DEFORMATIONS OF THE GALILEAN ALGEBRA

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ABSTRACT

We compute all the infinitesimal deformations of the Galilean algebra with and without central extension; as well as their integrability properties. Among the four parameter family of infinitesimal deformations of the unextended algebra, we find the Newton algebras, the Euclidean algebra $E(4)$, the Poincaré algebra, the de Sitter algebras, and $SO(5)$. For the centrally extended algebra we find, in particular, an infinitesimal deformation containing a Poincaré subalgebra (although the embedding is not the natural one); and centrally extended versions of the Newton algebras.

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§1 Introduction

The theory of Lie algebra deformations[1],[2],[3] provides us with a systematic procedure which is an inverse to the more common Lie algebra contractions[4],[5]. In practice, contractions are associated with physical parameters, which enter in the commutation relations of a representation of a Lie algebra, tending to some physically meaningful limit. For example, the low velocity limit ($c \to \infty$) of the Poincaré algebra yields[1] the Galilean algebra; and the Poincaré algebra is itself the flat space limit ($\kappa \to 0$) of the de Sitter algebras. It is more useful, however, to look at the inverse problem. Suppose that we are given a Lie algebra which, to the best of our empirical knowledge, is an exact symmetry of the physical system under consideration. It may be, however, that this symmetry is only approximate and hence—either to discover new symmetry principles or to suggest empirical tests which can probe the exactness of the symmetry— one would like to know all the possible algebras which are "close" in some sense to the given one. These are precisely those algebras to which the given algebra can be deformed. The advantage of the deformation approach lies in that deformations can be searched for systematically by computing Lie algebra cohomology groups. Equivalently, one could classify all isomorphism classes of Lie algebras of a given dimension and then compute all the possible contractions, but as the dimension grows this problem becomes computationally untractable whereas the computation of cohomology groups is still feasible due, in great part, to theorems like the one of Hochschild and Serre[7], which allows one to exploit the semisimple part of the algebra in question to simplify the calculations tremendously.

The possible deformations of the Poincaré algebra have been known for some time. It was proven by Levy-Nahas in **[3]** that the only algebras to which the

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[1] More precisely, as it is shown in **[6]** , the limit $c \to \infty$ of the Poincaré algebra yields either the Galilean algebra (rescaling the boosts and the space translations) or the Carroll algebra (rescaling the boosts and the time translations).

Poincaré algebra can be deformed are the de Sitter algebras $SO(4,1)$ and $SO(3,2)$. It had been proven previously by Sharp in [**8**] that this was the case among the semi-simple Lie algebras.

In this paper we analyze the question starting from the Galilean algebra. Since both the Galilean algebra as well as its centrally extended version are physically interesting we find all the Lie algebras to which these algebras can be deformed. Whereas our results for the centrally extended algebra are new, the ones for the unextended case can be read from the work of Bacry and Lévy-Leblond[**6**] and of Bacry and Nuyts[**9**] who classify all the ($3+1$ dimensional) kinematic Lie algebras under the constraint of space isotropy. Kinematic Lie algebras are those real Lie algebras generated by the ten elements $\{M_{ij}, K_i, P_i, P_0\}$ where i, j run from 1 to 3 and $M_{ij} = \frac{1}{2}\epsilon_{ijk} g^{kl} J_l$; and the constraint of space isotropy merely fixes the transformation laws of the generators under space rotations: K_i, J_i, P_i are vectors and P_0 is a scalar. Although we make no isotropy assumption on the deformations, we notice that as a consequence of the semi-simplicity of the rotation subalgebra (and hence its rigidity under deformations) there are no deformations of the Galilean algebra which are not isotropic.

In summary, we find that there are four infinitesimal deformations of the unextended Galilean algebra yielding, among many other algebras, the Newton algebras, the Euclidean algebra $E(4)$, the Poincaré algebra as well as diverse real forms of B_2: $SO(3,2)$, $SO(4,1)$, and $SO(5)$. For the centrally extended algebra there are three infinitesimal deformations one of which corresponds to centrally extended versions of the Newton algebras; and one of which contains a Poincaré subalgebra although the embedding is not the natural one. We also investigate the integrability properties of these infinitesimal deformations.

