x)$, our line element reduces to

$$ds^2 = -\frac{1}{x^{\frac{2}{3}}} dt^2 + dx^2 + x^{\frac{4}{3}} \left(dy^2 + dz^2\right) , \tag{5.25}$$

which is precisely the line element obtained for the true vacuum region.

Nevertheless, a complication arises. We recall that the initial configuration is symmetric about the origin, by construction. In our case, it is clear that our metric is, indeed, symmetric about the origin, but it possesses a physical singularity at the point $x = 0$. We would like to get rid of this problem by redefining (i.e., translating) $x$ such that the symmetric point is now non-singular. In order to do this we must first look at the shape of our metric functions. Because of symmetry, we would like the derivative of our metric functions to be zero at the point of symmetry. Clearly from (5.23) (having set $k_1 = 0$) this would only occur at $\lambda x = \frac{n\pi}{2}$, where $n$ is an integer. However, the Kretschmann invariant S, defined in the previous section, is found to be

$$S = \frac{64\lambda^4}{27} \left(\frac{1 + 2\sin^4 \lambda x}{\sin^4 \lambda x}\right) . \tag{5.26}$$

Clearly, it is singular at $\lambda x = n\pi$. Therefore, we conclude that in order to preserve symmetry and, at the same, avoid any singularities in our region of false vacuum, we must translate $x$ by $\frac{\pi}{2\lambda}$, and we must restrict our region of false vacuum to a slab[41] of width $< \frac{\pi}{\lambda}$. Translating $x$ in this fashion merely switches sines and cosines in (5.24), since $dx$ is not affected by translations.

[41] Exactly how small this region is can be seen by noticing that $\lambda$ is proportional to $\sqrt{\rho_o}$, where $\rho_o$ is the energy density of the false vacuum state of the Higgs field. By dimensional analysis we find that the energy density is proportional to $M^4$, where M is the unification energy. In the SU(5) GUT, this energy is on the order of $10^{14}$ GeV, and therefore the size of our slab, being inversely proportional to $\lambda$, is on the order of $10^{-42}$cm.

Consequently, the final form for the metric on the region of false vacuum becomes after some rescaling of the coordinates

$$ds^2 = -\cos^{-\frac{2}{3}} \lambda x \left( \frac{\sin \lambda x}{\lambda} \right)^2 dt^2 + dx^2 + \cos^{\frac{4}{3}} \lambda x \left( dy^2 + dz^2 \right) . \tag{5.27}$$

# 6 Junction of the two spaces

Having found the static solutions for both the false vacuum and the true vacuum regions, we must join the two spaces in order to be able to investigate the time evolution of our system. As can be seen in Figure 2, there exists a region between the true and false vacuum in which the energy density decreases from the initial value $V(\phi) = \rho_o$ to the final value $V(\phi) = 0$.

In this region we are free to unite the metrics in any way that we can, given that our goal is to examine any solvable model for such a symmetric configuration, and that there is no physically "correct" $\frac{\partial\phi}{\partial x}$ or $V(\phi)$.

Toward this goal it will be useful to perform a transformation of coordinates on our original metrics (5.11) and (5.27). Consider first the metric for the true vacuum region given by (5.11). Defining $\xi \equiv x^{\frac{2}{3}}$ we can rewrite the metric in the following fashion

$$ds^2_{\mathrm{TV}} = -\frac{1}{\xi}dt^2 + \xi d\xi^2 + \xi^2\left(dy^2 + dz^2\right). \tag{6.1}$$

Similarly, consider the metric given by (5.27) for the false vacuum region. Letting $\xi \equiv \cos^{\frac{2}{3}}\lambda x$ we can then rewrite (5.27) in the following way

$$ds^2_{\mathrm{FV}} = -\frac{1-\xi^3}{\lambda^2\xi}dt^2 + \frac{\frac{9}{4}\xi}{\lambda^2(1-\xi^3)}d\xi^2 + \xi^2\left(dy^2 + dz^2\right). \tag{6.2}$$

