GEOMETRIC BRST QUANTIZATION

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ABSTRACT

We formulate BRST quantization in the language of geometric quantization. We extend the construction of the classical BRST cohomology theory to reduce not just the Poisson algebra of smooth functions, but also any projective Poisson module over it. This construction is then used to reduce the sections of the prequantum line bundle. We find that certain polarizations (*e.g.* Kähler) induce a polarization of the ghosts which simplifies the form of the quantum BRST operator. In this polarization—which can always be chosen—the quantum BRST operator contains only linear and trilinear ghost terms. We also prove a duality theorem for the quantum BRST cohomology.

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§1 INTRODUCTION

Despite the fact that the BRST method made its first appearance as an accidental symmetry of the "quantum" action in perturbative lagrangian field theory[1],[2] it soon became apparent that it took its simplest and most elegant form in the hamiltonian formulation of gauge theories[3], where it is used in the homological reduction of the Poisson algebra of smooth functions of a symplectic manifold on which one has defined a set of first class constraints. Indeed, at a more abstract level, the BRST construction finds its most general form in the theory of "constrained" Poisson algebras[4],[5]; although it is that more concrete realization which is physically relevant, and hence the one we shall study in this paper.

The motivation behind this paper is the desire to define a quantization procedure which would exploit the naturality of the BRST construction in the symplectic context. Geometric quantization[6] is just such a quantization procedure, since it can be defined purely in terms of symplectic data and hence seems tailored for this purpose.

The problem of defining a BRST quantization procedure has been recently analyzed in the literature[7]. In **[7]** the authors discuss the BRST quantization in the case of constraints arising from the action of an algebra and they focus only on the ghost part assuming that the quantization of the ghost and matter parts are independent. In this paper we show that this is not always the case in geometric quantization, since the polarization of the matter forces, in some cases, a particular polarization of the ghosts. Also general properties of BRST quantized theories have recently been studied in **[8]** in the context of Fock space representations and in **[9]** in a slightly more general context. See also the recent paper **[10]** .

The organization of this paper is as follows. In Section 2 we review the symplectic reduction of a symplectic manifold (M, Ω) by a coisotropic submanifold M_o. This material is standard in the mathematics literature and thus we only mention the main facts. We remark that symplectic reduction, as well as cohomology, is a basically a subquotient. This becomes the dominant theme of the paper.

In Section 3 we study the first step of the subquotient: the restriction to the subspace. Suppose $i : M_o \to M$ is a closed embedded submanifold of codimension k corresponding to the zero set (assumed regular) of a smooth function $\Phi : M \to \mathbb{R}^k$. We define two Koszul-like complexes associated to this embedding, which will play a central rôle in the constructions of the BRST cohomology theories. The first yields a free acyclic resolution for $C^\infty(M_o)$ thought of as a $C^\infty(M)$-module. We give a novel proof of the acyclicity of this complex in which we introduce a double complex completely analogous to the Čech-de Rham complex introduced by Weil in order to prove the de Rham theorem. We call it the Čech-Koszul complex. Now let $E \longrightarrow M$ be a vector bundle. The second Koszul complex corresponds to an acyclic resolution of the smooth sections of the pull back bundle $i^{-1}E \longrightarrow M_o$. The acyclicity of this complex follows from the acyclicity of the first complex and the fact that the module of smooth sections of any vector bundle is (finitely generated) projective over the ring of smooth functions.

In Section 4 we tackle the second step of the subquotient: the quotient of the subspace. We define two cohomology theories associated to the foliation determined by the null distribution of $i^*\Omega$ on M_o. The first is a de Rham-like cohomology theory of differential forms (co)tangent to the leaves of the foliation (vertical forms) relative to the exterior derivative along the leaves of the foliation (vertical derivative). If the foliation fibers onto a smooth manifold $\widetilde{M}$—the symplectic quotient of M by M_o—the zeroth cohomology is naturally isomorphic to $C^\infty(\widetilde{M})$. The second cohomology theory is analogous to the first except that the differential forms take values in a vector bundle over M_o which admits a representation of the vectors tangent to the foliation; *e.g.* any bundle on which one can define the notion of a Lie derivative or which admits a connection relative to which the directions spanned by the vectors tangent to the foliation are flat. The zeroth cohomology is then a finitely generated projective module over $C^\infty(\widetilde{M})$ which corresponds to the module of smooth sections of some vector bundle over $\widetilde{M}$. We then lift these cohomology theories via the Koszul resolutions obtained in Section 3 to cohomology theories (BRST) in certain bigraded complexes. The existence of these cohomology theories must be proven since the vertical derivative does not lift to a differential operator, *i.e.* its square is not zero. However its square is chain homotopic to zero (relative to the Koszul differential) and the acyclicity of the Koszul resolution allows us to construct the desired differential.

In Section 5 we review the basics of Poisson superalgebras and define the notion of a Poisson module. We have not seen Poisson modules defined anywhere but we feel our definition is the natural one. We then show that the BRST cohomologies constructed in Section 4 are naturally expressed in the context of Poisson superalgebras and Poisson modules. This allows us to prove that not only the ring and module structures are preserved under BRST cohomology but, more importantly, the Poisson structures also correspond.

We are then ready to begin applying the constructions of the previous sections to the geometric quantization program. We have divided this into two sections: one discussing prequantization and the other polarization. In Section 6 we discuss how prequantum data is induced via symplectic reduction. If (M, Ω) is prequantizable then there is a hermitian line bundle with compatible connection such that its curvature is given by $-2\pi\sqrt{-1}\Omega$. Its pull back onto M_o is also a hermitian line bundle with compatible connection whose curvature is $-2\pi\sqrt{-1}i^*\Omega$. Since the flat directions of this connection coincide with the directions tangent to the null foliation, the smooth sections of this line bundle admit a representation of the vector fields tangent to the foliations and thus we can build a BRST cohomology theory as in Sections 4 and 5. We prove that all the prequantum data gets induced à la BRST except for the inner product, which involves integration. We discuss the reasons why.

In Section 7 we discuss how to induce polarizations. We find that in many cases—pseudo-Kähler polarizations, G action with G semisimple—a choice of polarization for M forces a polarization of the ghost part. This is to be compared with [**7**] where the quantization of the ghost and the matter parts are done separately. In particular one can always choose a polarization for the ghost part in which the quantum BRST operator simplifies enormously since it only contains linear and trilinear ghost terms.

In Section 8 and modulo some minor technicalities which we discuss there, we prove a duality theorem for the quantum BRST cohomology. Some of the technicalities are connected to the potential infinite dimensionality of the spaces involved.

Finally in Section 9 we conclude by discussing some of the results and some open problems.

NOTE ADDED: After completion of this work we became aware of the paper by Johannes Huebschmann **[11]** where he discusses geometric quantization from an algebraic stand point. He identifies Poisson algebras as a special kind of Rinehart's (A,R)–algebras. Under this identification our notion of Poisson module agrees with his. He also treats reduction but not in its more general case and without using BRST. In this purely algebraic context reduction can be taking into account by an algebraic extension of BRST à la Stasheff. We are currently investigating this extension. We wish to thank Jim Stasheff for making us aware of this work and Johannes Huebschmann for sending us the preprint.

§2 SYMPLECTIC REDUCTION

In this section we review the reduction of a symplectic manifold by a coisotropic submanifold. In particular we are interested in the case where the submanifold is the zero locus of a set of first class constraints. The interested reader can consult **[12]** for a very readable exposition of this subject.

Let (M,Ω) be a $2n$-dimensional symplectic manifold and $i : M_o \to M$ a coisotropic submanifold. That is, for all $m \in M_o$, $T_m M_o^\perp \subseteq T_m M_o$, where $T_m M_o^\perp = \{X \in T_m M \mid \Omega(X,Y) = 0\ \forall Y \in T_m M_o\}$. Because $d\Omega = 0$, the distribution $m \mapsto T_m M_o^\perp$ is involutive and it is known as the characteristic or null distribution of $i^*\Omega$. If its dimension is constant then, by Frobenius theorem, M_o

is foliated by maximal connected submanifolds having the characteristic distribution as its tangent space. There is a natural surjective map π from M_0 to the space of leaves $\widetilde{M}$ sending each point in M_o to the unique leaf containing it. If the foliation is fibrating the space of leaves inherits a smooth structure making π a smooth surjection. If that is the case there is a unique symplectic structure $\widetilde{\Omega}$ on $\widetilde{M}$ obeying $\pi^*\widetilde{\Omega} = i^*\Omega$. The resulting symplectic manifold $(\widetilde{M}, \widetilde{\Omega})$ is called the symplectic reduction of M by M_o.

In this paper we will focus on a very specific kind of coisotropic submanifold. Let $\{\phi_i\}$ be a set of k smooth functions on M and let J denote the ideal they generate in $C^\infty(M)$. That is, $f \in J$ if and only if $f = \sum_i \phi_i h_i$ for some $h_i \in C^\infty(M)$. This set of function is said to be first class if J is a Lie subalgebra of $C^\infty(M)$ under Poisson bracket. This is clearly equivalent to the existence of smooth functions $\{f_{ij}{}^k\}$ such that $\{\phi_i\, ,\, \phi_j\} = \sum_k f_{ij}{}^k \phi_k$. Assembling the ϕ_i together into one smooth function $\Phi : M \to \mathbb{R}^k$, we define $M_o \equiv \Phi^{-1}(0)$. If 0 is a regular value of Φ—*i.e.* the tangent map $d\Phi$ is surjective—M_o is a closed embedded coisotropic submanifold of M. Let X_i denote the hamiltonian vector field associated to ϕ_i. The characteristic distribution of M_o is spanned by the $\{X_i\}$. If the $\{\phi_i\}$ are constraints of a dynamical system whose phase space is M then the leaves of the foliation determined by the $\{X_i\}$ are the "gauge orbits"and if the foliation fibers the space of orbits $\widetilde{M}$ is the reduced phase space.

A very important special case of this construction comes about when the constraints arise from a hamiltonian group action. Let G be a connected Lie group acting on M via symplectomorphisms. To each element X in the Lie algebra $\mathfrak{g}$ of G we associate a Killing vector field $\widetilde{X}$ on M which is symplectic. To each symplectic vector field there is associated a closed 1-form on M: $i(\widetilde{X})\,\Omega$. If this form is exact then the vector field is hamiltonian. If all the Killing vector fields are hamiltonian the G-action is called hamiltonian. In this case we can associate to each vector $X \in \mathfrak{g}$ a hamiltonian function ϕ_X such that $d\phi_X + i(\widetilde{X})\,\Omega = 0$. Dual to this construction is the moment map $\Phi : M \to \mathfrak{g}^*$ defined by $\langle\Phi(m), X\rangle = \phi_X(m)$, for all $m \in M$. If a certain cohomological obstruction is overcome the hamiltonian functions $\{\phi_X\}$ close under Poisson bracket: $\{\phi_X\,,\,\phi_Y\} = \phi_{[X\,,Y]}$. If this is the case, the moment map is equivariant: intertwining between the G-actions on M and the coadjoint action on $\mathfrak{g}^*$.

Assume that we have a hamiltonian G-action on M giving rise to an equivariant moment map and furthermore suppose that $\mathbf{0} \in \mathfrak{g}^*$ is a regular value of the moment map. Denote $\Phi^{-1}(\mathbf{0})$ by M_o. Then M_o is a closed embedded coisotropic submanifold and for $m \in M_o$, $T_m M_o^\perp$ is precisely the subspace spanned by the Killing vectors. In this case the leaves of the foliation are just the orbits of the G-action. If the G-action on M_o is free and proper then the space of orbits inherits the structure of a smooth symplectic manifold in the way described above. If the G-action on M_o is not free but only locally free (*i.e.* the isotropy is discrete) then the space of orbits is a symplectic orbifold.

It is interesting to notice that the reduced symplectic manifold is always a "subquotient" of M. That is, first we restrict to a submanifold (M_o) and then we project onto the space of leaves of the null foliation. This is to be compared with cohomology which is also a subquotient of the cochains: first we restrict to the subspace of the cocycles and then we project by factoring out the coboundaries. It is therefore not surprising that one can set up a cohomology theory on M which recovers the symplectic quotient $\widetilde{M}$. This is precisely what the classical BRST cohomology achieves[3].

§3 THE KOSZUL CONSTRUCTION

In this section we describe algebraically the "restriction" part of the symplectic reduction procedure. We will look at a method which will allow us to, in effect, work with objects on M_o without actually having to restrict ourselves to M_o. For example suppose that we are interested in describing the functions on M_o without going to M_o, *i.e.* working with functions on M. It turns out that any smooth function on M_o extends to a smooth function on M and the difference of any two such extensions vanishes on M_o. Hence if we let $I(M_o)$ denote the (multiplicative) ideal of $C^\infty(M)$ consisting of functions which vanish at M_o, we have the following isomorphism

$$C^\infty(M_o) \cong C^\infty(M)/I(M_o) \ . \tag{3.1}$$

This is still not very good since $I(M_o)$ is not a very manageable object. It will turn out that $I(M_o)$ is precisely the ideal J generated by the constraints. Still this is not very good because we would rather work with the constraints themselves and not with the ideal they generate. The solution of this problem relies on a construction due to Koszul. We will see that there is a differential complex (the Koszul complex)

$$\cdots \longrightarrow K^2 \longrightarrow K^1 \longrightarrow C^\infty(M) \longrightarrow 0 \ , \tag{3.2}$$

whose homology in positive dimensions is zero and in zero dimension is precisely $C^\infty(M_o)$. We shall refer to this fact as the "quasi-acyclicity" of the Koszul complex. It will play a fundamental rôle in all our constructions.

We will also construct a similar complex for the smooth sections $\Gamma(E)$ of any vector bundle over M which will yield the smooth sections $\Gamma(E_o)$ of the pull back bundle of E onto M_o. This will be the basic construction in our description of prequantization.

We will first discuss the construction on $\mathbb{R}^m$ and later we will globalize.

Lemma 3.3. *Let $\mathbb{R}^m$ be given coordinates $(y,x) = (y^1,\ldots,y^k,x^1,\ldots,x^{m-k})$. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a smooth function such that $f(\mathbf{0},x) = 0$. Then there exist k smooth functions $h_i : \mathbb{R}^m \to \mathbb{R}$ such that $f = \sum_{i=1}^k \phi_i\, h_i$, where the ϕ_i are the functions defined by $\phi_i(y,x) = y^i$.*

Proof: Notice that

$$f(y,x) = \int_0^1 dt\, \frac{d}{dt} f(ty,x)$$

$$
\begin{aligned}
&= \int_0^1 dt \sum_{i=1}^k y^i \, (D_i\, f)(ty, x) \\
&= \sum_{i=1}^k y^i \int_0^1 dt \, (D_i\, f)(ty, x) \\
&= \sum_{i=1}^k \phi_i(y, x) \int_0^1 dt \, (D_i\, f)(ty, x) \ ,
\end{aligned}
$$

where D_i is the i^{th} partial derivative. Defining

$$
h_i(y, x) \stackrel{\text{def}}{=} \int_0^1 dt \, (D_i\, f)(ty, x) \tag{3.4}
$$

the proof is complete. ■

Therefore, if we let $P \subset \mathbb{R}^m$ denote the subspace defined by $y^i = 0$ for all i, the ideal of $C^\infty(\mathbb{R}^m)$ consisting of functions which vanish on P is precisely the ideal generated by the functions ϕ_i.

Definition 3.5. Let R be a ring[1]. A sequence $(\phi_1, \ldots, \phi_k)$ of elements of R is called **regular** if for all $j = 1, \ldots, k$, ϕ_j is not a zero divisor in R/I_{j-1}, where I_j is the ideal generated by $\phi_1, \ldots, \phi_j$ and $I_0 = \mathbf{0}$. In other words, if $f \in R$ and for any $j = 1, \ldots, k$, $\phi_j\, f \in I_{j-1}$ then $f \in I_{j-1}$ to start out with. In particular, ϕ_1 is not identically zero.