This paper is organized as follows. In section 2 we review the basic facts about deformations of Lie algebras. Lack of space prohibits a more detailed account, but the reader is urged to look at the beautiful treatment to be found in [**2**] . In section 3 we discuss the factorization theorem of Hochschild and Serre which simplifies many of the calculations. We also give two brief applications of this theorem: the determination of the possible central extensions of the Galilean algebra, and the theorem of Levy-Nahas on the deformations of the Poincaré algebra. Finally in section 4 we determine the infinitesimal deformations of the Galilean algebra with and without central extension and discuss their integrability domains.

§2 Deformations of Lie Algebras

Let $\mathfrak{g}$ be a finite dimensional real Lie algebra and $\mathfrak{g}[[t]]$ the space of formal power series in t with coefficients in $\mathfrak{g}$. By a deformation of $\mathfrak{g}$ we mean a skew-symmetric bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}[[t]]$ which satisfies the Jacobi identity formally[2] order by order in t. In other words we can think of a deformation as a new bracket $[\,,\,]_t$ defined by

$$[X\,,\,Y]_t = [X\,,\,Y] + \sum_{n=0}^{\infty} t^n\, C_n(X,Y)\ , \tag{2.1}$$

for all $X, Y \in \mathfrak{g}$ and where the C_n are cochains in $C^2(\mathfrak{g};\mathfrak{g})$. The Jacobi identity imposes certain conditions on these cochains. In particular, the linear term C_1 has to be a cocycle. Conversely, if C_1 is any cocycle in $C^2(\mathfrak{g};\mathfrak{g})$ we can begin to define a deformation by

$$[X\,,\,Y]_t = [X\,,\,Y] + t\, C_1(X,Y)\ . \tag{2.2}$$

The fact that C_1 is a cocycle guarantees that the Jacobi identity is satisfied up to terms of order t^2. Therefore we call C_1 an **infinitesimal deformation**. In general not all infinitesimal deformations are the linear term of a deformation. Those which are, are called **integrable**. To see what stands in the way of an

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[2] Notice that we do not impose any convergence properties on the series. We are dealing therefore with formal deformations. It will turn out, however, that the deformations found in this paper are all polynomial and, therefore, trivially convergent.

infinitesimal deformation giving rise to a deformation, we look at the terms or order t^2 in the Jacobi identity for the above infinitesimal deformation. One finds the following:

$$C_1(X, C_1(Y, Z)) + C_1(Y, C_1(Z, X)) + C_1(Z, C_1(X, Y)) \ , \tag{2.3}$$

which, since C_1 is a cocycle, can be seen to be a cocycle in $C^3(\mathfrak{g};\mathfrak{g})$. If and only if it is a also a coboundary, say $-dC_2$, can we continue the deformation as follows

$$[X\ ,\ Y]_t = [X\ ,\ Y] + t\, C_1(X, Y) + t^2\, C_2(X, Y) \ , \tag{2.4}$$

guaranteeing that the Jacobi identity is satisfied up to terms of order t^3 and higher. Looking at the terms of order t^3 we again find a cocycle in $C^3(\mathfrak{g};\mathfrak{g})$ and so on. Hence the obstruction to the integrability of an infinitesimal deformation is an infinite sequence of cocycles in $C^3(\mathfrak{g};\mathfrak{g})$ whose cohomology classes all have to vanish. These classes appear very naturally within the framework of Nijenhuis and Richardson[**2**] who define the structure of a graded Lie algebra on the cohomology $H(\mathfrak{g};\mathfrak{g})$. We refer the reader to their paper for the details.