Now, we can re-scale $\xi$ such that $\xi \to \chi = (\frac{3}{2\lambda})^{\frac{2}{3}}\xi$. This way and after some redefinitions of what we mean by $t$, $y$ and $z$, we can write the metric as follows

$$ds^2_{\mathrm{FV}} = -\frac{1-(\frac{2\lambda}{3})^2\chi^3}{\chi}dt^2 + \frac{\chi}{1-(\frac{2\lambda}{3})^2\chi^3}d\chi^2 + \chi^2\left(dy^2 + dz^2\right). \tag{6.3}$$

Rewriting $\lambda^2$ as $\frac{3}{4}\kappa\rho_o$ we can rewrite the metric in its final form

$$ds^2_{\mathrm{FV}} = -\frac{1-\frac{\kappa\rho_o}{3}\chi^3}{\chi}dt^2 + \frac{\chi}{1-\frac{\kappa\rho_o}{3}\chi^3}d\chi^2 + \chi^2\left(dy^2 + dz^2\right). \tag{6.4}$$

One obvious advantage that springs from this coordinate transformation is the fact that the transition from (6.4) to (6.1) as $\rho_o \to 0$ is made manifest. It must be remarked that this coordinate transformation maps our entire manifold onto a much smaller area. The metric given by (6.4) is seen to be singular at $\chi = (\frac{3}{\kappa\rho_o})^{\frac{1}{3}}$. However, computing the Kretschmann invariant yields the result that the only physical singularity of this metric occurs at $x = 0$. This seems to violate our initial requirement for regularity and symmetry at the origin,[42] but it must be remarked that what was the origin in our previous case is not the origin now. Our metric is now symmetric about the point $\chi = (\frac{3}{\kappa\rho_o})^{\frac{1}{3}}$, and this does not violate our initial requirement since the singularity previously found has just been seen to be an artifice of our choice of coordinates.

However, $g_{xx}$ and $g_{tt}$ change sign at this coordinate singularity. This is a baffling result which will be explored in further research in the near future. However, for the moment we will have to content ourselves with a very small region of false vacuum.

From these new expressions for the metrics stems an interesting fact. We notice that the functions corresponding to the $A(x,t)$ and $B(x,t)$ of (3.8) in (6.4-5) satisfy the following relation $AB = 1$. We also observe that the function corresponding to $C(x,t)$ remains the same in both regions, i.e. $C(x,t) = x^2$. Should this last function remain invariant in the interface, the analysis is tremendously simplified. The question is, thus, whether we have enough residual gauge freedom to impose such a constraint. The Einstein field equations generally contain certain gauge invariance analogous to the Maxwell equations in classical electrodynamics. We have ten independent equations —since the Ricci tensor is a 4-dimensional symmetric tensor— and ten unknown functions: the ten independent components of the metric —another 4-dimensional symmetric tensor. However there are four differential equations, known as the *Bianchi identities* , which offer four more equations to our system. Thus we have the freedom to impose four conditions on our coordinates. In our case, however, due to the symmetries we imposed as our initial *Ansatz* these numbers are not the same; although we *do* have certain gauge invariance. In the following section we shall investigate the possibility of choosing a gauge such that $C(x,t)$ remains invariant in all regions. A preliminary remark is in order nonetheless. We chose our most general plane-symmetric metric of the form (3.8). Here we had three unknown functions, since that was as far as Killing's equation (and our diagonalization of the $x$-$t$ submatrix) took us. However it must be remarked that by restricting our attention to the $x$-$t$ part of the metric we can reduce $-A(x,t)dt^2 + B(x,t)dx^2$ to $D(x,t)(dx^2 - dt^2)$ since every 2-dimensional space is conformally flat —the Weyl (conformal) tensor can be shown to vanish for dimensions less than three. So in general, we only need two out of the three functions. Of course, by not restricting our choice of gauge in the beginning we were able to obtain simpler field equations and we have the hope of being able to impose $C(x,t) = x^2$.

[42] See the end of section 5.