Proposition 3.6. *Let $\mathbb{R}^m$ be given coordinates $(y, x) = (y^1, \ldots, y^k, x^1, \ldots, x^{m-k})$. Then the sequence (ϕ_i) in $C^\infty(\mathbb{R}^m)$ defined by $\phi_i(y, x) = y^i$ is regular.*

Proof: First of all notice that ϕ_1 is not identically zero. Next suppose that $(\phi_1, \ldots, \phi_j)$ is regular. Let P_j denote the hyperplane defined by $\phi_1 = \cdots = \phi_j = 0$. Then by Lemma 3.3 , $C^\infty(P_j) = C^\infty(\mathbb{R}^m)/I_j$. Let $[f]_j$ denote the class of a $f \in C^\infty(\mathbb{R}^m)$ modulo I_j. Then ϕ_{j+1} gives rise to a function $[\phi_{j+1}]_j$ in $C^\infty(P_j)$

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[1] All rings explicitly considered in this paper will be assumed to be commutative and with unit.

which, if we think of P_j as coordinatized by $(y^{j+1}, \ldots, y^k, x^1, \ldots, x^{m-k})$, turns out to be defined by

$$[\phi_{j+1}]_j \, (y^{j+1}, \ldots, y^k, x^1, \ldots, x^{m-k}) = y^{j+1} \; . \tag{3.7}$$

This is clearly not identically zero and, therefore, the sequence $(\phi_1 \ldots, \phi_{j+1})$ is regular. By induction we are done. ■

We now come to the definition of the Koszul complex. Let R be a ring and let $\Phi = (\phi_1, \ldots, \phi_k)$ be a sequence of elements of R. We define a complex (in the sense of homological algebra) $K(\Phi)$ as follows: $K^0(\Phi) = R$ and for $p > 0$, $K^p(\Phi)$ is defined to be the free R module with basis $\{e_{i_1} \wedge \cdots \wedge e_{i_p} \mid 0 < i_1 < \cdots < i_p \leq k\}$.

Define a map $\delta_K : K^p(\Phi) \to K^{p-1}(\Phi)$ by $\delta_K e_i = \phi_i$ and extending to all of $K(\Phi)$ as an R-linear antiderivation. That is, δ_K is identically zero on $K^0(\Phi)$ and

$$\delta_K(e_{i_1} \wedge \cdots \wedge e_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1} \phi_{i_j} \, e_{i_1} \wedge \cdots \wedge \widehat{e_{i_j}} \wedge \cdots \wedge e_{i_p} \; , \tag{3.8}$$

where a $\widehat{\ }$ adorning a symbol denotes its omission. It is trivial to verify that $\delta_K^2 = 0$, yielding a complex

$$0 \longrightarrow K^k(\Phi) \xrightarrow{\delta_K} K^{k-1}(\Phi) \longrightarrow \cdots \longrightarrow K^1(\Phi) \longrightarrow R \longrightarrow 0 \; , \tag{3.9}$$

called the Koszul complex.

The following theorem is a classical result in homological algebra whose proof is completely straight-forward and can be found, for example, in **[13]** .

Theorem 3.10. *If $(\phi_1, \ldots, \phi_k)$ is a regular sequence in R then the homology of the Koszul complex is given by*

$$H^p(K(\Phi)) \cong \begin{cases} \mathbf{0} & \text{for } p > 0 \\ R/J & \text{for } p = 0 \end{cases} \; , \tag{3.11}$$

where J is the ideal generated by the ϕ_i.

Therefore the complex $K(\Phi)$ provides an acyclic resolution (known as the Koszul resolution) for the R-module R/J. Therefore if $R = C^\infty(\mathbb{R}^m)$ and Φ is the sequence $(\phi_1, \ldots, \phi_k)$ of Proposition 3.6 , the Koszul complex gives an acyclic resolution of $C^\infty(\mathbb{R}^m)/J$ which by Lemma 3.3 is just $C^\infty(P_k)$, where P_k is the subspace defined by $\phi_1 = \cdots = \phi_k = 0$.

We now globalize this construction. Let M be our original symplectic manifold and $\Phi : M \to \mathbb{R}^k$ be the function whose components are the constraints, *i.e.* $\Phi(m) = (\phi_1(m), \ldots, \phi_k(m))$. We assume that 0 is a regular value of Φ so that $M_o \equiv \Phi^{-1}(0)$ is a closed embedded submanifold of M. Therefore for each point $m \in M_o$ here exists an open set $U \in M$ containing m and a chart $\Psi : U \to \mathbb{R}^m$ such that Φ has components $(\phi_1, \ldots, \phi_k, x^1, \ldots, x^{m-k})$ and such that the image under Φ of $U \cap M_o$ corresponds exactly to the points $(\underbrace{0, \ldots, 0}_{k}, x^1, \ldots, x^{m-k})$. Let $\mathcal{U}$ be an open cover for M consisting of sets like these. Of course, there will be some sets $V \in \mathcal{U}$ for which $V \cap M_o = \varnothing$.

To motivate the following construction let's understand what is involved in proving, for example, that the ideal J generated by the constraints coincides with the ideal $I(M_o)$ of smooth functions which vanish on M_o. It is clear that $J \subset I(M_o)$. We want to show the converse. That is, if f is a smooth function vanishing on M_o then there are smooth functions h^i such that $f = \sum_i h^i \phi_i$. This is always true locally. That is, restricted to any set $U \in \mathcal{U}$ such that $U \cap M_o \neq \varnothing$, Lemma 3.3 implies that there will exist functions $h^i_U \in C^\infty(U)$ such that on U

$$f_U = \sum_i \phi_i \, h^i_U \ , \tag{3.12}$$

where f_U denotes the restriction of f to U. If, on the other hand, $V \in \mathcal{U}$ is such that $V \cap M_o = \varnothing$, then not all of the ϕ_i vanish and the statement is also true. There is a certain ambiguity in the choice of h^U_i. In fact, if δ_K denotes the Koszul differential we notice that (3.12) can be written as $f_U = \delta_K h_U$, where $h_U = \sum_i h^i_U e_i$ is a Koszul 1-cochain on U. Therefore, the ambiguity in h_U is precisely a Koszul 1-cocycle on U, but by Theorem 3.10 , the Koszul complex on U is quasi-acyclic and hence every 1-cocycle is a 1-coboundary. What we would like to show is that this ambiguity can be exploited to choose the h_U in such a way that $h_U = h_V$ on all non-empty overlaps $U \cap V$. This condition is precisely the condition for h_U to be a Čech 0-cocycle. In order to analyze this problem it is useful to make use of the machinery of Čech cohomology with coefficients in a sheaf. For a review of the necessary material we refer the reader to **[14]** ; and, in particular, to their discussion of the Čech-de Rham complex. Our construction is very close in spirit to that one: in fact, it should properly be called the Čech-Koszul complex.

Let $\mathcal{E}_M$ denote the sheaf of germs of smooth functions on M and let $\mathcal{K} = \bigoplus_p \mathcal{K}^p$ denote the free sheaf of $\mathcal{E}_M$-modules which appears in the Koszul complex: $\mathcal{K}^p = \bigwedge^p \mathbb{V} \otimes \mathcal{E}_M$, where $\mathbb{V}$ is the vector space with basis $\{e_i\}$. Let $C^p(\mathcal{U};\mathcal{K}^q)$ denote the Čech p-cochains with coefficients in the Koszul subsheaf $\mathcal{K}^q$. This becomes a double complex under the two differentials

$$\check{\delta} : C^p(\mathcal{U};\mathcal{K}^q) \to C^{p+1}(\mathcal{U};\mathcal{K}^q) \qquad \text{“Čech”}$$

and

$$\delta_K : C^p(\mathcal{U};\mathcal{K}^q) \to C^p(\mathcal{U};\mathcal{K}^{q-1}) \qquad \text{“Koszul”}$$

which clearly commute, since they are independent. We can therefore define the “total” complex $CK^n = \bigoplus_{p-q=n} C^p(\mathcal{U};\mathcal{K}^q)$ and the “total” differential $D = \check{\delta} + (-1)^p \delta_K$ on $C^p(\mathcal{U};\mathcal{K}^q)$. The total differential has total degree one $D : CK^n \to CK^{n+1}$ and moreover obeys $D^2 = 0$. Since the double complex is bounded, *i.e.* for each n, CK^n is the direct sum of a finite number of $C^p(\mathcal{U};\mathcal{K}^q)$'s, there are two spectral sequences converging to the total cohomology. We now proceed to compute them. In doing so we will find it convenient to depict our computations graphically. The original double complex is depicted by the following diagram.

$C^0(\mathcal{U};\mathcal{K}^2)$	$C^1(\mathcal{U};\mathcal{K}^2)$	$C^2(\mathcal{U};\mathcal{K}^2)$	
$C^0(\mathcal{U};\mathcal{K}^1)$	$C^1(\mathcal{U};\mathcal{K}^1)$	$C^2(\mathcal{U};\mathcal{K}^1)$	
$C^0(\mathcal{U};\mathcal{K}^0)$	$C^1(\mathcal{U};\mathcal{K}^0)$	$C^2(\mathcal{U};\mathcal{K}^0)$	

Upon taking cohomology with respect to the horizontal differential (*i.e.* Čech cohomology) and using the fact that the sheaves $\mathcal{K}^q$ are fine, being free modules over the structure sheaf $\mathcal{E}_M$, we get

$K^2(\Phi)$	0	0	
$K^1(\Phi)$	0	0	
$K^0(\Phi)$	0	0	

where $K^p(\Phi) \cong \bigwedge^p \mathbb{V} \otimes C^\infty(M)$ are the spaces in the Koszul complex on M. Taking vertical cohomology yields the Koszul cohomology

$H^2(K(\Phi))$	0	0	
$H^1(K(\Phi))$	0	0	
$H^0(K(\Phi))$	0	0	

Since the next differential in the spectral sequence necessarily maps across columns it must be identically zero. The same holds for the other differentials and we see that the spectral sequence degenerates at the E_2 term. In particular the total cohomology is isomorphic to the Koszul cohomology:

$$H^n_D \cong H^n(K(\Phi)) \ . \tag{3.13}$$

To compute the other spectral sequence we first start by taking vertical cohomology, *i.e.* Koszul cohomology. Because of the choice of cover $\mathcal{U}$ we can use Theorem 3.10 and Lemma 3.3 to deduce that the vertical cohomology is given by

0	0	0	
0	0	0	
$C^0(\mathcal{U};\mathcal{E}_M/\mathcal{J})$	$C^1(\mathcal{U};\mathcal{E}_M/\mathcal{J})$	$C^2(\mathcal{U};\mathcal{E}_M/\mathcal{J})$	

where $\mathcal{E}_M/\mathcal{J}$ is defined by the exact sheaf sequence

$$0 \to \mathcal{J} \to \mathcal{E}_M \to \mathcal{E}_M/\mathcal{J} \to 0 \ , \tag{3.14}$$

where $\mathcal{J}$ is the subsheaf of $\mathcal{E}_M$ consisting of germs of smooth functions belonging to the ideal generated by the ϕ_i. Because of our choice of cover, Lemma 3.3 implies that $\mathcal{J}(U)$ agrees, for all $U \in \mathcal{U}$, with those smooth functions vanishing on $U \cap M_o$, and hence we have an isomorphism of sheaves $\mathcal{E}_M/\mathcal{J} \cong \mathcal{E}_{M_o}$, where $\mathcal{E}_{M_o}$ is the sheaf of germs of smooth functions on M_o. Next we notice that $\mathcal{E}_{M_o}$ is a fine sheaf and hence all its Čech cohomology groups vanish except the zeroth one. Thus the E_2 term in this spectral sequence is just

0	0	0	
0	0	0	
$C^\infty(M_o)$	0	0	

Again we see that the higher differentials are automatically zero and the spectral sequence collapses. Since both spectral sequences compute the same cohomology we have the following corollary.

Corollary 3.15. *If 0 is a regular value for $\Phi : M \to \mathbb{R}^k$ the Koszul complex $K(\Phi)$ gives an acyclic resolution for $C^\infty(M_o)$. In other words, the cohomology of the Koszul complex is given by*

$$H^p(K(\Phi)) \cong \begin{cases} 0 & \text{for } p > 0 \\ C^\infty(M_o) & \text{for } p = 0 \end{cases}, \tag{3.16}$$

where $M_o \equiv \Phi^{-1}(0)$.

Notice that, in particular, this means that the ideal J generated by the constraints is precisely the ideal consisting of functions vanishing on M_o. This is because $C^\infty(M_o) \cong C^\infty(M)/I(M_o)$ since M_o is a closed embedded submanifold. On the other hand, Corollary 3.15 implies that $C^\infty(M_o) \cong C^\infty(M)/J$. Hence the equality between the two ideals.

It may appear overkill to use the spectral sequence method to arrive at Corollary 3.15 . In fact it is not necessary and the reader is urged to supply a proof using the "tic-tac-toe" methods in **[14]** . This way one gains some valuable intuition on this complex. In particular, one can show that way that the sequence Φ is regular in $C^\infty(M)$ and that $J = I(M_o)$ without having to first prove Corollary 3.15 . Lack of space prevents us from exhibiting both computations and the spectral sequence computation is decidedly shorter.

Suppose now that we want to do the same for sections of vector bundles rather than with functions. Suppose that $E \longrightarrow M$ is a vector bundle on M and that the $E_o \longrightarrow M_o$ is the restriction of the bundle to M_o. More formally, E_o is the pullback bundle $i^{-1}E$ via the natural inclusion $i : M_o \to M$. It turns out that this can be done at almost no extra cost. For this we have to introduce a generalization of the Koszul complex.

Let R be a ring and E an R-module. We can then define a complex $K(\Phi; E)$ associated to any sequence $(\phi_1, \ldots, \phi_k)$ by just tensoring the Koszul complex $K(\Phi)$ with E, that is, $K^p(\Phi; E) = K^p(\Phi) \otimes_R E$ and extending δ_K to $\delta_K \otimes \mathbf{1}$. Let $H(K(\Phi); E)$ denote the cohomology of this complex. It is naturally an R-module.

It is easy to show that if E and F are R-modules, then there is an R-module isomorphism

$$H(K(\Phi); E \oplus F) \cong H(K(\Phi); E) \oplus H(K(\Phi); F) \ . \tag{3.17}$$

Hence, if $F = \bigoplus_\alpha R$ is a free R-module then

$$H(K(\Phi); F) \cong \bigoplus_\alpha H(K(\Phi)) \ . \tag{3.18}$$

In particular if Φ is a regular sequence then the generalized Koszul complex with coefficients in a free R-module is quasi-acyclic. Now let P be a projective module, *i.e.* P is a summand of a free module. Then let N be an R-module such that $P \oplus N = F$, F a free R-module. Then

$$H(K(\Phi); F) \cong H(K(\Phi); P) \oplus H(K(\Phi); N) \ , \tag{3.19}$$

which, along with the quasi-acyclicity of $H(K(\Phi); F)$, implies the quasi-acyclicity of $H(K(\Phi); P)$. How about $H^0(K(\Phi); P)$? By definition

$$H^0(K(\Phi); P) \cong R/J \otimes_R P \cong P/JP \ . \tag{3.20}$$

Therefore we have the following algebraic result

Theorem 3.21. *If $\Phi = (\phi_1, \ldots, \phi_k)$ is a regular sequence in R, and P is a projective R-module, then the homology of the Koszul complex with coefficients in P is given by*

$$H^p(K(\Phi); P) \cong \begin{cases} 0 & \text{for } p > 0 \\ P/JP & \text{for } p = 0 \end{cases} \ , \tag{3.22}$$

where J is the ideal generated by the ϕ_i.

The relevance of this construction is that the smooth sections of any vector bundle over M have the structure of a (finitely generated) projective $C^\infty(M)$-module. More precisely, let $E \xrightarrow{\pi} M$ be a complex vector bundle of rank r over M and let $\Gamma(E)$ denote the space of smooth sections. It is clear that $\Gamma(E)$ is a module over $C^\infty(M)$ where multiplication is defined pointwise using the linear structure on each fiber. It is straight-forward to prove that $\Gamma(E)$ is a free rank r $C^\infty(M)$-module if and only if E is a trivial bundle. It can be shown[15] that M has a finite cover trivializing E. Since on each set of the cover, E is trivial we see that $\Gamma(E)$ is finitely generated: just take as a set of generators the local sections on each cover multiplied by the appropriate elements of a partition of unity subordinate to the cover.

It can also be shown[15] that given a vector bundle $E \xrightarrow{\pi} M$ there exists another vector bundle $F \xrightarrow{\rho} M$ such that their Whitney sum $E \oplus F$ is trivial. Therefore $\Gamma(E \oplus F) \cong \Gamma(E) \oplus \Gamma(F)$ is a free $C^\infty(M)$ module and we see that $\Gamma(E)$ is a direct summand of a free module. In summary, $\Gamma(E)$ is a finitely generated projective $C^\infty(M)$-module.

After these remarks Theorem 3.21 and Corollary 3.15 provide an immediate corollary.