On the other hand not all infinitesimal deformations are "essential". Suppose that C_1 is a coboundary. That is $C_1 = -dB_1$, for some B_1 in $C^1(\mathfrak{g};\mathfrak{g})$. Then we define the map T by $T(X) = X + t\, B_1(X)$. It is then trivial to verify that up to terms of order t^2

$$[T(X)\ ,\ T(Y)]_t = T([X\ ,\ Y]) \ . \tag{2.5}$$

Conversely such a map T exists only if C_1 is a coboundary. Hence infinitesimal deformations such that C_1 is a coboundary will be called **trivial**. This allows us to introduce an equivalence relation on the set of infinitesimal deformations. Two infinitesimal deformations are considered equivalent if their difference is trivial. Hence the equivalence classes of infinitesimal deformations are in bijective correspondence with the cohomology group $H^2(\mathfrak{g};\mathfrak{g})$.

Hence we see that there are two crucial cohomology groups in the theory of deformations of Lie algebras: $H^2(\mathfrak{g};\mathfrak{g})$, which contains the non-trivial infinitesimal deformations; and $H^3(\mathfrak{g};\mathfrak{g})$ which contains the obstructions to the integrability of the infinitesimal deformations. In general, unless one can show that it vanishes, it is not necessary nor useful to compute $H^3(\mathfrak{g};\mathfrak{g})$ since only certain classes have to be checked. We do however need to compute $H^2(\mathfrak{g};\mathfrak{g})$ and in the next section we will describe a method due to Hochschild and Serre[**7**] which makes the computations rather straight-forward.

Notice that a semi-simple algebra is **rigid** in the sense that it admits no non-trivial deformations.

§3 The Factorization Theorem of Hochschild and Serre

In [**7**] Hochschild and Serre proved a factorization theorem that in many cases simplifies the calculation of Lie algebra cohomology groups. Let $\mathfrak{g}$ be a finite dimensional real Lie algebra and $\mathfrak{h}$ an ideal such that the quotient Lie algebra $\mathfrak{s} = \mathfrak{g}/\mathfrak{h}$ is semisimple. Let $\mathfrak{m}$ denote a $\mathfrak{g}$-module. Then the ideal $\mathfrak{h}$ defines a filtration of the cochains $C(\mathfrak{g};\mathfrak{m})$ whose spectral sequence degenerates at the E_2 term yielding the following isomorphism

$$H^n(\mathfrak{g};\mathfrak{m}) \cong \bigoplus_{i=0}^{n} H^{n-i}(\mathfrak{s};\mathbb{R}) \otimes H^i(\mathfrak{h};\mathfrak{m})^{\mathfrak{s}} \ , \qquad (3.1)$$

where $^{\mathfrak{s}}$ denotes $\mathfrak{s}$-invariants. Since $\mathfrak{s}$ is semisimple, it acts reducibly on the cochains $C(\mathfrak{h};\mathfrak{m})$ and hence the invariant cohomology can be computed from the invariant cochains.

Moreover, using the Whitehead lemmas we know that $H^1(\mathfrak{s};\mathbb{R}) = H^2(\mathfrak{s};\mathbb{R}) = \mathbf{0}$. If, in addition, $\mathfrak{s}$ is simple then $H^3(\mathfrak{s};\mathbb{R}) \cong \mathbb{R}$. Hence for $\mathfrak{s}$ simple, the first few

$H(\mathfrak{g};\mathfrak{g})$'s are as follows

$$
\begin{aligned}
H^0(\mathfrak{g};\mathfrak{g}) &\cong Z(\mathfrak{g}) \\
H^1(\mathfrak{g};\mathfrak{g}) &\cong H^1(\mathfrak{h};\mathfrak{g})^{\mathfrak{s}} \\
H^2(\mathfrak{g};\mathfrak{g}) &\cong H^2(\mathfrak{h};\mathfrak{g})^{\mathfrak{s}} \\
H^3(\mathfrak{g};\mathfrak{g}) &\cong H^3(\mathfrak{h};\mathfrak{g})^{\mathfrak{s}} \oplus Z(\mathfrak{g}) \ ,
\end{aligned}
$$

where $Z(\mathfrak{g})$ denotes the center of $\mathfrak{g}$.