## 6.1 Gauge Invariance

Recall the most general form of our metric, as given by (3.8)

$$ds^2 = -\mathrm{A}(x,t)dt^2 + \mathrm{B}(x,t)dx^2 + \mathrm{C}(x,t)\left(dy^2 + dz^2\right) . \tag{6.5}$$

We are interested in finding out the possibility of choosing a gauge (i.e. transforming our coordinates) in such a way that our metric looks like

$$ds^2 = -\mathrm{A}'(x',t')dt'^2 + \mathrm{B}'(x',t')dx'^2 + x'^2\left(dy^2 + dz^2\right) . \tag{6.6}$$

We will to accomplish this in two steps. First we will perform the following transformations

$$x \mapsto x' = \sqrt{\mathrm{C}(x,t)} ,$$

and

$$t \mapsto t' = \mathrm{T}(x,t) .$$

This clearly is always possible. The transformed metric will generally look as follows

$$ds^2 = -\mathrm{A}'(x',t')dt'^2 + \mathrm{B}'(x',t')dx'^2 + 2\mathrm{D}(x',t')dx'dt' + x'^2\left(dy^2 + dz^2\right) . \tag{6.7}$$

The question remains whether we can get rid of the cross term by redefining only $t'$, since we are already satisfied with the definition of $x'$. Mathematically, we need a transformation

$$t' \mapsto \tau = \tau(x',t')$$

such that the cross term disappears. Let us define $\tau$ by

$$d\tau = f(x',t')\left[\mathrm{A}'(x',t')dt' - \mathrm{D}(x',t')dx'\right] , \tag{6.8}$$

where $f(x',t')$ is chosen such that $d\tau$ is a perfect differential. That is,

$$f(x',t')\mathrm{A}'(x',t') = \frac{\partial\tau}{\partial t'} \quad , \quad -f(x',t')\mathrm{D}(x',t') = \frac{\partial\tau}{\partial x'} . \tag{6.9}$$

Clearly, this choice of $\tau$ and $f$ will diagonalize the metric. The condition (6.9) on $f$ can be expressed as a partial differential equation, using the fact that partial derivatives commute,

$$\frac{\partial}{\partial x'}\left(f\mathrm{A}'\right) + \frac{\partial}{\partial t'}\left(f\mathrm{D}\right) = 0 . \tag{6.10}$$

This equation can always be solved, if only like an initial value problem. [43] Consequently, we have shown that we *do* have enough residual gauge invariance to force such a coordinate transformation.

[43] S. Weinberg, *Op. cit.* p.336.

## 6.2 Generalized Constraint Equation

At the beginning of this section we observed that for the true and false vacuum regions the metric functions A, B satisfied a curious relationship, namely AB = 1. Of course we don't expect this relationship to hold in the interface, however it seems very likely that they could satisfy a relationship of the sort AB = $f$, where $f$ is a calculable function which has values of 1 in both our regions of interest. We shall attempt to solve for $f$ and for the functions A, B in the interface region, where $\phi_x$, V($\phi$) are not specified.

Consider the following line element

$$ds^2 = -\mathrm{A}dt^2 + \frac{f}{\mathrm{A}}dx^2 + x^2\left(dy^2 + dz^2\right) . \tag{6.11}$$

The Einstein field equations are given by (4.11) with vanishing first time derivatives, $\mathrm{C} = x^2$ and $\mathrm{B} = \frac{f}{\mathrm{A}}$, although this last relation will be left implicit for the sake of clarity. The resulting equations are

$$\frac{\mathrm{B}_{tt}}{2\mathrm{A}} - \frac{\mathrm{A}_{xx}}{2\mathrm{A}} + \frac{\mathrm{A}_x^2}{4\mathrm{A}^2} - -\frac{\mathrm{A}_x}{\mathrm{A}x} + \frac{\mathrm{A}_x\mathrm{B}_x}{4\mathrm{AB}} - \kappa\mathrm{B}\mathrm{V}(\phi) = 0\,, \tag{6.12$a$}$$

$$-\frac{\mathrm{B}_{tt}}{2\mathrm{A}} + \frac{\mathrm{A}_{xx}}{2\mathrm{A}} - \frac{\mathrm{A}_x^2}{4\mathrm{A}^2} - \frac{\mathrm{A}_x\mathrm{B}_x}{4\mathrm{AB}} - \frac{\mathrm{B}_x}{\mathrm{B}x} + \kappa\left(\phi_x^2 + \mathrm{B}\mathrm{V}(\phi)\right) = 0\,, \tag{6.12$b$}$$

$$-\frac{\mathrm{B}_x}{2\mathrm{B}x} + \frac{1}{x^2} + \frac{\mathrm{A}_x}{2\mathrm{A}x} + \kappa\mathrm{B}\mathrm{V}(\phi) = 0\,. \tag{6.12$c$}$$