Corollary 3.23. *Let 0 be a regular value for $\Phi : M \to \mathbb{R}^k$ and let $E \xrightarrow{\pi} M$ be a smooth vector bundle over M. Then the Koszul complex $K(\Phi; \Gamma(E))$ gives an acyclic resolution of $\Gamma(E)/J\Gamma(E)$.*

Our next and final result of this section concerns the object $\Gamma(E)/J\Gamma(E)$. Let $i : M_o \to M$ denote the natural inclusion. Then if $E \xrightarrow{\pi} M$ is a vector bundle over M denote by $i^{-1}E \longrightarrow M_o$ the pull-back bundle via i. It will follow from the following theorem that $\Gamma(E)/J\Gamma(E)$ is isomorphic to $\Gamma(i^{-1}E)$. But first we need some remarks of a more general nature.

Let $\psi : \widetilde{M} \to M$ be a smooth map between differentiable manifolds. It induces a ring homomorphism

$$\psi^* : C^\infty(M) \to C^\infty(\widetilde{M}) \tag{3.24}$$

defined by $\psi^* f = f \circ \psi$ for $f \in C^\infty(M)$. This makes any $C^\infty(\widetilde{M})$-module (in particular $C^\infty(\widetilde{M})$ itself) into a $C^\infty(M)$-module, by *restriction of scalars*: multiplication by $C^\infty(M)$ is effected by precomposing multiplication by $C^\infty(\widetilde{M})$ with ψ^*.

Now let $\widetilde{E}\xrightarrow{\widetilde{\pi}}\widetilde{M}$ and $E\xrightarrow{\pi}M$ be vector bundles of the same rank with the property that there is a bundle map given by the following commutative diagram

$$\begin{array}{ccc} \widetilde{E} & \xrightarrow{\varphi} & E \\ \Big\downarrow{\scriptstyle\widetilde{\pi}} & & \Big\downarrow{\scriptstyle\pi} \\ \widetilde{M} & \xrightarrow{\psi} & M \end{array} \tag{3.25}$$

(*i.e.* φ is smooth fiber-preserving) with the property that φ restricts to a linear isomorphism on the fibers. Then we may form the following $C^\infty(M)$-module

$$C^\infty(\widetilde{M}) \otimes_{C^\infty(M)} \Gamma(E) \tag{3.26}$$

which can be made into a $C^\infty(\widetilde{M})$-module by *extension of scalars*: left multiplication by $C^\infty(\widetilde{M})$. Define a map $\varphi^\sharp : \Gamma(E) \to \Gamma(\widetilde{E})$ by

$$(\varphi^\sharp \sigma)(\widetilde{m}) = (\varphi_{\widetilde{m}})^{-1} \left[\sigma(\psi(\widetilde{m}))\right] \ , \tag{3.27}$$

for all $\widetilde{m} \in \widetilde{M}$ and $\sigma \in \Gamma(E)$. Then the following can be easily proven[15]

Theorem 3.28. *With the above notation, there exists an isomorphism of $C^\infty(\widetilde{M})$ modules*

$$C^\infty(\widetilde{M}) \otimes \Gamma(E) \longrightarrow \Gamma(\widetilde{E}) \ , \tag{3.29}$$

defined by $f \otimes \sigma \mapsto f \cdot \varphi^\sharp \sigma$ and where the tensor product is over $C^\infty(M)$. ■

In the case we are interested in we have the following commutative bundle diagram

$$\begin{array}{ccc} i^{-1}E & \stackrel{j}{\longrightarrow} & E \\ \big\downarrow{\scriptstyle \pi_o} & & \big\downarrow{\scriptstyle \pi} \\ M_o & \stackrel{i}{\longrightarrow} & M \end{array} \tag{3.30}$$

By Theorem 3.28 , we have that

$$\Gamma(i^{-1}E) \cong C^\infty(M_o) \otimes_{C^\infty(M)} \Gamma(E) \ . \tag{3.31}$$

But $C^\infty(M_o) \cong C^\infty(M)/J$, whence

$$\begin{aligned} \Gamma(i^{-1}E) &\cong C^\infty(M)/J \otimes_{C^\infty(M)} \Gamma(E) \\ &\cong \Gamma(E)/J\Gamma(E) \ , \end{aligned} \tag{3.32}$$

where the last isomorphism is standard.

Therefore we conclude this section with the following important corollary.

Corollary 3.33. *If* $\mathbf{0}$ *is a regular value of* $\Phi : M \to \mathbb{R}^k$. *Then the Koszul complex* $K(\Phi; \Gamma(E))$ *gives an acyclic resolution of the module of smooth sections of the pullback of the bundle* $E \stackrel{\pi}{\longrightarrow} M$ *via the natural inclusion* $i : \Phi^{-1}(\mathbf{0}) \to M$.

Proof: This is a direct consequence of Corollary 3.23 and the isomorphism of (3.32) . ■

§4 BRST COHOMOLOGY

In this section we complete the construction of the algebraic equivalent of symplectic reduction by first defining a cohomology theory that describes the passage of M_o to $\widetilde{M}$ and then, in keeping with our philosophy of not having to work on M_o, we extend it to a cohomology theory (BRST) which allows us to work with $\widetilde{M}$ from objects defined on M.

To help fix the ideas we shall first discuss functions. A smooth function on $\widetilde{M}$ pulls back to a smooth function on M_o which is constant on the fibers. Conversely, any smooth function on M_o which is constant on the fibers defines a smooth function on $\widetilde{M}$. Since the fibers are connected (after all they arise as integral submanifolds of a distribution) a function is constant on the fibers if and only if it is locally constant. Since the hamiltonian vector fields $\{X_i\}$ associated to the constraints $\{\phi_i\}$ form a global basis of the tangent space to the fibers, a function f on M_o is locally constant on the fibers if and only if $X_i\, f = 0$ for all i. In an effort to build a cohomology theory and in analogy to the de Rham theory, we pick a global basis $\{\omega^i\}$ for the cotangent space to the fibers such that they are dual to the $\{X_i\}$, *i.e.* $\omega^i(X_j) = \delta^i_j$. We then define the **vertical derivative** d_V on functions as

$$d_V f = \sum_i (X_i f)\, \omega^i \qquad \forall\, f \in C^\infty(M_o)\ . \tag{4.1}$$

Let $\Omega_V(M_o)$ denote the exterior algebra generated by the $\{\omega^i\}$ over $C^\infty(M_o)$. We will refer to them as **vertical forms**. We can extend d_V to a derivation

$$d_V : \Omega^p_V(M_o) \to \Omega^{p+1}_V(M_o) \tag{4.2}$$

by defining

$$d_V \omega^i = -\frac{1}{2} \sum_{j,k} f_{jk}{}^i\, \omega^j \wedge \omega^k\ , \tag{4.3}$$

where the $\{f_{ij}{}^k\}$ are the functions appearing in the Lie bracket of the hamiltonian vector fields associated to the constraints: $[X_i\, ,\, X_j] = \sum_k f_{ij}{}^k\, X_k$; or, equivalently, in the Poisson bracket of the constraints themselves: $\{\phi_i\, ,\, \phi_j\} = \sum_k f_{ij}{}^k\, \phi_k$.

Notice that the choice of $\{\omega^i\}$ corresponds to a choice of connection on the fiber bundle $M_o \stackrel{\pi}{\longrightarrow} \widetilde{M}$. Let $\mathcal{V}$ denote the subbundle of TM_o spanned by the $\{X_i\}$. It can be characterized either as $\ker \pi_*$ or as $TM_o^\perp$. A connection is then a choice of complementary subspace $\mathcal{H}$ such that $TM_o = \mathcal{V} \oplus \mathcal{H}$. It is clear that a choice of $\{\omega^i\}$ implies a choice of $\mathcal{H}$ since we can define $X \in \mathcal{H}$ if and only if $\omega^i(X) = 0$ for all i. If we let pr_V denote the projection $TM_o \to \mathcal{V}$ it is then clear that acting on vertical forms, $d_V = \mathrm{pr}_V^* \circ d$, where d is the usual exterior derivative on M_o.

It follows therefore that $d_V^2 = 0$. We call its cohomology the **vertical cohomology** and we denote it as $H_V(M_o)$. It turns out that it can be computed[16] in terms of the de Rham cohomology of the typical fiber in the fibration $M_o \xrightarrow{\pi} \widetilde{M}$. In particular, from its definition, we already have that

$$H_V^0(M_o) \cong C^\infty(\widetilde{M}) \ . \tag{4.4}$$

However this is not the end of the story since we don't want to have to work on M_o but on M. The results of the previous section suggest that we use the Koszul construction. Notice that $\Omega_V(M_o)$ is isomorphic to $\bigwedge \mathbb{R}^k \otimes C^\infty(M_o)$ where $\mathbb{R}^k$ has basis $\{\omega^i\}$. The Koszul complex gives a resolution for $C^\infty(M_o)$. Therefore extending the Koszul differential as the identity on $\bigwedge \mathbb{R}^k$ we get a resolution for $\Omega_V(M_o)$. We find it convenient to think of $\mathbb{R}^k$ as $\mathbb{V}^*$, whence the resolution of $\Omega_V(M_o)$ is given by

$$\cdots \longrightarrow \bigwedge \mathbb{V}^* \otimes \mathbb{V} \otimes C^\infty(M) \stackrel{\mathbf{1}\otimes\delta_K}{\longrightarrow} \bigwedge \mathbb{V}^* \otimes C^\infty(M) \longrightarrow 0 \ . \tag{4.5}$$

This gives rise to a bigraded complex $\mathbb{K} = \bigoplus_{c,b} \mathbb{K}^{c,b}$, where

$$\mathbb{K}^{c,b} \equiv \bigwedge^c \mathbb{V}^* \otimes \bigwedge^b \mathbb{V} \otimes C^\infty(M) \ , \tag{4.6}$$

under the Koszul differential $\delta_K : \mathbb{K}^{c,b} \to \mathbb{K}^{c,b-1}$. The Koszul cohomology of this bigraded complex is zero for $b > 0$ by (3.18) , and for $b = 0$ it is isomorphic to the vertical forms, where the vertical derivative is defined. To make contact with the usual notation, elements of $\bigwedge \mathbb{V}^*$ (respectively, $\bigwedge \mathbb{V}$) are known as **ghosts** (respectively, **antighosts**).

The purpose of the BRST construction is to lift the vertical derivative to $\mathbb{K}$. That is, to define a differential δ_1 on $\mathbb{K}$ which anticommutes with the Koszul differential, which induces the vertical derivative upon taking Koszul cohomology, and which obeys $\delta_1^2 = 0$. This would mean that the total differential $D = \delta_K + \delta_1$ would obey $D^2 = 0$ acting on $\mathbb{K}$ and its cohomology would be isomorphic to the vertical cohomology. This is possible only in the case of a group action, *i.e.* when the linear span of the constraints closes under Poisson bracket. In general this is not possible and we will be forced to add further δ_i's to D to ensure $D^2 = 0$. The need to include these extra terms was first pointed out by Fradkin and Fradkina in [**17**] .

We find it convenient to define $\delta_0 = (-1)^c \delta_K$ on $\mathbb{K}^{b,c}$. We define δ_1 on functions and ghosts (*i.e.* $\{\omega^i\}$) as the vertical derivative[1]

$$\begin{aligned}\delta_1 f &= \sum_i (X_i f)\, \omega^i \\ &= \sum_i \{\phi_i\, ,\, f\}\, \omega^i \end{aligned} \tag{4.7}$$

and

$$\delta_1\, \omega^i = -\frac{1}{2} \sum_{j,k} f_{jk}{}^i\, \omega^j \wedge \omega^k\ . \tag{4.8}$$

We can then extend it as a derivation to all of $\bigwedge \mathbb{V}^* \otimes C^\infty(M)$. Notice that it trivially anticommutes with δ_0 since it stabilizes $\bigwedge \mathbb{V}^* \otimes C^\infty(M)$ where δ_0 acts trivially. We now define it on antighosts (*i.e.* $\{e_i\}$) in such a way that it commutes with δ_0 everywhere. This does not define it uniquely but a convenient choice is

$$\delta_1 e_i = \sum_{j,k} f_{ji}{}^k\, \omega^j \wedge e_k\ . \tag{4.9}$$

---

[1] Notice that the vertical derivative is defined on M_o and hence has no unique extension to M. The choice we make is the simplest and the one that, in the case of a group action, corresponds to the Lie algebra coboundary operator.

Notice that $\delta_1^2 \neq 0$ in general, although it does in the case where the ${f_{ij}}^k$ are constant. However since it anticommutes with δ_0 it does induce a map in δ_0 (*i.e.* Koszul) cohomology which precisely agrees with the vertical derivative d_V, which does obey $d_V^2 = 0$. Hence δ_1^2 induces the zero map in Koszul cohomology. This is enough (see algebraic lemma below) to deduce the existence of a derivation $\delta_2 : \mathbb{K}^{c,b} \to \mathbb{K}^{c+2,b+1}$ such that $\delta_1^2 + \{\delta_0\,,\,\delta_2\} = 0$, where $\{\,,\,\}$ denotes the anticommutator. This suggests that we define $D_2 = \delta_0 + \delta_1 + \delta_2$. We see that

$$D_2^2 = \delta_0^2 \oplus \{\delta_0\,,\,\delta_1\} \oplus (\delta_1^2 + \{\delta_0\,,\,\delta_2\}) \oplus \{\delta_1\,,\,\delta_2\} \oplus \delta_2^2\;, \tag{4.10}$$

where we have separated it in terms of different bidegree and arranged them in increasing c-degree. The first three terms are zero but, in general, the other two will not vanish. The idea behind the BRST construction is to keep defining higher $\delta_i : \mathbb{K}^{c,b} \to \mathbb{K}^{c+i,b+i-1}$ such that their partial sums $D_i = \delta_0 + \cdots + \delta_i$ are nilpotent up to terms of higher and higher c-degree until eventually $D_k^2 = 0$. The proof of this statement will follow by induction from the quasi-acyclicity of the Koszul complex, but first we need to introduce some notation that will help us organize the information.

Let us define $F^p\mathbb{K} = \bigoplus_{c\geq p}\bigoplus_b \mathbb{K}^{c,b}$. Then $\mathbb{K} = F^0\mathbb{K} \supseteq F^1\mathbb{K} \supseteq \cdots$ is a filtration of $\mathbb{K}$. Let $\operatorname{Der}\mathbb{K}$ denote the derivations (with respect to the $\wedge$ product) of $\mathbb{K}$. We say that a derivation has bidegree (i,j) if it maps $\mathbb{K}^{c,b} \to \mathbb{K}^{c+i,b+j}$. $\operatorname{Der}\mathbb{K}$ is naturally bigraded

$$\operatorname{Der}\mathbb{K} = \bigoplus_{i,j} \operatorname{Der}^{i,j}\mathbb{K}\;, \tag{4.11}$$

where $\operatorname{Der}^{i,j}\mathbb{K}$ consists of derivations of bidegree (i,j). This decomposition makes $\operatorname{Der}\mathbb{K}$ into a bigraded Lie superalgebra under the graded commutator:

$$[\,,\,] : \operatorname{Der}^{i,j}\mathbb{K} \times \operatorname{Der}^{k,l}\mathbb{K} \to \operatorname{Der}^{i+k,j+l}\mathbb{K}\;. \tag{4.12}$$

We define $F^p\operatorname{Der}\mathbb{K} = \bigoplus_{i\geq p}\bigoplus_j \operatorname{Der}^{i,j}\mathbb{K}$. Then $F\operatorname{Der}\mathbb{K}$ gives a filtration of $\operatorname{Der}\mathbb{K}$ associated to the filtration $F\,\mathbb{K}$ of $\mathbb{K}$.

The remarks immediately following (4.10) imply that $D_2^2 \in F^3 \mathrm{Der}\,\mathbb{K}$. Moreover, it is trivial to check that $[\delta_0\,,\, D_2^2] \in F^4 \mathrm{Der}\,\mathbb{K}$. In fact,

$$[\delta_0\,,\, D_2^2] = [D_2\,,\, D_2^2] - [\delta_1\,,\, D_2^2] - [\delta_2\,,\, D_2^2] \tag{4.13}$$

where the first term vanishes because of the Jacobi identity and the last two terms are clearly in $F^4\mathrm{Der}\,\mathbb{K}$. Therefore the part of D_2^2 in $F^3\mathrm{Der}\,\mathbb{K}/F^4\mathrm{Der}\,\mathbb{K}$ is a δ_0-chain map: that is, $[\delta_0\,,\,\{\delta_1\,,\,\delta_2\}] = 0$. Since it has non-zero b-degree, the quasi-acyclicity of the Koszul complex implies that it induces the zero map in Koszul cohomology. By the following algebraic lemma (see below), there exists a derivation δ_3 of bidegree $(3,2)$ such that $\{\delta_0\,,\,\delta_3\} + \{\delta_1\,,\,\delta_2\} = 0$. If we define $D_3 = \sum_{i=0}^{3} \delta_i$, this is equivalent to $D_3^2 \in F^4\mathrm{Der}\,\mathbb{K}$. But by arguments identical to the ones above we deduce that $[\delta_0\,,\, D_3^2] \in F^5\,\mathrm{Der}\,\mathbb{K}$, and so on. It is not difficult to formalize these arguments into an induction proof of the following theorem:

Theorem 4.14. *We can define a derivation $D = \sum_{i=0}^{k} \delta_i$ on $\mathbb{K}$, where δ_i are derivations of bidegree $(i, i-1)$, such that $D^2 = 0$.*

Finally we come to the proof of the algebraic lemma used above.