Central Extensions of the Galilean Algebra

As a trivial application of the factorization theorem we compute the central extensions of the Galilean algebra; that is, $H^2(\mathfrak{g};\mathbb{R})$. The Galilean algebra is a real Lie algebra generated by $\{M_{ij}, K_i, P_i, P_0\}$ where i,j run from 1 to 3 and $M_{ij} = -M_{ji}$. The Lie bracket is given by

$$
\begin{aligned}
[M_{ij}\ ,\ M_{kl}] &= g_{jk}\, M_{il} + g_{il}\, M_{jk} - g_{ik}\, M_{jl} - g_{jl}\, M_{ik} \\
[M_{ij}\ ,\ K_k] &= g_{jk}\, K_i - g_{ik}\, K_j \\
[M_{ij}\ ,\ P_k] &= g_{jk}\, P_i - g_{ik}\, P_j \\
[K_i\ ,\ P_0] &= P_i \ ,
\end{aligned}
$$

and all other brackets are zero. Let $\mathfrak{h}$ denote the ideal generated by $\{K_i, P_i, P_0\}$. Then $\mathfrak{s}$ is the subalgebra generated by $\{M_{ij}\}$. By the factorization theorem, $H^2(\mathfrak{g};\mathbb{R}) \cong H^2(\mathfrak{h};\mathbb{R})^{\mathfrak{s}}$. The space of $\mathfrak{s}$-invariant 2-cochains is one-dimensional spanned by $K^{i*} \wedge P^{j*} g_{ij}$, where K^{i*} is the canonical dual vector to K_i and the same for P^{i*}. It is clearly a cocycle and not a coboundary since the only $\mathfrak{s}$-invariant 1-cochain is P^{0*}, which is a cocycle. Therefore there is only one non-trivial central extension. Let's denote by c its generator. Then the extended Galilean algebra is supplemented with the extra term $[K_i\ ,\ P_j] = g_{ij}\, c$ as is well-known.

Deformations of the Poincaré Algebra

As a final trivial application of the factorization theorem we determine the deformations of the Poincaré algebra. This was first done by Levy-Nahas in **[3]**

. The Poincaré algebra is spanned by $\{M_{ab}, P_a\}$, where a, b run from 1 to 4 and $M_{ab} = -M_{ba}$. The Lie bracket is given by

$$\begin{aligned} [M_{ab}\,,\, M_{cd}] &= g_{bc}\, M_{ad} + g_{ad}\, M_{bc} - g_{ac}\, M_{bd} - g_{bd}\, M_{ac} \\ [M_{ab}\,,\, P_c] &= g_{bc}\, P_a - g_{ac}\, P_b\ , \end{aligned}$$

all other brackets being zero. From this it already follows that $Z(\mathfrak{g}) = \mathbf{0}$. We choose $\mathfrak{h}$ to be translation ideal. Then $\mathfrak{s}$ can be identified with the Lorentz subalgebra. There are no Lorentz invariant elements in $\mathfrak{h}$. The only Lorentz invariant cochain in $C^1(\mathfrak{h}; \mathfrak{g})$ is $P^{a*} \otimes P_a$ which is clearly a cocycle but not a coboundary by the previous remark. There are only two linearly independent Lorentz invariant cochains in $C^2(\mathfrak{h}; \mathfrak{g})$. They are $P^{a*} \wedge P^{b*} \otimes M_{ab}$ and $P^{a*} \wedge P^{b*} \otimes \epsilon_{abcd} g^{ce} g^{df} M_{ef}$. The first one is a cocycle but the second one is not. Therefore there is a unique non-trivial infinitesimal deformation of the Poincaré algebra. There is only one linearly independent Lorentz invariant cochain in $C^3(\mathfrak{h}; \mathfrak{g})$. It is $P^{a*} \wedge P^{b*} \wedge P^{c*} \otimes \epsilon_{abcd} g^{de}\, P_e$. It is not just a cocycle but also a coboundary. Hence there are no obstructions to integrability and the unique non-trivial infinitesimal deformation is integrable. It turns out that the deformed algebra needs no terms of order t^2 since the obstruction cocycle vanishes identically and hence we are left with the deformed algebra

$$\begin{aligned} [M_{ab}\,,\, M_{cd}] &= g_{bc}\, M_{ad} + g_{ad}\, M_{bc} - g_{ac}\, M_{bd} - g_{bd}\, M_{ac} \\ [M_{ab}\,,\, P_c] &= g_{bc}\, P_a - g_{ac}\, P_b \\ [P_a\,,\, P_b] &= t M_{ab}\ . \end{aligned}$$

Notice that by rescaling P_a we can always reduce t down to a sign, but without complexifying we cannot reabsorb the sign. These algebras correspond to the de Sitter and anti de Sitter Lie algebras depending on the sign of t. Both of these algebras are simple and hence rigid, admitting no further deformations.