The constraint equation (4.12) now becomes

$$-\frac{2}{x^2} + \frac{2\mathrm{B}_x}{\mathrm{B}x} - \kappa\phi_x^2 - 2\kappa\mathrm{B}\mathrm{V}(\phi) = 0\,. \tag{6.13}$$

We can rewrite (6.13) in the following fashion

$$\frac{d\mathrm{B}}{dx} - \left(\tfrac{1}{2}\kappa x\phi_x^2 + \frac{1}{x}\right)\mathrm{B} = \kappa x\mathrm{V}(\phi(x))\mathrm{B}^2\,. \tag{6.14}$$

This equation is solved in a straight-forward manner in the *naive* assumption that $\phi_x^2$ is independent of the metric function B. However we will see in the "thin wall" limit that this is not the case.

## 6.3 "Thin wall" limit

In Figure 2 we can distinguish three separate regions. The middle region (region II) is what constitutes the wall. In general we would be interested in a wall of finite thickness, however in order to examine the simpler solutions first, we would like to let the thickness of the wall vanish.

It is fairly obvious that in the limit $\delta \to 0$, $2\delta$ being the thickness of the wall, $V(\phi(x))$ and $\phi_x^2$ will obtain the abrupt shapes depicted in Figure 3.

A problem arises immediately. At $x = x_o$, $\phi_x$ is a Dirac $\delta$-function. However what appears in our equations is $\phi_x^2$. This is obviously ill-defined, even inside an integral. In order to avoid these divergences we will look at this problem under a different light. Let us consider the mixed energy-momentum tensor for the Higgs field[44]

$$T^\mu_\nu = \partial^\mu\phi\partial_\nu\phi - \delta^\mu_\nu\left(\tfrac{1}{2}\partial_\rho\phi\partial^\rho\phi + V(\phi)\right) . \tag{6.15}$$

Due to the divergenceless of the Einstein tensor, the energy momentum tensor is locally conserved. This condition can be expressed in a covariant fashion as

$$D_\mu T^{\mu\nu} \equiv \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}T^{\mu\nu}\right) + \Gamma^\nu_{\mu\rho}T^{\mu\rho} = 0\,, \tag{6.16}$$

Given that the metric tensor is covariantly divergenceless and hence it commutes with the covariant derivative operation, we can rewrite the conservation law for the mixed tensor in the following way

$$D_\mu T^\mu_\nu = \frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}T^\mu_\nu\right) - \Gamma^\rho_{\mu\nu}T^\mu_\rho = 0\,. \tag{6.17}$$

Let us concentrate in the $T^\mu_t$ componenents. The above conservation law can be expressed in the following way

$$\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}T^\mu_t\right) - \Gamma^\rho_{\mu t}T^\mu_\rho = 0\,. \tag{6.18}$$

Looking at the last term and using (4.3) as the definition of $\Gamma^\rho_{\mu\lambda}$, we obtain

$$T^\mu_\rho\Gamma^\rho_{\mu t} = \tfrac{1}{2}\left[\partial_\mu g_{\lambda t} + \partial_t g_{\lambda\mu} - \partial_\lambda g_{\mu t}\right]T^{\mu\lambda}\,. \tag{6.19}$$

Since our initial metric is static, the middle term vanishes, and since $T^{\mu\lambda}$ is symmetric under the interchange of $\mu\lambda$ whereas the factor multiplying it is antisymmetric, the whole term is canceled. We can now express our local conservation law as

$$\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}T^\mu_t\right) = 0\,. \tag{6.20}$$

[44] The following analysis is due to Alan Guth.