Lemma 4.15. *Let*

$$\cdots \longrightarrow \mathbb{K}_2 \xrightarrow{\delta_0} \mathbb{K}_1 \xrightarrow{\delta_0} \mathbb{K}_0 \to 0 \tag{4.16}$$

denote the Koszul complex where $\mathbb{K}_b = \bigoplus_c \mathbb{K}^{c,b}$. Let $d : \mathbb{K}_b \to \mathbb{K}_{b+i}$, $(i \geq 0)$ be a derivation which commutes with δ_0 and which induces the zero map on cohomology. Then there exists a derivation $K : \mathbb{K}_b \to \mathbb{K}_{b+i+1}$ such that $d = \{\delta_0\,,\, K\}$.

Proof: Since $C^\infty(M)$ is an $\mathbb{R}$-algebra it is, in particular, a vector space. Let $\{f_\alpha\}$ be a basis for it. Then, since $\delta_0\, f_\alpha = 0$, $\delta_0\, d\, f_\alpha = 0$. Since d induces the zero map in cohomology, there exists λ_α such that $d\, f_\alpha = \delta_0\, \lambda_\alpha$. Define $K\, f_\alpha = \lambda_\alpha$. Similarly, since $\delta_0\, d\, \omega^i = 0$, there exists μ^i such that $d\,\omega^i = \delta_0\, \mu^i$. Define $K\,\omega^i = \mu^i$. Since $C^\infty(M)$ and the $\{\omega^i\}$ generate $\mathbb{K}_0$, we can extend K to all of $\mathbb{K}_0$ as a derivation and, by construction, in such a way that on $\mathbb{K}_0$, $d = \{\delta_0\,,\, K\}$. Now, $\delta_0\, d\, e_i = d\, \delta_0\, e_i$.

But since $\delta_0\, e_i \in \mathbb{K}_0$, $\delta_0\, d\, e_i = \delta_0\, K\, \delta_0\, e_i$. Therefore $\delta_0\,(d\, e_i - K\, \delta_0\, e_i) = 0$. Since $d\, e_i \in \mathbb{K}^{i+1}$ for some $i \geq 0$, the quasi-acyclicity of the Koszul complex implies that there exists ξ_i such that $d\, e_i - K\, \delta_0\, e_i = \delta_0\, \xi_i$. Define $K\, e_i = \xi_i$. Therefore, $d\, e_i = \{\delta_0\, ,\, K\}\, e_i$. We can now extend K as a derivation to all of $\mathbb{K}$. Since d and $\{\delta_0\, ,\, K\}$ are both derivations and they agree on generators, they are equal. ■

Defining the total complex $\mathbb{K} = \bigoplus_n \mathbb{K}^n$, where $\mathbb{K}^n = \bigoplus_{c-b=n} \mathbb{K}^{c,b}$, we see that $D : \mathbb{K}^n \to \mathbb{K}^{n+1}$. Its cohomology is therefore graded, that is, $H_D = \bigoplus_n H^n_D$. D is known as the BRST operator and its cohomology is the classical BRST cohomology. The total degree is known as the **ghost number**. We now compute the classical BRST cohomology. Notice that since all terms in D have non-negative filtration degree with respect to $F\,\mathbb{K}$, there is a spectral sequence associated to this filtration which converges to the cohomology of D. The E_1 term is the cohomology of the associated graded object $\mathrm{Gr}^p\mathbb{K} \equiv F^p\mathbb{K}/F^{p+1}\mathbb{K}$, with respect to the induced differential. The induced differential is the part of D of c-degree 0, that is, δ_0. Therefore the E_1 term is given by

$$E_1^{c,b} \cong \bigwedge{}^c \mathbb{V}^* \otimes H^b(K(\Phi)) \;. \tag{4.17}$$

That is, $E_1^{c,0} \cong \Omega^c_V(M_o)$ and $E_1^{c,b>0} = 0$.

The E_2 term is the cohomology of E_1 with respect to the induced differential d_1. Tracking down the definitions we see that d_1 is induced by δ_1 and hence it is just the vertical derivative d_V. Therefore, $E_2^{c,0} \cong H^c_V(M_0)$ and $E_2^{c,b>0} = 0$. Notice, however, that the spectral sequence is degenerate at this term, since the higher differentials $d_2, d_3, \ldots$ all have b-degree different from zero. Therefore we have proven the following theorem.

Theorem 4.18. *The classical BRST cohomology is given by*

$$H^n_D \cong \begin{cases} 0 & \text{for } n < 0 \\ H^n_V(M_o) & \text{for } n \geq 0 \end{cases} \;. \tag{4.19}$$

In particular, $H^0_D \cong C^\infty(\widetilde{M})$.

Now suppose that $E \longrightarrow M$ is a vector bundle over M whose smooth sections $\Gamma(E)$ afford a representation of the Lie algebra structure of $C^\infty(M)$ given by the Poisson brackets. That is, we have an action

$$\begin{aligned} C^\infty(M) \times \Gamma(E) &\to \Gamma(E) \\ (f, \sigma) &\mapsto f \times \sigma \end{aligned} \tag{4.20}$$

such that for all $f, g \in C^\infty(M)$ and $\sigma \in \Gamma(E)$

$$f \times (g \times \sigma) - g \times (f \times \sigma) = \{f\,,\,g\} \times \sigma\ . \tag{4.21}$$

Furthermore we demand that this action be a derivation with respect to the usual action of $C^\infty(M)$ on $\Gamma(E)$ given by pointwise multiplication. That is, for all $f, g \in C^\infty(M)$ and $\sigma \in \Gamma(E)$

$$f \times (g\,\sigma) = \{f\,,\,g\}\,\sigma + g\,(f \times \sigma)\ \ . \tag{4.22}$$

For example, if E admits a flat connection ∇ then we can define

$$f \times \sigma \equiv \nabla_{X_f} \sigma\ , \tag{4.23}$$

where X_f is the hamiltonian vector field associated to f. The fact that ∇ is flat implies that

$$\nabla_X \nabla_Y - \nabla_Y \nabla_X = \nabla_{[X\,,Y]}\ , \tag{4.24}$$

and hence it gives a representation. Similarly if we have a notion of Lie derivative on $\Gamma(E)$, we can define

$$f \times \sigma \equiv \mathcal{L}_{X_f} \sigma\ . \tag{4.25}$$

Let $E_o \longrightarrow M_o$ denote the pullback of E via the inclusion $i : M_o \to M$. We can now define **vertical cohomology with coefficients in E_o** as follows. Define the

E_o-valued vertical forms $\Omega_V(E_o)$ by

$$\Omega_V(E_o) \equiv \Omega_V(M_o) \otimes_{C^\infty(M_o)} \Gamma(E_o) \ . \tag{4.26}$$

We define ∇_V by

$$\nabla_V \, \sigma = \sum_i (\phi_i \times \sigma) \, \omega^i \tag{4.27}$$

and

$$\nabla_V \, \omega^i = -\frac{1}{2} \sum_{j,k} f_{jk}{}^i \, \omega^j \wedge \omega^k \ , \tag{4.28}$$

for all $\sigma \in \Gamma(E_o)$. We then extend it to all of $\Omega_V(E_o)$ as a derivation. Just as for d_V, it is easy to verify that $\nabla_V^2 = 0$. We denote its cohomology by $H_V(E_o)$. Moreover notice that for all E_o-valued vertical forms θ

$$\nabla_V \, (f \, \theta) = d_V \, f \wedge \theta + f \, \nabla_V \, \theta \ . \tag{4.29}$$

Therefore, $H_V^0(E_o)$ becomes a $C^\infty(\widetilde{M})$-module (under pointwise multiplication) after the identification of $C^\infty(\widetilde{M})$ with $H_V^0(M_o)$. To see this notice that if $d_V \, f = 0$ and $\nabla_V \, \sigma = 0$, for some $\sigma \in \Gamma(E_o)$, $\nabla_V \, (f \, \sigma) = 0$. Moreover, it is easy to verify that this module is finitely generated and projective. Hence, by general arguments[15], it is the module of sections of some vector bundle over $\widetilde{M}$.

Now using the Koszul resolution for $\Gamma(E_o)$ given by Corollary 3.33 we would like to lift ∇_V to a differential D_∇ on the complex $\mathbb{K}(E) = \bigoplus_{c,b} \mathbb{K}^{c,b}(E)$ where

$$\mathbb{K}^{c,b}(E) \equiv \bigwedge^c \mathbb{V}^* \otimes \bigwedge^b \mathbb{V} \otimes \Gamma(E) \ . \tag{4.30}$$

This follows virtually identical steps as for the case of functions. We define

$$\nabla_0 \equiv (-1)^c \delta_K \otimes \mathbf{1} \qquad \text{on} \quad \mathbb{K}^{c,b}(E) \ . \tag{4.31}$$

Then, just as before, we define ∇_1 by

$$\nabla_1 \, \sigma = \sum_i (\phi_i \times \sigma) \, \omega^i \ , \tag{4.32}$$

$$\nabla_1 \omega^i = -\frac{1}{2}\sum_{j,k} f_{jk}{}^i\, \omega^j \wedge \omega^k \ , \tag{4.33}$$

and

$$\nabla_1 e_i = \sum_{j,k} f_{ji}{}^k\, \omega^j \wedge e_k \ . \tag{4.34}$$

In this way $\nabla_0^2 = 0$ and $\{\nabla_0\,, \nabla_1\} = 0$. Therefore filtering $\mathbb{K}(E)$ in the same way as we filtered $\mathbb{K}$ and following isomorphic arguments to those leading to Theorem 4.14 we prove the following theorem.

Theorem 4.35. *We can define a derivation $D_\nabla = \sum_{i=0}^k \nabla_i$ on $\mathbb{K}(E)$, where ∇_i are derivations of bidegree $(i, i-1)$, such that $D_\nabla^2 = 0$.*

Another spectral sequence argument isomorphic to the one yielding Theorem 4.18 allows us to compute the cohomology of D_∇.

Theorem 4.36. *The cohomology of D_∇ is given by*

$$H^n_{D_\nabla} \cong \begin{cases} 0 & \text{for } n < 0 \\ H^n_V(E_o) & \text{for } n \geq 0 \end{cases} . \tag{4.37}$$

§5 POISSON ALGEBRAS AND POISSON MODULES

So far in the construction of the BRST complex no use has been made of the Poisson structure of the smooth functions on M. In this section we remedy the situation. It turns out that the complex $\mathbb{K}$ introduced in the last section is a Poisson superalgebra and the BRST operator D can be made into a Poisson derivation. Moreover the complex $\mathbb{K}(E)$ has a natural Poisson module structure over $\mathbb{K}$ under which D and D_∇ correspond. It will then follow that in cohomology all constructions based on the Poisson structures will be preserved. This is especially important in the context of geometric quantization since all objects there can be defined purely in terms of the Poisson algebra structure of the smooth functions. In this section we review the concepts associated to Poisson algebras and Poisson modules. We define the relevant Poisson structures in $\mathbb{K}$ and in $\mathbb{K}(E)$ and explore its consequences.

Recall that a **Poisson superalgebra** is a $\mathbb{Z}_2$-graded vector space $P = P_0 \oplus P_1$ together with two bilinear operations preserving the grading:

$$\begin{aligned} P \times P &\to P \\ (a, b) &\mapsto ab \end{aligned} \qquad \text{(multiplication)}$$

and

$$\begin{aligned} P \times P &\to P \\ (a, b) &\mapsto [a\, ,\, b] \end{aligned} \qquad \text{(Poisson bracket)}$$

obeying the following properties

(P1) P is an associative supercommutative superalgebra under multiplication:

$$\begin{aligned} a(bc) &= (ab)c \\ ab &= (-1)^{|a||b|}\, ba\ ; \end{aligned}$$

(P2) P is a Lie superalgebra under Poisson bracket:

$$\begin{aligned} [a\, ,\, b] &= (-1)^{|a||b|}\, [b\, ,\, a] \\ [a\, ,\, [b\, ,\, c]] &= [[a\, ,\, b]\, ,\, c] + (-1)^{|a||b|}\, [b\, ,\, [a\, ,\, c]]\ ; \end{aligned}$$

(P3) Poisson bracket is a derivation over multiplication:

$$[a\, ,\, bc] = [a\, ,\, b]c + (-1)^{|a||b|}\, b[a\, ,\, c]\ ;$$

for all $a, b, c \in P$ and where $|a|$ equals 0 or 1 according to whether a is even or odd, respectively.

The algebra $C^\infty(M)$ of smooth functions of a symplectic manifold (M,Ω) is clearly an example of a Poisson superalgebra where $C^\infty(M)_1 = 0$. On the other hand, if $\mathbb{V}$ is a finite dimensional vector space and $\mathbb{V}^*$ its dual, then the exterior algebra $\bigwedge(\mathbb{V} \oplus \mathbb{V}^*)$ posseses a Poisson superalgebra structure. The associative multiplication is given by exterior multiplication $(\wedge)$ and the Poisson bracket is defined for $u, v \in \mathbb{V}$ and $\alpha, \beta \in \mathbb{V}^*$ by

$$[\alpha\, ,\, v] = \langle \alpha, v\rangle \qquad [v\, ,\, w] = 0 = [\alpha\, ,\, \beta]\ , \tag{5.1}$$

where $\langle , \rangle$ is the dual pairing between $\mathbb{V}$ and $\mathbb{V}^*$. We then extend it to all of $\bigwedge(\mathbb{V} \oplus \mathbb{V}^*)$ as an odd derivation. Therefore the classical ghosts/antighosts in BRST possess a Poisson algebra structure. In **[7]** it is shown that this Poisson bracket is induced from the supercommutator in the Clifford algebra $\mathrm{Cl}(\mathbb{V} \oplus \mathbb{V}^*)$ with respect to the non-degenerate inner product on $\mathbb{V} \oplus \mathbb{V}^*$ induced by the dual pairing.

To show that $\mathbb{K}$ is a Poisson superalgebra we need to discuss tensor products. Given two Poisson superalgebras P and Q, their tensor product $P \otimes Q$ can be given the structure of a Poisson superalgebra as follows. For $a, b \in P$ and $u, v \in Q$ we define

$$(a \otimes u)(b \otimes v) = (-1)^{|u||b|}\, ab \otimes uv \tag{5.2}$$

$$[a \otimes u\, ,\, b \otimes v] = (-1)^{|u||b|}\, \left([a\, ,\, b] \otimes uv + ab \otimes [u\, ,\, v]\right)\ . \tag{5.3}$$

The reader is invited to verify that with these definitions (P1)-(P3) are satisfied. From this it follows that $\mathbb{K} = C^\infty(M) \otimes \bigwedge(\mathbb{V} \oplus \mathbb{V}^*)$ becomes a Poisson superalgebra.

Now let P be a Poisson superalgebra and $M = M_0 \oplus M_1$ a $\mathbb{Z}_2$-graded vector space. We will call M a P**-module**, or a **Poisson module over** P, if there exist two bilinear operations preserving the grading

$$P \times M \to M$$

$$(a, m) \mapsto a \cdot m$$

and

$$P \times M \to M$$
$$(a, m) \mapsto a \times m$$

obeying the following properties

(M1) $\cdot$ makes M a module over the associative structure of P:

$$a \cdot (b \cdot m) = (ab) \cdot m \ ;$$

(M2) $\times$ makes M into a module over the Lie superalgebra structure of P:

$$a \times (b \times m) - (-1)^{|a||b|}\, b \times (a \times m) = [a\, ,\, b] \times m \ ;$$

(M3) $a \times (b \cdot m) = [a\, ,\, b] \cdot m + (-1)^{|a||b|}\, b \cdot (a \times m)$; where $a, b \in P$ and $m \in M$.