§4 Deformations of the Galilean Algebra

In this section we look at the deformations of Galilean algebra with and without central extension.

Unextended Galilean Algebra

The factorization theorem tells us that in oder to compute $H^2(\mathfrak{g};\mathfrak{g})$ we merely need to compute $H^2(\mathfrak{h};\mathfrak{g})^{\mathfrak{s}}$. The method is straight forward. We first isolate the $\mathfrak{s}$-invariant cochains and then determine which of these are cocycles and coboundaries. This then gives us the dimension of the cohomology group as well as representative cocycles from each class. We let $Z(\mathfrak{h};\mathfrak{g})^{\mathfrak{s}}$ and $B(\mathfrak{h};\mathfrak{g})^{\mathfrak{s}}$ determine the $\mathfrak{s}$-invariant cocycles and coboundaries respectively, the following table summarizes the results.

Space	Dimension
$C^0(\mathfrak{h};\mathfrak{g})^{\mathfrak{s}}$	1
$Z^0(\mathfrak{h};\mathfrak{g})^{\mathfrak{s}}$	0
$H^0(\mathfrak{h};\mathfrak{g})^{\mathfrak{s}}$	0
$B^1(\mathfrak{h};\mathfrak{g})^{\mathfrak{s}}$	1
$C^1(\mathfrak{h};\mathfrak{g})^{\mathfrak{s}}$	7
$Z^1(\mathfrak{h};\mathfrak{g})^{\mathfrak{s}}$	3
$H^1(\mathfrak{h};\mathfrak{g})^{\mathfrak{s}}$	2
$B^2(\mathfrak{h};\mathfrak{g})^{\mathfrak{s}}$	4
$C^2(\mathfrak{h};\mathfrak{g})^{\mathfrak{s}}$	16
$Z^2(\mathfrak{h};\mathfrak{g})^{\mathfrak{s}}$	8
$H^2(\mathfrak{h};\mathfrak{g})^{\mathfrak{s}}$	4
$B^3(\mathfrak{h};\mathfrak{g})^{\mathfrak{s}}$	8

In particular there is a four parameter family of non-trivial infinitesimal deformations generated by the following cocycles

$$
\begin{aligned}
C_1^1 &= K^{i*} \wedge K^{j*} \otimes \epsilon_{ijk} g^{kl} P_l \\
C_1^2 &= \frac{1}{2} K^{i*} \wedge K^{j*} \otimes M_{ij} - P^{i*} \wedge K^{j*} \otimes g_{ij} P_0 \\
C_1^3 &= P^{0*} \wedge P^{i*} \otimes K_i \\
C_1^4 &= P^{0*} \wedge P^{i*} \otimes P_i + P^{0*} \wedge K^{i*} \otimes K_i \ .
\end{aligned}
$$

The most general non-trivial infinitesimal deformation is therefore a linear combination $\sum_{a=1}^4 t_a \, C_1^a$. To investigate the integrability of the infinitesimal deformations we must first compute the obstruction cocycles and then determine which ones are coboundaries. The first obstruction is the 3-cocycle $J_2 = \sum_{a,b=1}^4 t_a t_b J_2^{ab}$ where

$$
J_2^{ab}(X,Y,Z) = C_1^a(X, C_1^b(Y,Z)) + \text{cyclic permutations} \ . \tag{4.1}
$$