To make this law global, we merely integrate it over a hypersurface Ω at fixed time

$$\int_\Omega d^3x\sqrt{g}\left[\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}\mathrm{T}^\mu_t)\right] = 0\,. \tag{6.21}$$

Let us now define $\mathrm{T}^i_t \equiv \mathrm{J}^i$. Our integral becomes

$$\int_\Omega d^3x\left[\partial_t(\sqrt{g}\mathrm{T}^t_t) - \nabla\cdot(\sqrt{g}\mathbf{J})\right] = 0\,, \tag{6.22}$$

and using the divergence theorem we obtain

$$\frac{d}{dt}\int_\Omega d^3x\sqrt{g}\mathrm{T}^t_t - \int_{\partial\Omega}\sqrt{g}\mathbf{J}\cdot d^2\mathbf{S} = 0\,. \tag{6.23}$$

The second term will vanish for a sufficiently large Ω, since it represents the flux of energy through its boundary ∂Ω. Thus we conclude that the quantity $\int_\Omega d^3x\sqrt{g}\mathrm{T}^t_t$ is a conserved quantity. From (6.15) we find $\mathrm{T}^t_t$ to be

$$\mathrm{T}^t_t = -\tfrac{1}{2}\frac{1}{\mathrm{B}}\phi_x^2 - \mathrm{V}(\phi)\,. \tag{6.24}$$

Since $\mathrm{V}(\phi)$ is finite, the only singular part of this expression is that proportional to $\phi_x^2$. Hence we define

$$\left(\mathrm{T}^t_t\right)^{\mathrm{sing}} \equiv -\tfrac{1}{2}\frac{1}{\mathrm{B}}\phi_x^2\,. \tag{6.25}$$

Since $\mathrm{T}^t_t$ is generally the (Hamiltonian) energy density, we will define the energy of the "wall" by

$$\mathrm{E}_{\mathrm{wall}} \equiv \int_\mathrm{V} d^3x\sqrt{g}(\mathrm{T}^t_t)^{\mathrm{sing}}\,. \tag{6.26}$$

Using (6.6) as our form of the metric, we obtain

$$\mathrm{E}_{\mathrm{wall}} \equiv \int_\mathrm{V} dx\,dy\,dz\sqrt{\mathrm{AB}x^4}\left(-\tfrac{1}{2}\frac{1}{\mathrm{B}}\phi_x^2\right)\,. \tag{6.27}$$

Since nothing depends on $y, z$ we shall perform their integration first, yielding

$$\mathrm{E}_{\mathrm{wall}} \equiv \tfrac{1}{2}\mathrm{S}\int dx\,x^2\sqrt{\frac{\mathrm{A}}{\mathrm{B}}}\phi_x^2\,, \tag{6.28}$$

where S is the oriented area of the wall. Defining σ to be the surface energy density of the wall, we will define $\mathrm{E}_{\mathrm{wall}}$ to equal

$$\mathrm{E}_{\mathrm{wall}} \equiv \sigma\mathrm{S}x_o^2\sqrt{\mathrm{A}(x_o)}\,. \tag{6.29}$$

For this to hold, it is clear that $\phi_x^2$ must be

$$\phi_x^2 = 2\sigma\sqrt{B}\,\delta(x - x_o)\ . \tag{6.30}$$

As we mentioned earlier, $\phi_x^2$ does depend on the metric function and it is now clear that our solution to the differential equation would have been incorrect. We will now proceed to solve the equations correctly.

In the thin wall limit, $\phi_x^2$ and $V(\phi)$ can be expressed as follows

$$\lim_{\delta\to 0} V(\phi(x)) = \rho_o(1 - \theta(x - x_o)) \tag{6.31}$$

$$\lim_{\delta\to 0} \phi_x^2 = 2\sigma\sqrt{B}\delta(x - x_o)\ . \tag{6.32}$$

where $\theta(x - x_o)$ is the unit step function. Hence, equations (6.12) now look like

$$\frac{B_{tt}}{2A} - \frac{A_{xx}}{2A} + \frac{A_x^2}{4A^2} - -\frac{A_x}{Ax} + \frac{A_xB_x}{4AB} - \kappa B\rho_o(1 - \theta(x - x_o)) = 0\ , \tag{6.33a}$$

$$-\frac{B_{tt}}{2A} + \frac{A_{xx}}{2A} - \frac{A_x^2}{4A^2} - \frac{A_xB_x}{4AB} - \frac{B_x}{Bx} + \kappa\left(2\sigma\sqrt{B}\delta(x - x_o) + B\rho_o(1 - \theta(x - x_o))\right) = 0\ , \tag{6.33b}$$

$$-\frac{B_x}{2Bx} + \frac{1}{x^2} + \frac{A_x}{2Ax} + \kappa B\rho_o(1 - \theta(x - x_o)) = 0\ . \tag{6.33c}$$