In particular a Poisson algebra becomes a Poisson module over itself after identifying $a \cdot b$ with ab and $a \times b$ with $[a\, ,\, b]$. Notice that equations (4.21) and (4.22) imply that $\Gamma(E)$ becomes a Poisson module over $C^\infty(M)$.

Just like the tensor product of two Poisson superalgebras can be made into a Poisson superalgebra, if M and N are Poisson modules over P and Q, respectively, their tensor product $M \otimes N$ becomes a $P \otimes Q$-module under the following operations:

$$(a \otimes u) \cdot (m \otimes n) = (-1)^{|u||m|}\, (a \cdot m) \otimes (u \cdot n) \tag{5.4}$$
$$a \otimes u \times m \otimes n = (-1)^{|u||m|}\, (a \times m \otimes u \cdot n + a \cdot m \otimes u \times n) \ , \tag{5.5}$$

for all $a \in P$, $u \in Q$, $m \in M$, and $n \in N$. Again the reader is invited to verify that with these definitions (M1)-(M3) are satisfied.

Since $\bigwedge(\mathbb{V} \oplus \mathbb{V}^*)$ is a Poisson module over itself and $\Gamma(E)$, for $E \longrightarrow M$ the kind of vector bundle discussed in the previous section, is a Poisson module over $C^\infty(M)$, their tensor product $\mathbb{K}(E)$ becomes a Poisson module over $\mathbb{K}$.

Now let P be a Poisson superalgebra which, in addition, is $\mathbb{Z}$-graded, that is, $P = \bigoplus_n P^n$ and $P^n P^m \subseteq P^{m+n}$ and $[P^n , P^m] \subseteq P^{m+n}$; and such that the $\mathbb{Z}_2$-grading is the reduction modulo 2 of the $\mathbb{Z}$-grading, that is, $P_0 = \bigoplus_n P^{2n}$ and $P_1 = \bigoplus_n P^{2n+1}$. We call such algebras graded Poisson superalgebras. Notice that P^0 is an even Poisson subalgebra of P.

For example, letting $\mathbb{K} = C^\infty(M) \otimes \bigwedge(\mathbb{V} \oplus \mathbb{V}^*)$ we can define $\mathbb{K}^n = \bigoplus_{c-b=n} \mathbb{K}^{c,b}$. This way $\mathbb{K}$ becomes a $\mathbb{Z}$-graded Poisson superalgebra. Although the bigrading is preserved by the exterior product, the Poisson bracket does not preserve it. In fact, the Poisson bracket obeys

$$[\,,\,] : \mathbb{K}^{i,j} \times \mathbb{K}^{k,l} \to \mathbb{K}^{i+k,j+l} \oplus \mathbb{K}^{i+k-1,j+l-1} \; . \tag{5.6}$$

We can also define the analogous concept of a graded Poisson module over a graded Poisson superalgebra P in the obvious way: $M = \bigoplus_n M^n$, where $M_0 = \bigoplus_n M^{2n}$ and $M_1 = \bigoplus_n M^{2n+1}$, and both actions of P on M respect the grading. Therefore $\mathbb{K}(E)$ becomes a graded Poisson module over $\mathbb{K}$.

By a Poisson derivation of degree k we will mean a linear map $D : P^n \to P^{n+k}$ such that

$$D(ab) = (Da)b + (-1)^{k|a|}\, a(Db) \tag{5.7}$$

$$D[a\, , \, b] = [Da\, , \, b] + (-1)^{k|a|}\, [a\, , \, Db] \; . \tag{5.8}$$

The map $a \mapsto [Q\, , \, a]$ for some $Q \in P^k$ automatically obeys (5.7) and (5.8) . Such Poisson derivations are called inner. Whenever the degree derivation is inner, any Poisson derivation of non-zero degree is inner[**18**] as we now show. The degree derivation N is defined uniquely by $Na = na$ if and only if $a \in P^n$. In the case

$P = \mathbb{K}$, N is the ghost number operator which is an inner derivation $[G\,,\,\cdot]$, where $G = \sum_i \omega^i \wedge e_i$, where $\{e_i\}$ is a basis for $\mathbb{V}$ and $\{\omega^i\}$ denotes its canonical dual basis. Now if $a \in P^n$, and the degree of D is $k \neq 0$, it follows from (5.8) that

$$Da = \frac{-1}{k}[DG\,,\,a]\ , \tag{5.9}$$

and so D is an inner derivation.

If, furthermore, D should obey $D^2 = 0$, and be of degree 1, $Q = -DG$ would obey $[Q\,,\,Q] = 0$. To see this notice that for all $a \in P^n$

$$D^2 a = [Q\,,\,[Q\,,\,a]] = \frac{1}{2}[[Q\,,\,Q]\,,\,a] = 0\ .$$

But for $a = G$ we get that $[Q\,,\,Q] = 0$. In this case it is easy to verify that $\ker D$ becomes a Poisson subalgebra of P and $\operatorname{im} D$ is a Poisson ideal of $\ker D$. Therefore the cohomology space $H_D = \ker D/\operatorname{im} D$ naturally inherits the structure of a Poisson algebra. If let M be a P-module and define the endomorphism $\mathbb{D} : M \to M$ by

$$\mathbb{D}\,m = Q \times m \qquad \forall m \in M\ , \tag{5.10}$$

it follows from (M2) and the fact that $[Q\,,\,Q] = 0$ that $\mathbb{D}^2 = 0$. Furthermore, using (M1)-(M3), it follows that its cohomology, $H_{\mathbb{D}}$, inherits the structure of a graded Poisson module over H_D. In particular, $H^0_{\mathbb{D}}$ is a Poisson module over H^0_D.

The BRST operator D constructed in the previous section is a derivation over the exterior product. Nothing in the way it was defined guarantees that it is a Poisson derivation and, in fact, it need not be so. However one can show that the δ_i's — which were, by far, not unique — can be defined in such a way that the resulting D is a Poisson derivation, from which it would immediately follow that it is inner. It is easier, however, to show the existence of the element $Q \in \mathbb{K}^1$ such that $D = [Q\,,\,\cdot]$. We will show that there exists $Q = \sum_{i\geq 0} Q_i$, where $Q_i \in \mathbb{K}^{i+1,i}$, such that $[Q\,,\,Q] = 0$ and that the cohomology of the operator $[Q\,,\,\cdot]$ is isomorphic to that of D. This was first proven by Henneaux in **[19]** and later in a completely algebraic way by Stasheff in **[4]** . Our proof is a simplified version of this latter proof.

From the discussion previous to Theorem 4.18 we know that the only parts of D which affect its cohomology are δ_0, which is the Koszul differential, and δ_1 acting on the Koszul cohomology. Hence we need only make sure that the Q_i we construct realize these differentials. Notice that if $Q_i \in \mathbb{K}^{i+1,i}$, $[Q_i\,,\,\cdot]$ has terms of two different bidegrees $(i+1,i)$ and $(i,i-1)$. Hence the only term which can contribute to the Koszul differential is Q_0. There is a unique element $Q_0 \in \mathbb{K}^{1,0}$ such that $[Q_0\,,\,e_i] = \delta_0\, e_i = \phi_i$. This is given by

$$Q_0 = \sum_i \omega^i\, \phi_i \ . \tag{5.11}$$

Notice that

$$[Q_0\,,\,e_i] = \delta_0\, e_i = \phi_i \tag{5.12}$$

$$[Q_0\,,\,\omega^i] = \delta_0\, \omega^i = 0 \tag{5.13}$$

$$[Q_0\,,\,f] = (\delta_0 + \delta_1)\, f = \sum_i [\phi_i\,,\,f]\, \omega^i \ . \tag{5.14}$$

There is now a unique $Q_1 \in \mathbb{K}^{2,1}$ such that $[Q_1\,,\,\omega^i] = \delta_1\, \omega^i$, namely,

$$Q_1 = -\frac{1}{2} \sum_{i,j,k} f_{ij}{}^k\, \omega^i \wedge \omega^j \wedge e_k \ . \tag{5.15}$$

If we define $R_1 = Q_0 + Q_1$ we then have that

$$[R_1\,,\,e_i] = (\delta_0 + \delta_1)\, e_i \tag{5.16}$$

$$[R_1\,,\,\omega^i] = (\delta_0 + \delta_1)\, \omega^i \tag{5.17}$$

$$[R_1\,,\,f] = (\delta_0 + \delta_1 + \delta_2)\, f \ . \tag{5.18}$$

In particular, two things are imposed upon us: $\delta_2\, f$ and $\delta_1\, e_i$; the latter imposition agrees with the choice made in (4.9) .

Letting $F\mathbb{K}$ denote the filtration of $\mathbb{K}$ defined in the previous section, and using the notation in which, if $O \in \mathbb{K}$ is an odd element, O^2 stands for $\frac{1}{2}[O\,,\,O]$, the following are satisfied:

$$R_1^2 \in F^3\mathbb{K} \qquad \text{and} \qquad [Q_0\,,\,R_1^2] \in F^4\mathbb{K}\;. \tag{5.19}$$

That means that the part of R_1^2 which lives in $F^3\mathbb{K}/F^4\mathbb{K}$ is a δ_0-cocycle, since the $(0,-1)$ part of Q_0 is precisely δ_0. By the quasi-acyclicity of the Koszul complex it is a coboundary, say, $-\delta_0\,Q_2$ for some $Q_2 \in \mathbb{K}^{3,2}$. In other words, there exists $Q_2 \in \mathbb{K}^{3,2}$ such that if $R_2 = Q_0 + Q_1 + Q_2$, then $R_2^2 \in F^4\mathbb{K}$. If this is the case then

$$[Q_0\,,\,R_2^2] = [R_2\,,\,R_2^2] - [Q_1\,,\,R_2^2] - [Q_2\,,\,R_2^2]\;. \tag{5.20}$$

But the first term is zero because of the Jacobi identity and the last two terms are clearly in $F^5\mathbb{K}$ due to the fact that, from (5.6) ,

$$[F^p\mathbb{K}\,,\,F^q\mathbb{K}] \subseteq F^{p+q-1}\mathbb{K}\;. \tag{5.21}$$

Hence, $[Q_0\,,\,R_2^2] \in F^5\mathbb{K}$, from where we can deduce the existence of $Q_3 \in \mathbb{K}^{4,3}$ such that $R_3 = Q_0 + Q_1 + Q_2 + Q_3$ obeys $R_3^2 \in F^5\mathbb{K}$, and so on. It is easy to formalize this into an induction proof of the following theorem.

Theorem 5.22. *There exists $Q = \sum_i Q_i$, where $Q_i \in \mathbb{K}^{i+1,i}$ such that $[Q\,,\,Q] = 0$.*

Now let $D = [Q\,,\,\cdot]$. Then $D^2 = 0$ and repeating the proof of Theorem 4.18 we obtain the following.

Theorem 5.23. *The cohomology of D is given by*

$$H_D^n \cong \begin{cases} 0 & \text{for } n < 0 \\ H_V^n(M_o) & \text{for } n \geq 0 \end{cases}\;. \tag{5.24}$$

In particular, $H_D^0 \cong C^\infty(\widetilde{M})$.

From now on we will take $D = [Q\,,\,\cdot]$ to be the classical BRST operator.

Now we consider an arbitrary vector bundle $E \longrightarrow M$, whose smooth sections $\Gamma(E)$ form a Poisson module over $C^\infty(M)$. Then we can define $\mathbb{D}$ by (5.10) . Then $\mathbb{D}$ decomposes into $\mathbb{D} = \sum_i \nabla_i$, where $\nabla_i : \mathbb{K}^{c,b} \to \mathbb{K}^{c+i,b+i-1}$. We can recover the ∇_i from the Q_i by picking the contribution with the right bidegree. Using the explicit expressions for Q_0 and Q_1 it is easy to verify that ∇_0 and ∇_1 defined in this way agree precisely with the ones in (4.31) , (4.32) , (4.33) , and (4.34) .

This being enough to determine its cohomology, we deduce that the cohomology of $\mathbb{D}$ and that of the operator D_∇ of Theorem 4.35 are isomorphic. That is,

Theorem 5.25. *The cohomology of* $\mathbb{D}$ *is given by*

$$H^n_{\mathbb{D}} \cong \begin{cases} 0 & \text{for } n < 0 \\ H^n_V(E_o) & \text{for } n \geq 0 \end{cases} . \tag{5.26}$$

We conclude, therefore, that the classical BRST construction — both for functions and for sections — is completely compatible with the Poisson structure. Roughly speaking, the BRST construction feels right at home in the Poisson category.

§6 PREQUANTIZATION

Geometric quantization is an attempt to develop a mathematically consistent and invariant quantization scheme. It tries to overcome the problems of the more traditional "canonical" quantization. The canonical quantization of finite dimensional systems consists in finding a unitary irreducible representation of the Heisenberg algebra

$$[q^a\,,\,p_b] = \sqrt{-1}\hbar\,\delta^a_b\ , \tag{6.1}$$

where (q,p) are local coordinates for the phase space of the system we are quantizing. The Stone–Von Neumann theorem guarantees that there is essentially a unique such representation[2]. In this representation — taken, without loss of generality, to be $L^2(\mathbb{R}^n)$ — q^a is represented by the multiplication operator $\psi(q) \mapsto q^a\,\psi(q)$; and p_b is represented by $\psi(q) \mapsto -\sqrt{-1}\hbar\psi'(q)$. A classical observable $f(p,q)$ is then represented by $f(-\sqrt{1}\hbar\frac{\partial}{\partial q}, q)$.

This has two obvious problems. First, the operator $f(-\sqrt{1}\hbar\frac{\partial}{\partial q}, q)$ requires for its definition that we give an ordering prescription, since q and $\frac{\partial}{\partial q}$ do not commute. And second, the Heisenberg algebra is not general coordinate invariant, so the above prescription depends on the choice of coordinates.

It was Dirac who first noticed the similarity between the algebraic structures in both quantum and classical mechanics. He observed that the Poisson bracket seemed to be the classical analogue of the quantum commutator. The fact that the Poisson bracket has an invariant meaning in the phase space allowed Dirac to reformulate canonical quantization in an invariant fashion. The Poisson brackets (*i.e.* the symplectic structure) thus plays a fundamental rôle in the Dirac quantization approach. The Dirac quantization problem consists therefore in finding an irreducible representation of the Lie algebra (under Poisson bracket) of real smooth functions as "self-adjoint" operators in a Hilbert space with the properties that the constant function with value 1 shall be represented by the identity operator and that, if (q,p) is local chart forming a canonically conjugate pair (*i.e.* they obey the Heisenberg algebra), then they shall act irreducibly or at least, in case one wants to include internal degrees of freedom, with finite reducibility. These conditions are demanded, for example, by the Heisenberg uncertainty relation.

A celebrated theorem of Van Hove[20] forbids the existence of such a representation; although he showed that one could find an irreducible representation of some subalgebra if one dropped the last condition on canonically conjugate pairs.

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[2] Strictly speaking, the theorem guarantees the uniqueness up to unitary equivalence of the irreducible representations of the exponentiated (Weyl) form of the commutation relations.

The geometric quantization program of Kostant[21] and Souriau[22] provides an invariant method of constructing such representations. The first part of the method, called **prequantization**, consists of dropping the irreducibility condition and constructing a representation of the Lie algebra of smooth functions as "self-adjoint" operators in a Hilbert space, purely in terms of symplectic data. The second part of the construction, called **polarization**, will take care of making this representation irreducible and in the process restricting the class of functions which can be quantized. In this section we discuss prequantization. We will review how prequantum data gets induced under symplectic reduction, and how this is implemented à la BRST.

Let (M,Ω) be a symplectic manifold. Since $d\Omega = 0$, the symplectic form defines a class in the real de Rham cohomology group $H^2_{dR}(M;\mathbb{R})$. We say Ω is integral if this class lies in the image of the map

$$H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{R}) \cong H^2_{dR}(M;\mathbb{R}) \ . \tag{6.2}$$

If this is the case we speak of an integral symplectic manifold.