Adding (6.33a) and (6.33b) we obtain

$$\frac{A_x}{A} + \frac{B_x}{B} = 2\kappa\sigma x\sqrt{B}\delta(x - x_o)\ . \tag{6.34}$$

Substituting for $\frac{A_x}{A}$ into (6.33c) we obtain

$$\frac{dB}{dx} - \kappa\sigma x\delta(x - x_o)B^{\frac{3}{2}} - \frac{B}{x} = \kappa\rho_o x(1 - \theta(x - x_o))B^2\ . \tag{6.35}$$

We shall divide the real line into three regions:

1. $0 \le x < x_o$ Region I
2. $x = x_o$ Region II
3. $x > x_o$ Region III

First we notice that the left hand side of (6.34) is nothing but the logarithmic derivative of AB and hence of $f$. And in region I, $\phi_x^2 = 0$. Therefore,

$$AB = k_1\ . \tag{6.36}$$

We shall set $k_1 = 1$ without lack of generality. In region I, equation (6.35) becomes

$$\frac{d\mathrm{B}}{dx} - \frac{\mathrm{B}}{x} = \kappa\rho_o x\mathrm{B}^2 \,. \tag{6.37}$$

This is nothing but *Bernoulli's* equation and it is linearized by a change of variables $\Psi \equiv \mathrm{B}^{-1}$. Thus (6.37) becomes

$$\frac{d\Psi}{dx} + \frac{\Psi}{x} = -\kappa\rho_o x \,. \tag{6.38}$$

This is solved trivially solved for $\Psi$ and hence for B yielding the expected result

$$\mathrm{B}(x) = \frac{x}{k_2 - \frac{\kappa\rho_o}{3}x^3} \,, \tag{6.39}$$

and we shall let $k_2 = 1$. Using this solution and (6.36), the other metric function becomes

$$\mathrm{A}(x) = \frac{1 - \frac{\kappa\rho_o}{3}x^3}{x} \tag{6.40}$$

as expected.

In region III (6.34) yields the same result, namely

$$\mathrm{AB} = k_3 \,. \tag{6.41}$$

However (6.35) becomes much simpler

$$\frac{d\mathrm{B}}{dx} - \frac{\mathrm{B}}{x} = 0 \,, \tag{6.42}$$

or equivalently

$$\mathrm{B}(x) = k_4 x \,. \tag{6.43}$$

Together, these yield the form of the other metric function

$$\mathrm{A}(x) = \frac{k_3}{k_4 x} \,. \tag{6.44}$$

It now remains to match the two solutions in region II, i.e., at $x = x_o$. We shall do this by integrating our differential equations (6.34) and (6.35) from $x_o - \epsilon$ to $x_o + \epsilon$ and letting $\epsilon \to 0$.

Consider equation (6.34). It can be rewritten as

$$\frac{d}{dx}\ln(\mathrm{AB}) = 2\kappa\sigma x\delta(x - x_o)\sqrt{\mathrm{B}} \,. \tag{6.45}$$

Integrating as prescribed before

$$\lim_{\epsilon\to 0}\int_{x_o-\epsilon}^{x_o+\epsilon}\frac{d}{dx}\ln(\mathrm{AB})\,dx = 2\kappa\sigma\lim_{\epsilon\to 0}\int_{x_o-\epsilon}^{x_o+\epsilon} x\delta(x-x_o)\sqrt{\mathrm{B}}\,dx\,. \tag{6.46}$$