If (M,Ω) is an integral symplectic manifold then there exists at least one complex line bundle $E \longrightarrow M$ with a hermitian structure, *i.e.* a sesquilinear map

$$\langle,\rangle : \Gamma(E) \times \Gamma(E) \to C^\infty_{\mathbb{C}}(M) \ , \tag{6.3}$$

which is antilinear in the first factor and linear in the second; and with a connection

$$\nabla : \Gamma(E) \to \Omega^1(M) \otimes \Gamma(E) \ , \tag{6.4}$$

such that

(PQ1) $\langle,\rangle$ is parallel with respect to ∇; that is, for all $\sigma,\tau \in \Gamma(E)$,

$$d\langle\sigma,\tau\rangle = \langle\sigma,\nabla\tau\rangle + \langle\nabla\sigma,\tau\rangle \ ;$$

(PQ2) the symplectic form and the curvature 2-form of the connection are related by

$$\mathrm{curv}(\nabla) = -2\pi\sqrt{-1}\Omega \ .$$

The triple $(E, \nabla, \langle , \rangle)$ satisfying the above properties will be called prequantum data for the symplectic manifold (M, Ω). Hence integral symplectic manifolds are also known as prequantizable symplectic manifolds.

Let $d\mu_L$ denote the Liouville measure on M. This is the measure induced by the volume form proportional to $\underbrace{\Omega \wedge \cdots \wedge \Omega}_{n}$ for M a $2n$-dimensional manifold. This allows us to define an inner product on $\Gamma(E)$ by integrating the pointwise inner product with respect to this measure:

$$(\sigma, \tau) \equiv \int_M \langle \sigma, \tau \rangle \, d\mu_L \ . \tag{6.5}$$

Let $\Gamma_{L^2}(E)$ denote the Hilbert space completion of the subspace of $\Gamma(E)$ consisting of sections σ such that $\|\sigma\|^2 \equiv (\sigma, \sigma) < \infty$. This will become the prequantum Hilbert space. The prequantization map assigning to a smooth function f an operator $O(f)$ in $\Gamma_{L^2}(E)$ is the following

$$f \mapsto O(f) \equiv \nabla_{X_f} + 2\pi\sqrt{-1}\, f \ , \tag{6.6}$$

where X_f is the Hamiltonian vector field associated to f, that is, $i(X_f)\,\Omega + df = 0$. The prequantization map obeys the following

$$O(f)O(g)\sigma - O(g)O(f)\sigma = O(\{f\,,\,g\})\sigma \ , \tag{6.7}$$

$$O(f)(g\sigma) = \{f\,,\,g\}\sigma + g\,O(f)\sigma \ , \tag{6.8}$$

for all $\sigma \in \Gamma(E)$ and $f, g \in C^\infty(M)$, making $\Gamma(E)$ into a Poisson module over $C^\infty(M)$. Moreover each $O(f)$ is a skew-symmetric operator. That is, if $\sigma, \tau \in$

$\Gamma_{L^2}(E)$ are in the domain of $O(f)$ then

$$(O(f)\sigma, \tau) + (\sigma, O(f)\tau) = 0 \ . \tag{6.9}$$

If, in addition, X_f is a complete vector field, $O(f)$ has a skew-self-adjoint extension and generates, by Stone's theorem, a one parameter family of unitary operators in $\Gamma_{L^2}(E)$.

The prequantization map has the property that the only operator of the form $O(f)$ for some $f \in C^\infty(M)$ which commutes with all the other $O(g)$'s are the scalars, corresponding to $O(c)$ for c a constant function on M. Still this representation is highly reducible: roughly speaking it consists of integrable functions of both the momenta and the coordinates. Thus we need to cut down the size of $\Gamma_{L^2}(E)$. This process, known as polarization, will be the topic of the next section. In the rest of this section we show how prequantum data gets induced under symplectic reduction, and how this works in the BRST context.

In **[23]** Guillemin and Sternberg proved that in the case of a hamiltonian group action one could induce prequantum data on the reduced symplectic manifold. Their construction goes roughly as follows. Let M_o denote the constrained submanifold $\Phi^{-1}(0)$, Φ being the moment mapping in their case, and let $E_o \longrightarrow M_o$ denote the pull back bundle of E via the inclusion $i : M_o \to M$. They define a complex line bundle $\widetilde{E} \longrightarrow \widetilde{M}$, where $\widetilde{M} = M_o/G$, by defining its sheaf of sections to be the $\mathfrak{g}$-invariant sections[3] of E_o. On M_o, a section σ was $\mathfrak{g}$-invariant if and only if $\nabla_X \sigma = 0$ for all Killing vectors X. Hence if σ and τ are $\mathfrak{g}$-invariant sections, so is their pointwise inner product $\langle \sigma, \tau \rangle$, since

$$X\langle \sigma, \tau \rangle = \langle \nabla_X \sigma, \tau \rangle + \langle \sigma, \nabla_X \tau \rangle \ . \tag{6.10}$$

A connection $\widetilde{\nabla}$ is also constructed by constructing its connection 1-form patchwise.

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[3] Strictly speaking one must assume that the $\mathfrak{g}$ action on the sections of E_o lifts to a G action, to guarantee that the invariant sections correspond to the sections of some vector bundle over $\widetilde{M}$.

Finally, the inner product was defined integrating the pointwise inner product with the Liouville measure on $\widetilde{M}$.

We now proceed to see how the prequantum data gets induced à la BRST. We need not restrict ourselves to the case of a group action. First of all notice that the sections $\Gamma(E)$ of the prequantum line bundle form a Poisson module over $C^\infty(M)$. Therefore the discussion of the previous sections goes through unaltered. That is, we construct the graded complex $\mathbb{K}(E) = \bigoplus_n \mathbb{K}^n(E)$ and the derivation

$$D_\nabla : \mathbb{K}^n(E) \to \mathbb{K}^{n+1}(E) \ , \tag{6.11}$$

given by $Q \times \sigma$, where $Q \in \mathbb{K}^1$ is the element such that the classical BRST operator D is given by $[Q\,,\,\cdot]$. We then define $\widetilde{E} \longrightarrow \widetilde{M}$ by defining its space of smooth sections $\Gamma(\widetilde{E})$ as $H^0_{D_\nabla}$. It then becomes a $H^0_D \cong C^\infty(\widetilde{M})$ module as shown in the previous section. Since the prequantization map is precisely one of the maps incorporated in the Poisson module structure of $\Gamma(\widetilde{E})$ we have induced all the prequantum data except for the inner product. We will be able to induce a pointwise inner product but not an inner product. We will comment on the reasons why later on.

In order to induce a pointwise inner product on $H^0_{D_\nabla}$ it will be first of all necessary to define a pointwise inner product on $\mathbb{K}(E)$. To motivate this construction let us first understand in Poisson terms the invariance of the pointwise inner product of two invariant sections. This invariance follows from the following fact. Since $\langle,\rangle$ is $\mathbb{R}$-linear in both slots it induces a map

$$\langle,\rangle : \Gamma(E) \otimes \Gamma(E) \to C^\infty_{\mathbb{C}}(M) \ ,$$

which is a $C^\infty(M)$-module homomorphism (a homomorphism of Poisson modules over $C^\infty(M)$). That is, if $\sigma, \tau \in \Gamma(E)$ and $f \in C^\infty(M)$ then

$$[f\,,\,\langle\sigma,\tau\rangle] = \langle f \times \sigma, \tau\rangle + \langle \sigma, f \times \tau\rangle \ . \tag{6.12}$$

We would now like to extend $\langle , \rangle$ to a $\mathbb{K}$-module homomorphism

$$\langle\!\langle , \rangle\!\rangle : \mathbb{K}(E) \otimes \mathbb{K}(E) \to \mathbb{K}_{\mathbb{C}} \ . \tag{6.13}$$

This boils down to essentially defining a linear map

$$\langle , \rangle : \bigwedge(\mathbb{V} \oplus \mathbb{V}^*) \otimes \bigwedge(\mathbb{V} \oplus \mathbb{V}^*) \to \bigwedge(\mathbb{V} \oplus \mathbb{V}^*) \ , \tag{6.14}$$

satisfying, for all $\phi, \omega, \theta \in \bigwedge(\mathbb{V} \oplus \mathbb{V}^*)$, the following relations

$$\langle \phi \wedge \omega, \theta \rangle = \phi \wedge \langle \omega, \theta \rangle = (-1)^{|\phi||\omega|} \langle \omega, \phi \wedge \theta \rangle \ , \tag{6.15}$$

and

$$[\phi \, , \, \langle \omega, \theta \rangle] = \langle [\phi \, , \, \omega], \theta \rangle + (-1)^{|\phi||\omega|} \langle \omega, [\phi \, , \, \theta] \rangle \ . \tag{6.16}$$

There is one obvious candidate:

$$\langle \omega, \theta \rangle = \omega \wedge \theta \ . \tag{6.17}$$

However, although this pointwise inner product will turn out to play an important rôle when we discuss duality, it does not seem to be the natural pointwise inner product on $\Gamma(\widetilde{E})$. There are other variants of this inner product, differing from it in a degree dependent sign, which eliminate some of the $\pm$'s which will appear when we discuss duality. However these signs are not very relevant and for simplicity we will stick with this pointwise inner product.

At any rate, with this choice we have constructed a sesquilinear map

$$\langle\!\langle , \rangle\!\rangle : \mathbb{K}(E) \times \mathbb{K}(E) \to \mathbb{K}_{\mathbb{C}} \ , \tag{6.18}$$

which is invariant under the action of $\mathbb{K}$. It is then clear that, if $Z(E)$ and $B(E)$ stand for the D_∇ cocycles and coboundaries respectively and Z and B stand for the D cocycles and coboundaries respectively, the mapping $\langle\langle , \rangle\rangle$ obeys

$$Z(E) \times Z(E) \to Z \ , \tag{6.19}$$

$$Z(E) \times B(E) \to B \ , \tag{6.20}$$

$$B(E) \times Z(E) \to Z \ ; \tag{6.21}$$

from where it follows that it induces a well defined map in cohomology. In particular, since it is graded, it induces a map

$$\widetilde{\langle , \rangle} : H^0_{D_\nabla} \times H^0_{D_\nabla} \to H^0_D \otimes \mathbb{C} \ , \tag{6.22}$$

which, under the relevant identifications, becomes a pointwise inner product

$$\widetilde{\langle , \rangle} : \Gamma(\widetilde{E}) \times \Gamma(\widetilde{E}) \to C^\infty_{\mathbb{C}}(\widetilde{M}) \ . \tag{6.23}$$

This is the best that can be done about the inner product under the present circumstances. There is no inner product on $\mathbb{K}(E)$ which induces, by evaluating it on D_∇ cocycles, the prequantum inner product on $\widetilde{M}$. The reason is the following. The inner product consists of integrating the pointwise inner product with respect to the Liouville measure. It is impossible that one can evaluate the inner product of sections of the prequantum bundle on $\widetilde{M}$ by merely picking representative sections on M and evaluating the inner product there. The reason being that functions on $\widetilde{M}$ are represented by functions on M whose restriction to M_o are constant on the leaves of the null foliation. But M_o has Liouville measure zero in M and hence two functions which agree on M_o but which disagree at will away from M_o have different integrals. Therefore the inner product would not be independent of the representatives. By tensoring the sections of the prequantum line bundle with half-forms (see **[6]**) the BRST cohomology of this new complex yields objects whose pointwise inner product can be integrated on $\widetilde{M}$ but, again, the integral does not lift to M.

§7 POLARIZATION

In this section we discuss the second step in the geometric quantization program. As we remarked in the previous section, the representation constructed via prequantization is highly reducible and we must cut down its size. To help fix the ideas, let us discuss the familiar example of M a cotangent bundle, say T^*N. In this case the symplectic form is exact and hence the prequantum line bundle is trivial. We can therefore choose a global non-vanishing section and hence identify the space of sections with the complex valued functions themselves. However this space is far too big: it contains functions of both position and momentum. We would like to end up with functions of just position. It is then clear what we must do. We must pick the subspace of the functions which are independent of the momentum, *i.e.* they are constant on the fibers of the bundle $T^*N \xrightarrow{\pi} N$. In other words, they are the functions which are annihilated by the vector fields tangent to the fibers. These vector fields span an integrable distribution of TT^*N, being exactly the kernel of the derivative of the projection $\pi_* : TT^*N \to TN$. Moreover this distribution is lagrangian, since locally it is spanned by $\{\frac{\partial}{\partial p_a}\}$ in a basis where the symplectic form is $\sum_a dq^a \wedge dp_a$. This distribution, $\ker \pi_*$, is the canonical real polarization (see later) of T^*N. A general polarization will consist of a suitable generalization of this object. Of course, in general, a symplectic manifold need not have a canonical polarization. In this sense, cotangent bundles are special.

We begin then by defining polarizations. By a **polarization** of a symplectic manifold (M, Ω) we will mean an involutive lagrangian complex subbundle of the complexified tangent bundle of M. In other words, let $F = \{m \mapsto F_m \subset T_m^{\mathbb{C}} M\}$ be a smooth involutive distribution such that F_m is a complex lagrangian subspace of $T_m^{\mathbb{C}} M \cong \mathbb{C} \otimes_{\mathbb{R}} T_m M$, made into a complex symplectic vector space by extending the symplectic form $\mathbb{C}$-linearly to a new symplectic form $\Omega_{\mathbb{C}}$. Then F is a polarization of the symplectic manifold (M, Ω) and (M, Ω, F) is called a **polarized symplectic manifold**.

Notice that if F is a polarization, so is $\overline{F}$. A polarization is **real** if $F = \overline{F}$. This is the case if and only if[24] $F = \mathbb{C}\otimes V$ for some involutive lagrangian subbundle V of TM. The canonical polarization of a cotangent bundle gives rise, upon complexification, to a real polarization. On the other extreme, a polarization is **totally complex** if $F\cap\overline{F} = 0$. In this case, $T^{\mathbb{C}}M \cong F\oplus\overline{F}$. Therefore $F\cap TM = F\cap\sqrt{-1}TM = 0$ and hence F is the graph of a bundle isomorphism $TM \to \sqrt{-1}TM$. That is, $F_m = \{X + \sqrt{-1}J_mX \mid X \in T_mM\}$ for some isomorphism $J_m : T_mM \to T_mM$. Since F_m is a complex linear subspace we deduce that $J^2 = -\mathbf{1}$ and so it is a complex structure. Moreover since F is lagrangian, for all $X, Y \in TM$, we have

$$\begin{aligned} 0 &= \Omega_{\mathbb{C}}(X + \sqrt{-1}JX, Y + \sqrt{-1}JY) \\ &= \Omega(X,Y) - \Omega(JX,JY) + \sqrt{-1}\left(\Omega(X,JY) + \Omega(JX,Y)\right) \ . \end{aligned} \tag{7.1}$$

From the real part of this equation we deduce that J is a symplectomorphism and from the imaginary part that $g(X,Y) \equiv \Omega(X,JY)$ is symmetric. Moreover it follows that J is orthogonal with respect to g. In fact, for all $X, Y \in TM$

$$\begin{aligned} g(JX,JY) &= \Omega(JX,J^2Y) \\ &= \Omega(Y,JX) \\ &= g(Y,X) \\ &= g(X,Y) \ . \end{aligned}$$

Also, for all $X, Y \in TM$,

$$[X+\sqrt{-1}JX\,,\,Y+\sqrt{-1}JY] = [X\,,\,Y] - [JX\,,\,JY] + \sqrt{-1}\left([X\,,\,JY] + [JX\,,\,Y]\right) \ , \tag{7.2}$$

which, since F is involutive, implies that

$$J[X\,,\,Y] - J[JX\,,\,JY] = [X\,,\,JY] + [JX\,,\,Y] \ . \tag{7.3}$$

In other words, the Nijenhuis tensor vanishes and, by the Newlander-Nirenberg theorem, (M,J) is a complex manifold whose holomorphic vector fields correspond to the sections $\Gamma(F)$ of F. Therefore, (M, Ω, F) becomes a pseudo-Kähler manifold, becoming Kähler only when g is positive definite. In this latter case we say F is a **positive definite** polarization.

Let (M, Ω, F) be a polarized symplectic manifold. Let us define A_F to be the following class of functions

$$A_F = \{f \in C^\infty_{\mathbb{C}}(M) \mid \overline{X} f = 0 \ , \ \forall\, X \in \Gamma(F)\} \ . \tag{7.4}$$

Alternatively we can characterize these functions as follows.

Proposition 7.5. *A_F consists precisely of those functions in $C^\infty_{\mathbb{C}}(M)$ whose associated hamiltonian vector fields are in $\Gamma(\overline{F})$.*

Proof: By definition, a function $f \in C^\infty_{\mathbb{C}}(M)$ belongs to A_F if and only if $\overline{X} f = 0$ for all $X \in \Gamma(F)$. But $\overline{X} f = df(\overline{X}) = \Omega_{\mathbb{C}}(\overline{X}, X_f)$. Hence $f \in A_F$ if and only if $X_f \in \Gamma(\overline{F}^\perp)$. But $\overline{F}$ is lagrangian, so that $\overline{F}^\perp = \overline{F}$. Thus $f \in A_F$ if and only if $X_f \in \Gamma(\overline{F})$. ■

Corollary 7.6. *A_F is an abelian Poisson subalgebra of $C^\infty_{\mathbb{C}}(M)$.*

Proof: If $f, g \in A_F$ then for all $X \in \Gamma(F)$, $\overline{X} f = \overline{X} g = 0$. Therefore $\overline{X}(fg) = (\overline{X} f)g + f(\overline{X} g) = 0$, so $fg \in A_F$. Moreover, $\{f \ , \ g\} = \Omega_{\mathbb{C}}(X_f, X_g)$ which is zero by the previous Proposition and the fact that $\overline{F}$ is lagrangian. ■

In some cases, *e.g.* F a totally complex polarization, A_F is a maximal abelian subalgebra of $C^\infty_{\mathbb{C}}(M)$. This, in fact, is sometimes taken to be the algebraic definition of a polarization of a Poisson algebra[**25**]. Let us now define another class of functions

$$N_F = \{f \in C^\infty_{\mathbb{C}}(M) \mid [X_f \ , \overline{Y}] \in \Gamma(\overline{F}) \ , \ \forall\, Y \in \Gamma(F)\} \ . \tag{7.7}$$

That is, N_F consist of those functions whose hamiltonian vector fields lie in the normalizer of $\Gamma(\overline{F})$. Hence by general properties of normalizers N_F is a Lie subalgebra of $C^\infty_{\mathbb{C}}(M)$ and that $A_F \subset N_F$ is an abelian ideal. However, in general, N_F is not a Poisson subalgebra of $C^\infty_{\mathbb{C}}(M)$.

Now let $\Gamma(E)$ denote the smooth sections of the prequantum line bundle on M. We now define the polarized sections

$$\Gamma_F(E) = \{\sigma \in \Gamma(E) \mid \nabla_{\overline{X}}\sigma = 0 \ , \ \forall X \in \Gamma(F)\} \ . \tag{7.8}$$

If F is a totally complex polarization then E is a holomorphic line bundle and we can choose ∇ to be a holomorphic connection. In this case, $\Gamma_F(E)$ correspond to the holomorphic sections. We leave it as an exercise to verify that $\Gamma_F(E)$ is a Poisson module over A_F and a Lie submodule over N_F. In fact, N_F is the maximal Lie subalgebra of $C^\infty_{\mathbb{C}}(M)$ stabilizing $\Gamma_F(E)$.

The quantum Hilbert space in the geometric quantization program is precisely the Hilbert space completion of the polarized sections of finite norm. For general polarizations there may not be any polarized sections of finite norm. Indeed, suppose that P is not totally complex. Then the norm of a polarized section is invariant on the leaves of the foliation defined by the integrable distribution $D = P \cap \overline{P}$. If these leaves are non-compact then there may not be any sections of finite norm. This obstacle can be overcome via "half-form" quantization, which naturally yields objects which can be integrated in the space of leaves M/D. Half-form quantization consists in tensoring the pre-quantum line bundle with the bundle of half-forms in such a way that the point-wise inner product does not just yield a function to be integrated with respect to the Liouville form, but actually yields a form which can be integrated in the appropriate space. We will not discuss half-form quantization in this paper, except to note that our algebraic constructions extend naturally to this case since the module of smooth sections of the new pre-quantum bundle is also a Poisson module. Supposing that the polarization is totally complex but not positive definite we may still find that there are no polarized sections of finite norm. In fact, let $\{z^a\}$ be a local holomorphic chart which trivializes the pre-quantum line bundle and relative to which the symplectic form is

$$\Omega = \frac{i}{2} g_{ab} dz^a \wedge d\overline{z}^b \ . \tag{7.9}$$

The inner product, after identifying holomorphic sections with holomorphic functions, is just the usual L^2 inner product of holomorphic functions on an open set of $\mathbb{C}^n$ with measure $e^{-\frac{1}{2}g_{ab}z^a\bar{z}^b}dzd\bar{z}$. If g_{ab} is not positive definite, *i.e.* if the polarization is not positive definite, the exponential grows in the negative directions and, if the image of the chart is unbounded in those directions, there will be no holomorphic integrable functions. Based on this considerations we will often restrict ourselves to positive definite (totally complex) polarizations.

Having defined the Hilbert space as the space of polarized integrable sections, we notice that the only readily quantizable observables are the (real) functions in N_F. In order to quantize other observables one must resort to ways of pairing Hilbert spaces obtained from different polarizations. We will not discuss this topic here but rather refer the interested reader to any of the standard references on geometric quantization[**6**].

Now suppose that (M,Ω) admits a hamiltonian action of a connected Lie group G. A polarization F is G-**invariant** if the induced action of G on $T^{\mathbb{C}}M$ preserves F. Since G is connected, F is G-invariant if and only if for all Killing vectors X,

$$[X\,,\,Y] \in \Gamma(F) \qquad \forall\, Y \in \Gamma(F)\ . \tag{7.10}$$

In particular, this implies that

$$\{\phi_X\,,\,f\} \in A_F \qquad \forall\, f \in A_F\ , \tag{7.11}$$

from where it follows that A_F is a $\mathfrak{g}$-submodule of $C^\infty_{\mathbb{C}}(M)$.

In [**23**] Guillemin and Sternberg showed that a positive definite G-invariant polarization on (M,Ω) induces canonically a positive definite polarization $\widetilde{F}$ on the reduced manifold $(\widetilde{M},\widetilde{\Omega})$. Let F_o be the subbundle of $T^{\mathbb{C}}M_o$ given by $F_o = F \cap T^{\mathbb{C}}M_o$. The by G-invariance, F_o induces an involutive lagrangian subbundle of $T^{\mathbb{C}}\widetilde{M}$. Since F is G-invariant, the $\{\phi_X\}$ belong to N_F and hence the polarized sections become a $\mathfrak{g}$-module. The polarized sections $\Gamma_{\widetilde{F}}(\widetilde{M})$ are precisely the $\mathfrak{g}$-invariant polarized sections on M_o, that is, $\Gamma_{F_o}(M_o)^{\mathfrak{g}}$.

In [**7**] Kostant and Sternberg discussed BRST quantization for the case of a group action in the spirit that quantizing the "ghost" part ($\bigwedge(\mathfrak{g}^* \oplus \mathfrak{g})$) and the "matter" part ($C^\infty(M)$) could be done independently. The quantization of $\bigwedge(\mathfrak{g}^* \oplus \mathfrak{g})$ relied on the observation that the Clifford algebra $\mathrm{Cl}(\mathfrak{g} \oplus \mathfrak{g}^*)$ is naturally filtered and the associated graded object to this filtration is precisely $\bigwedge(\mathfrak{g}^* \oplus \mathfrak{g})$. That is, we have a canonical isomorphism of vector spaces $c : \mathrm{Cl}(\mathfrak{g}^* \oplus \mathfrak{g}) \to \bigwedge(\mathfrak{g}^* \oplus \mathfrak{g})$, which is to be thought of as "classical limit". For finite dimensional $\mathfrak{g}$, $\mathrm{Cl}(\mathfrak{g}^* \oplus \mathfrak{g})$ has, up to isomorphism, a unique irreducible module S and thus it can be identified with $\mathrm{End}\, S$. A quantization of $\bigwedge(\mathfrak{g}^* \oplus \mathfrak{g})$ was defined in [**7**] to be a map $\bigwedge(\mathfrak{g}^* \oplus \mathfrak{g}) \to \mathrm{End}\, S$ induced from a map $q : \bigwedge(\mathfrak{g}^* \oplus \mathfrak{g}) \to \mathrm{Cl}(\mathfrak{g}^* \oplus \mathfrak{g})$ which is an inverse of c. No natural inverse exists and the choice of q corresponds to an ordering prescription.

A model for S can be constructed by choosing a maximal isotropic subspace of $\mathfrak{g} \oplus \mathfrak{g}^*$, that is, a subspace $N \subset \mathfrak{g} \oplus \mathfrak{g}^*$ such that $N^\perp = N$, for $\perp$ the orthogonal complement with respect to the inner product on $\mathfrak{g} \oplus \mathfrak{g}^*$ induced by the dual pairing. For example one could take N to be $\mathfrak{g}$ or $\mathfrak{g}^*$. Having chosen a maximal isotropic subspace N , let P be a complementary maximal isotropic subspace, *i.e.* $\mathfrak{g} \oplus \mathfrak{g}^* = N \oplus P$. Then a model for S is given by

$$S \cong \bigwedge(\mathfrak{g}^* \oplus \mathfrak{g})/I(N) \cong \bigwedge P \; , \tag{7.12}$$

where $I(N)$ is the ideal generated by N.

The crucial observation is that $\bigwedge P$ is a maximal abelian subalgebra (under Poisson bracket) of $\bigwedge(\mathfrak{g}^* \oplus \mathfrak{g})$ and, in analogy with the case of a symplectic manifold, can be considered to be the polarized functions and, in this case, the polarized sections. Hence a choice of maximal isotropic subspace N is equivalent to a choice of polarization.

We will see that in many cases, *e.g.* the case of a totally complex polarization, a particular polarization is imposed on $\bigwedge(\mathfrak{g}^* \oplus \mathfrak{g})$. In this context, therefore, the quantization of the "matter" and the "ghosts" part go hand in hand.

In analogy with the group case, we define a polarization F in the symplectic manifold (M, Ω) to be **invariant** with respect to a set of first class constraints $\{\phi_i\}$ if and only if $\phi_i \in N_F$ for all i. In other words, if $[X_i\,,\,Y] \in \Gamma(F)$ for all $Y \in \Gamma(F)$. Let F be an invariant polarization. We then consider the subspace of $\mathbb{K}(E)$ consisting of polarized sections:

$$\bigwedge(\mathbb{V}^* \oplus \mathbb{V}) \otimes \Gamma_F(E)\ . \tag{7.13}$$

We would like to consider the prequantum BRST operator D_∇ acting on this subspace and define the physical Hilbert space as its cohomology (at least for ghost number zero). However the BRST operator may not leave this subspace invariant. For example, let $e_k \otimes \sigma$ be an element in $\mathbb{V} \otimes \Gamma_F(E)$. Then

$$D_\nabla(e_k \otimes \sigma) = \sum_i Q_i \times (e_k \otimes \sigma)\ . \tag{7.14}$$

If this is to belong to $\bigwedge(\mathbb{V}^* \oplus \mathbb{V}) \otimes \Gamma_F(E)$ each piece with a different bidegree must belong to $\bigwedge(\mathbb{V}^* \oplus \mathbb{V}) \otimes \Gamma_F(E)$. In particular the $(0,0)$ piece must be a polarized section. The $(0,0)$ piece is just $\phi_k\,\sigma$, which is a polarized section if and only if $\phi_k \in A_F$.

If the polarization is totally complex A_F corresponds to holomorphic functions. Since ϕ_k is a real non-constant function it cannot be holomorphic. Hence $\phi_k\,\sigma$ is not a polarized section and the BRST operator does not stabilize $\bigwedge(\mathbb{V}^* \oplus \mathbb{V}) \otimes \Gamma_F(E)$. If we divide out by the ideal generated by $\mathbb{V}$ then this term is not present. So we are forced to polarize the ghosts in such a way that only $\bigwedge\mathbb{V}^*$ appears. We thus define the complex $\mathbb{K}_F(E)$ by

$$\mathbb{K}_F(E) = \bigwedge\mathbb{V}^* \otimes \Gamma_F(E)\ . \tag{7.15}$$

It is then clear that the only part of D_∇ which survives is ∇_1 since it has b-degree zero. It clearly obeys $\nabla_1^2 = 0$, since it was only $\mathbb{V}$ which gave us a hard time at that.

The quantum BRST cohomology — for the case of a totally complex polarization — is therefore the cohomology of the complex

$$\cdots \longrightarrow \bigwedge^c \mathbb{V}^* \otimes \Gamma_F(E) \xrightarrow{\nabla_1} \bigwedge^{c+1} \mathbb{V}^* \otimes \Gamma_F(E) \longrightarrow \cdots \ . \tag{7.16}$$

In the case of a group action ∇_1 agrees with the Lie algebra coboundary operator and hence the quantum BRST cohomology is precisely the Lie algebra cohomology with coefficients in the module of polarized sections: $H(\mathfrak{g}; \Gamma_F(E))$.

There in one striking fact about this result. No part of the quantum BRST operator carries the information concerning the restriction to M_o; since, in fact, the only part with this information was the Koszul differential ∇_0. A heuristic explanation can be given as follows. In the case of a totally complex polarization, $\Gamma_F(E)$ are holomorphic sections. By the uniqueness of analytic continuation, a holomorphic section on M in completely determined once you know it on a real submanifold M_o; in other words, there is a unique extension to a holomorphic section on M.

Another instance when a polarization is imposed on the ghost part is in the case a semisimple group action and F is a polarization for which A_F is a maximal abelian subalgebra of $C^\infty_{\mathbb{C}}(M)$. Let $\mathfrak{h}$ denote the subalgebra of $\mathfrak{g}$ defined by

$$X \in \mathfrak{h} \Leftrightarrow \phi_X \in A_F \ . \tag{7.17}$$

Notice that $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$ since the map $X \mapsto \phi_X$ is a Lie algebra morphism and A_F is abelian. But moreover $\mathfrak{h}$ is an ideal. To see this notice that if $X \in \mathfrak{h}$, $Y \in \mathfrak{g}$ then for all $f \in A_F$

$$\begin{aligned} \{\phi_{[X\,,Y]}\,, f\} &= \{\{\phi_X\,, \phi_Y\}\,, f\} \\ &= \{\phi_X\,, \{\phi_Y\,, f\}\} + \{\{\phi_X\,, f\}\,, \phi_Y\} \ . \end{aligned} \tag{7.18}$$

Since F is invariant $\{\phi_Y\,, f\} \in A_F$ and both terms are Poisson brackets of elements in A_F and because A_F is abelian they vanish. Since $\phi_{[X\,,Y]}$ commutes with A_F, it must, by maximality, be in A_F itself. Therefore $[X\, , \, Y] \in \mathfrak{h}$ and hence it is an ideal. If $\mathfrak{g}$ is semisimple it cannot have any abelian ideals and thus $\mathfrak{h}$ is zero. Therefore no ϕ_X belongs to A_F and, once again, D_∇ does not stabilize $\bigwedge(\mathfrak{g}^* \oplus \mathfrak{g}) \otimes \Gamma_F(E)$ unless we polarize the ghost part in such a way that only $\bigwedge\mathfrak{g}^* \otimes \Gamma_F(E)$ is left.

In the case of a general polarization it is not clear that a specific polarization of the ghost part is forced, but in any case we may always choose to polarize the ghosts in such a way that the resulting quantum BRST complex is (7.16) . With this choice of polarization the explicit form of the ∇_i for $i \neq 1$ is irrelevant. This makes the quantization much simpler than what one might have originally expected. Therefore we choose to work with this polarization for the remainder of this section.

We end this section with a brief comment about the inner product. We remarked in the previous section how it seems impossible to calculate the inner product on $\widetilde{M}$ by evaluating the inner product on M on suitable representatives, due to the geometric nature of the inner product. It may seem, at least with the chosen polarization, that this problem could be obviated since holomorphic sections are uniquely characterized by their behavior on M_o. Still, if the gauge orbits are not compact the integrable polarized sections on M and $\widetilde{M}$ may not correspond. Of course, one choose a quantization prescription in which one always compute everything on M, but it is then not clear how to show that this quantum theory agrees with the theory in $\widetilde{M}$.

There are other problems surrounding the inner product. Ideally, in the purest BRST philosophy, we would like to be able to compute everything pertaining to objects in $\widetilde{M}$ by choosing representative objects in M (cocycles of the relevant BRST cohomology) and computing with them in M. This requires certain compatibility conditions, namely that the cohomology inherits the relevant structure from the cochains. In the case of the polarized sections, we would like the inner product to be induced and we see that this does not occur in general, because of the problem with the integration. But even without the integral, we have the problem of defining a reasonable pointwise inner product. In the last section we gave one which descends to a pointwise inner product in cohomology, but it has a major drawback. The norm of the polarized sections (elements in the zeroth BRST cohomology) is always zero, since the inner product pairs cocycles of complementary ghost number. We have not been able to find a pointwise inner product which descends to cohomology and which is non-zero when restricted to just cocycles of zero ghost number.