This can be seen to yield

$$\ln(\mathrm{A}_+\mathrm{B}_+)\Big|_{x=x_o} - \ln(\mathrm{A}_-\mathrm{B}_-)\Big|_{x=x_o} = 2\kappa\sigma x_o\sqrt{\mathrm{B}(x_o)}\,. \tag{6.47}$$

where the + and − refer to regions III and I respectively. Using our previous results (6.39-40) and (6.43-44), we obtain

$$\ln k_3 = 2\kappa\sigma x_o\sqrt{\mathrm{B}(x_o)}\,. \tag{6.48}$$

Integrating (6.35) in a similar fashion we obtain the following

$$\mathrm{B}_+\Big|_{x=x_o} - \mathrm{B}_-\Big|_{x=x_o} = \kappa\sigma x_o \mathrm{B}^{\frac{3}{2}}(x_o)\,. \tag{6.49}$$

Substituting for B

$$k_4 x_o - \frac{x_o}{1-\frac{\kappa\rho_o}{3}x_o^3} = \kappa\sigma x_o \mathrm{B}^{\frac{3}{2}}(x_o)\,. \tag{6.50}$$

Solving for $\mathrm{B}(x_o)$ we obtain

$$\mathrm{B}(x_o) = (\kappa\sigma)^{\frac{2}{3}}\left[\frac{k_4(1-\frac{\kappa\rho_o}{3}x_o^3)-1}{1-\frac{\kappa\rho_o}{3}x_o^3}\right]^{\frac{2}{3}}\,. \tag{6.51}$$

We shall now study the continuity characteristics of our solutions. For B to be continuous, we require

$$\mathrm{B}_-(x_o) = \mathrm{B}_+(x_o) = \mathrm{B}(x_o)\,,$$

or

$$k_4 x_o = \frac{x_o}{1-\frac{\kappa\rho_o}{3}x_o^3} = (\kappa\sigma)^{\frac{2}{3}}\left[\frac{k_4(1-\frac{\kappa\rho_o}{3}x_o^3)-1}{1-\frac{\kappa\rho_o}{3}x_o^3}\right]^{\frac{2}{3}}\,. \tag{6.52}$$

Clearly there exists no constant $k_4$ such that these three relationships are simultaneously satisfied. Consequently, we must deduce that B is not continuous across the wall.

Let us now examine the continuity characteristics of A. Equation (6.48) indicates that choosing $k_3$ such that

$$k_3 = \exp\left(2x_o(\sigma\kappa)^{\frac{2}{3}}\left[\frac{k_4(1-\frac{\kappa\rho_o}{3}x_o^3)-1}{1-\frac{\kappa\rho_o}{3}x_o^3}\right]^{\frac{1}{3}}\right) \tag{6.53}$$

could, in principle make A continuous across the wall. However, we would like our metric to approach Minkowski space for very large $x$. Thus we would like AB = 1, or equivalently $k_3 = 1$. We see that if $k_4 = \frac{1}{1-\frac{\kappa\rho_o}{3}x_o^3}$, this last condition is indeed satisfied. However this choice of $k_4$ makes $B(x_o) = 0$ as given by (6.51). The question arises whether we can let this happen.

We are dividing our 4-dimensional manifold $\mathscr{M}$ into two submanifolds of the same dimensionality, and in doing so we separate them by a 3-dimensional hypersurface $\Sigma$. It is also crucial to notice that the separation causes discontinuities and $\delta$-functions in our energy-momentum tensor. Therefore it would, in principle, be expected to have discontinuities in the component of the metric normal to $\Sigma$. However, discontinuities in any other component would be physically inadmissible,[45] given that the geometry of $\Sigma$ must be well defined at all points. In our case the only other component that could have a discontinuity is A, since the $y$-$z$ component of our metric is continuous by construction. Since our solution presents A continuous, it shows no indication of being physically inadmissible. Consequently we conclude that we have arrived at a possible set of initial conditions.

[45] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1972) p.553.

# 7 Conclusion

To summarize, we have obtained a valid set of initial conditions which will serve as a static, initial configuration for an analysis of the time evolution of the system. This is precisely the aspect of the problem in which we are eventually interested. It was due to unexpected problems in the setting up of the initial conditions that an analysis of the dynamics could not be presented in this thesis. The problem is left to be explored further in the near future.