§8 DUALITY IN QUANTUM BRST COHOMOLOGY

Duality theorems in BRST cohomology have recently attracted some attention. In [**8**] we proved a duality theorem for the quantum BRST cohomology in the context of a Fock space representation with a mild finiteness hypothesis. The results did not depend on the Fock space context but we did assume the existence of a non-degenerate inner product on the space of states with respect to which the BRST operator was (anti-)hermitian. This assumption, we believed, was well founded on physical grounds. From this duality we then proved also a duality on operators. In [**9**] Marc Henneaux obtained similar duality theorems but with weaker assumptions. In particular he dispensed with the notion of an inner product to prove the operator duality. He only needs an non-degenerate inner product in order to turn this around and obtain a duality for the states. The finiteness assumption, however, is still key in his proofs.

For the classical BRST cohomology and in the case of compact orientable gauge orbits, a duality theorem was proven in [**16**] which is a manifestation of the Poincaré duality on the fibers. This result had been conjectured already in [**9**] at least for the case of a trivial bundle $M_o \longrightarrow \widetilde{M}$.

There is a peculiar difference between the classical and quantum versions of the duality in BRST cohomology. In the quantum case there are states with both negative and positive ghost numbers and the duality is a symmetry between the cohomologies at $+n$ and $-n$ ghost numbers:

$$H^n_{\text{quantum}} \cong H^{-n}_{\text{quantum}} \; ; \tag{8.1}$$

whereas in the classical case — due to the vanishing of BRST cohomology for negative ghost number — the duality is a symmetry between complementary dimensions

$$H^n_{\text{classical}} \cong H^{k-n}_{\text{classical}} \; . \tag{8.2}$$

We also saw in the last section how the quantum BRST cohomology (at least for our choice of polarization) had no states of negative ghost number. Hence it seems at first sight that we will not be able to obtain a duality similar to that in (8.1) . The resolution of this apparent paradox is that the classical and quantum ghost numbers don't quite correspond. We will see that for the ghost number operator to be antihermitian — a fact assumed in [**8**] — we must shift the ghost number by $-\frac{k}{2}$, *i.e.* we must normal order. In this case and assuming that the polarization is positive definite (notice that this already implies that it is totally complex) and modulo some conditions on the constraints we will recover an isomorphism à la (8.1) . However shifting the ghost number operator by a constant affects the ghost number of the states but not the one of the operators. In particular there will be operators which are BRST invariant and which have negative ghost number. These do not come from quantizing BRST invariant observables, since classical BRST cohomology is zero for negative ghost number. In fact, as remarked in [**9**] , they seem to come from distributions rather than from smooth functions.

From now on we assume that the polarization is positive definite. As remarked in the last section this forces a polarization on the ghost part in such a way that only the ∇_1 part of the BRST operator remains. The idea behind the proof of the duality is very simple and it is similar in spirit to that in [**8**] . We will define a positive definite inner product on the complex $\mathbb{K}_F(E)$. This will allow us to define a formal adjoint of the BRST operator and a BRST laplacian. Then we will prove a Hodge decomposition for the BRST complex and from this the duality will follow.

We now define the inner product. The inner product splits as a product of two inner products: the one on the "matter" part and the one on the "ghost" part. We will choose the one on the matter part to be the prequantum inner product: the integral of the pointwise inner product with respect to the Liouville measure. For the ghost part we will take the following. Fix a non-zero "volume" form $\varepsilon \in \bigwedge^k \mathbb{V}^*$. Since $\bigwedge^k \mathbb{V}^*$ is one dimensional any other k-form is proportional to ε. We define the Berezin integral on $\bigwedge \mathbb{V}^*$ by

$$\int_{\text{Ber}} \omega = 0 \;, \qquad \forall\, \omega \in \bigwedge^{p\neq k} \mathbb{V}^* \tag{8.3}$$

$$\int_{\text{Ber}} \lambda\varepsilon = \lambda \;, \qquad \forall\, \lambda \in \mathbb{R} \;. \tag{8.4}$$

We then define the inner product of forms by

$$\langle \omega, \theta \rangle = (-i)^{p(k-p)} \int_{\text{Ber}} \omega \wedge \theta \;, \tag{8.5}$$

for $\omega \in \bigwedge^p \mathbb{V}^*$ and $i = \sqrt{-1}$. The reason for the factors of i is the hermiticity of the inner product on $\mathbb{K}_F(E)$:

$$(\omega \otimes \sigma, \theta \otimes \tau) = (-i)^{p(k-p)} \int_M \langle \sigma, \tau \rangle \, d\mu_L \cdot \int_{\text{Ber}} \omega \wedge \theta \;, \tag{8.6}$$

for $\omega \in \bigwedge^p \mathbb{V}^*$. Notice that this is zero unless $\theta \in \bigwedge^{k-p} \mathbb{V}^*$. Let us define an element of $\mathbb{K}_F(E)$ to be **integrable** if it belongs to $\bigwedge \mathbb{V}^* \otimes \Gamma_F(E)_{\text{fin}}$, where $\Gamma_F(E)_{\text{fin}}$ denotes the subspace of the polarized sections of finite norm.

In order for the ghost number operator to be skew-adjoint we will have to redefine it by shifting it by an appropriate constant. The ghost number operator is given by $N = \sum_{i=1}^{k} \varepsilon(\omega^i)\imath(e_i)$, where ε and $\imath$ are the exterior and interior product operations on $\bigwedge \mathbb{V}^*$. In particular notice that $\imath(e_i)$ is a derivation which can be "integrated by parts" in the Berezin integral. We shift N by a constant c and notice that, if $N_c \equiv N - c$, then for all $\omega \otimes \sigma, \theta \otimes \tau \in \mathbb{K}_F(E)$ integrable, it is easy to show that

$$(N_c\, \omega \otimes \sigma, \theta \otimes \tau) = (k - 2c)(\omega \otimes \sigma, \theta \otimes \tau) - (\omega \otimes \sigma, N_c\, \theta \otimes \tau) \ ; \tag{8.7}$$

from where it follows that for $c = \frac{k}{2}$ we obtain a skew-adjoint ghost number operator.

The BRST operator for this choice of polarization is given by the ∇_1 piece:

$$\sum_i \varepsilon(\omega^i)\, O(\phi_i) - \frac{1}{2} \sum_{i,j,k} f_{ij}{}^k \varepsilon(\omega^i)\varepsilon(\omega^j)\imath(e_k) \ . \tag{8.8}$$

Let $\omega \otimes \sigma \in \mathbb{K}_F^p(E)$, $\theta \otimes \tau \in \mathbb{K}_F(E)$ be integrable. Then after a straight-forward calculation we find

$$(\nabla_1\, \omega \otimes \sigma, \theta \otimes \tau) = \pm \left[(\omega \otimes \sigma, \nabla_1\, \theta \otimes \tau) + \sum_{i,j} (\omega \otimes \sigma, f_{ij}{}^j \varepsilon(\omega^i) \theta \otimes \tau) \right] \ , \tag{8.9}$$

where the explicit dependence of $\pm$ on p, k is of no consequence. We therefore see that if $\sum_{i,j} f_{ij}{}^j = 0$ for all i, then $\nabla_1^\dagger = \pm \nabla_1$, and therefore an inner product is induced in cohomology. To see this notice that the only condition needed for the inner product to descend to cohomology is that the cobundaries be orthogonal to the cocycles, that is, $\ker \nabla_1 \subseteq \ker \nabla_1^\dagger$. For the case of a group action, the condition $\sum_{i,j} f_{ij}{}^j = 0$ is precisely that $\operatorname{tr} \operatorname{ad}(e_i) = 0$, that is, that $\mathfrak{g}$ be unimodular. This condition is equivalent to the existence of a bi-invariant metric on the group manifold. The condition that the inner product descends to cohomology is, of course, very desirable on physical grounds since gauge related states should be physically equivalent. In this case, however, this inner product is not the physical inner product, and we make this choice, not on physical grounds, but to be able to prove duality.

Therefore from now on we assume that for all i, $\sum_{i,j} f_{ij}{}^{j} = 0$. In this case, $\nabla_1^\dagger = \pm\nabla_1$ acting on integrable elements of $K_F(E)$. We now define a Hodge-type operator. For definiteness and with no loss in generality, let us fix the following volume form on $\bigwedge\mathbb{V}^*$: $\varepsilon = \omega^1\wedge\cdots\wedge\omega^k$. We define the operator $\star : \bigwedge^p\mathbb{V}^* \to \bigwedge^{k-p}\mathbb{V}^*$ as follows:

$$\star(1) = \varepsilon \; ; \tag{8.10}$$

$$\star(\varepsilon) = 1 \; ; \tag{8.11}$$

$$\star(\omega^{i_1}\wedge\cdots\wedge\omega^{i_p}) = \pm\omega^{i_{p+1}}\wedge\cdots\wedge\omega^{i_k} \; ; \tag{8.12}$$

where $(i_1,\ldots,i_k)$ is a permutation of $(1,\ldots,k)$ and $\pm$ refers to the sign of the permutation. It is trivially verified that $\star^2 = (-1)^{p(k-p)}$ on $\bigwedge^p\mathbb{V}^*$. Let us then define $\bar\star = (i)^{p(k-p)}\star$ on $\bigwedge^p\mathbb{V}^*$. It clearly satisfies $\bar\star^2 = 1$. This allows to redefine the inner product $(,)$ in such a way that it is now positive definite. In fact, let's define a new inner product

$$(\Psi,\Xi)_\star \equiv (\Psi,\bar\star\Xi) \; , \tag{8.13}$$

for $\Psi,\Xi\in\mathbb{K}_F(E)$. If $\omega\otimes\sigma,\theta\otimes\tau\in\mathbb{K}_F(E)$ the new inner product becomes

$$(\omega\otimes\sigma,\theta\otimes\tau)_\star = \int_M \langle\sigma,\tau\rangle\, d\mu_L \cdot \int_{\mathrm{Ber}} \omega\wedge\star\theta \; . \tag{8.14}$$

It follows that this inner product is positive definite. To see this notice that the inner product on the sections is positive definite by construction. And to show that the other part is positive definite, we need only exhibit an orthonormal basis. Let $I = (i_1,\ldots,i_p)$ denote a sequence $1\le i_1 < \cdots < i_n \le k$. Then if we define $\omega^I = \omega^{i_1}\wedge\cdots\wedge\omega^{i_p}$, the collection of all such $\{\omega^I\}$ forms a basis for $\bigwedge\mathbb{V}^*$. On this basis, it is easy to see that

$$\int_{\mathrm{Ber}} \omega^I\wedge\star\omega^J = \delta^{IJ} = \begin{cases} 1 & \text{for } I=J \\ 0 & \text{otherwise} \end{cases} \; . \tag{8.15}$$

Moreover we see that the integrable elements of $\mathbb{K}_F(E)$ are precisely those elements of $\mathbb{K}_F(E)$ which have finite norm with respect to this new inner product. We let $\Gamma_{F,L^2}(E)$ denote the Hilbert space completion of this space with respect to this new inner product.

The BRST operator ∇_1 is generally unbounded so we have to be careful and specify its domain. From the definition of $\Gamma_{F,L^2}(E)$, it is clear that the BRST operator is densely defined, since it is defined on the integrable elements and the $\Gamma_{F,L^2}(E)$ is their closure.[4] Now let ∇_1^* denote the adjoint of ∇_1 with respect to the new inner product. Notice that on $\bigwedge \mathbb{V}^* \otimes \Gamma_F(E)_{\text{fin}}$

$$\nabla_1^* = \pm \bar{\star} \nabla_1 \bar{\star} \; . \tag{8.16}$$

In particular, notice that ∇_1^* is also densely defined. Therefore ∇_1 is a closeable operator with minimal closed extension $\overline{\nabla_1} \equiv (\nabla_1^*)^*$. We can therefore define the BRST operator to be the closure of ∇_1. To avoid cumbersome notation we will denote this also by ∇_1.

Since ∇_1^* is closed, $\ker \nabla_1^*$ is a closed subspace of $\Gamma_{F,L^2}(E)$ and hence we have the following orthogonal decomposition

$$\Gamma_{F,L^2}(E) = \ker \nabla_1^* \oplus (\ker \nabla_1^*)^\perp \; . \tag{8.17}$$

Now, since ∇_1 is closed, $(\ker \nabla_1^*)^\perp$ is precisely the closure of $\operatorname{im} \nabla_1$. Thus we have the following orthogonal decomposition

$$\Gamma_{F,L^2}(E) = \ker \nabla_1^* \oplus \overline{\operatorname{im} \nabla_1} \; . \tag{8.18}$$

Therefore we can decompose $\ker \nabla_1$ as an orthogonal direct sum

$$\ker \nabla_1 = \ker \nabla_1 \cap \ker \nabla_1^* \oplus \overline{\operatorname{im} \nabla_1} \; . \tag{8.19}$$

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[4] We are assuming here that the constraints are such that the BRST operator stabilizes $\Gamma_F(E)_{\text{fin}}$. This is always the case, for instance, if M admits a finite holomorphic atlas.[**26**]

Therefore we have the following isomorphism

$$H_{\text{quantum BRST}} \equiv \ker \nabla_1 / \overline{\operatorname{im} \nabla_1} \cong \ker \nabla_1 \cap \ker \nabla_1^* \ , \tag{8.20}$$

where we have defined the BRST cohomology in this way for convenience.[5]

Moreover this isomorphism is an isomorphism of graded (by ghost number) vector spaces. Because of (8.16) , $\bar{\star}$ stabilizes $\ker \nabla_1 \cap \ker \nabla_1^*$. Since it connects opposite ghost numbers and $\bar{\star}^2 = 1$, it is an isomorphism between the BRST cohomology spaces at opposite ghost number. Thus we have proven the following duality theorem.

Theorem 8.21. *If the constraints satisfy the condition* $\sum_{i,j} f_{ij}{}^j = 0$ *for all* i, *there is an isomorphism*

$$H^g_{\text{quantum BRST}} \cong H^{-g}_{\text{quantumBRST}} \ . \tag{8.22}$$

Notice that the problems of closure as well as the condition that the constraints stabilize the integrable subspace is unnecessary if M is assumed compact, since in this case the space of holomorphic sections is finite.

§9 CONCLUSIONS AND OPEN PROBLEMS

In this paper we have shown how all of the objects relevant to the geometric quantization program have an associated BRST cohomology theory describing their reduction via a symplectic quotient of the original symplectic manifold on which they are described. Moreover we have seen that the BRST cohomology preserves all Poisson structures.

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[5] Since $\operatorname{im} \nabla_1$ is, in general, infinite dimensional it is not necessarily closed. Factoring out by a non-closed subspace results in the natural surjection not being continuous and in the factor object not being complete.

The coherence between the BRST construction and the Poisson structures suggests that we can phrase this construction in purely algebraic terms in the category of "constrained" Poisson algebras. That is, given a Poisson algebra P with a distinguished set of elements $\Phi = \{\phi_i\}$ such that the ideal J they generate is a Lie subalgebra of P and such that the sequence Φ is regular we can construct a BRST cohomology theory which reduces P to another Poisson algebra. This is precisely the point of view taken by Stasheff[4],[5]. We can extend this further by homologically reducing Poisson modules of P. In the same way we show that a BRST theory exists which reduces the Poisson module into a Poisson module of the homological reduction of P. Moreover properties such as finitely generated and projective are preserved under reduction. We can also define algebraically a polarization of P as a maximal abelian Poisson subalgebra of P. However we have not been able to prove that maximal abelian subalgebras correspond under homological reduction. We have not been able to see how the algebraic analogues of inner products get induced, although the algebraic analogues of pointwise inner products can be induced.

The power of the algebraic approach lies in the fact that all the objects in the geometric quantization program can be defined purely algebraically without the assumption that the Poisson algebra P or its homological reduction are the Poisson algebras of symplectic manifolds. This is the case, for example, in Yang-Mills, where the reduced phase space has singularities[27]. We think that this particular formulation of BRST quantization can play a fundamental rôle in the quantization of such physically relevant systems.

There is another method of quantization which is based purely on the Poisson structure of the classical observables: the theory of deformations of Poisson algebras[28]. It would be interesting to see under what conditions, if any, the deformations of a Poisson algebra and of its homological reduction correspond.

Finally there are several obvious extensions to these constructions: to the case of supermanifolds in the sense of Kostant, to arbitrary symplectic Banach manifolds, and, more importantly, to the case of nonregular constraints[29]. The extensions to supermanifolds and to the non-regular case should be completely straight-forward; but the extension to infinite dimensional manifolds may pose problems having to do with the convergence of the spectral sequences used in Sections 3 and 4.

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