A straight-forward calculation yields

$$
\begin{aligned}
J_2 =& t_1 t_3 \, (\frac{1}{2} P^{0*} \wedge K^{i*} \wedge K^{j*} \otimes \epsilon_{ijk} g^{kl} K_l - P^{0*} \wedge K^{i*} \wedge P^{j*} \otimes \epsilon_{ijk} g^{kl} P_l) \\
&+ t_1 t_2 \, K^{i*} \wedge K^{j*} \wedge K^{k*} \otimes \epsilon_{ijk} P_0 - \frac{1}{2} t_1 t_4 P^{0*} \wedge K^{i*} \wedge K^{j*} \otimes \epsilon_{ijk} g^{kl} P_l \\
&+ t_2 t_3 (P^{i*} \wedge P^{j*} \wedge K^{k*} \otimes g_{jk} K_i - P^{0*} \wedge P^{i*} \wedge K^{j*} \otimes M_{ij}) \\
&+ t_2 t_4 (2 P^{0*} \wedge P^{i*} \wedge K^{j*} \otimes g_{ij} P_0 - P^{0*} \wedge K^{i*} \wedge K^{j*} \otimes M_{ij} \\
&+ P^{i*} \wedge P^{j*} \wedge K^{k*} \otimes g_{jk} P_i + P^{i*} \wedge K^{j*} \wedge K^{k*} \otimes g_{ij} K_k) \ .
\end{aligned} \tag{4.2}
$$

All terms except for those proportional to $t_1 t_2$ and $t_2 t_4$ are coboundaries. Hence this infinitesimal deformation is not integrable unless $t_1 = t_4 = 0$ or $t_2 = 0$.

In the first case, $t_1 = t_4 = 0$, we have that $J_2 = -dC_2$ where

$$
C_2 = \frac{1}{2} t_2 t_3 P^{i*} \wedge P^{j*} \otimes M_{ij} \ . \tag{4.3}
$$

Computing the obstruction to the integrability of this second order infinitesimal deformation we find that $J_3 = 0$ and hence this is already a deformation. Reabsorbing the deformation parameter t into the t_a's we have the following deformed algebra

$$
\begin{aligned}
[M_{ij}\,,\,M_{kl}] &= g_{jk}\,M_{il} + g_{il}\,M_{jk} - g_{ik}\,M_{jl} - g_{jl}\,M_{ik} \\
[M_{ij}\,,\,K_k] &= g_{jk}\,K_i - g_{ik}\,K_j \\
[M_{ij}\,,\,P_k] &= g_{jk}\,P_i - g_{ik}\,P_j \\
[K_i\,,\,P_0] &= P_i \\
[K_i\,,\,K_j] &= t_2 M_{ij} \\
[P_0\,,\,P_i] &= t_3 K_i \\
[P_i\,,\,P_j] &= t_2 t_3 M_{ij} \\
[P_i\,,\,K_j] &= -t_2 g_{ij} P_0 \ ,
\end{aligned}
$$

for any value of t_2 and t_3. For $t_3 = 0$ we obtain either the Euclidean algebra $E(4)$ or the Poincaré algebra depending on the sign of t_2, which via rescaling can be reduced to a sign itself. For t_2 and t_3 both non-zero we get, depending on their sign: $SO(5)$, $SO(4,1)$, or $SO(3,2)$. The correspondence is the usual one: $K_i \to M_{0i}$, $P_i \to M_{5i}$ and $P_0 \to M_{50}$. Then substituting these into the commutation relations we see that after some rescaling we can identify t_2 with $-g_{00}$ and $t_2 t_3$ with $-g_{55}$. Finally, when $t_2 = 0$ we obtain the Newton algebras which, depending on the sign of t_3, we call N_+ and N_-. For $t_3 > 0$ and after some rescaling we get the algebra N_+ which is defined by

$$
\begin{aligned}
[M_{ij}\,,\,M_{kl}] &= g_{jk}\,M_{il} + g_{il}\,M_{jk} - g_{ik}\,M_{jl} - g_{jl}\,M_{ik} \\
[M_{ij}\,,\,K_k] &= g_{jk}\,K_i - g_{ik}\,K_j \\
[M_{ij}\,,\,P_k] &= g_{jk}\,P_i - g_{ik}\,P_j \\
[P_0\,,\,K_i] &= -P_i \\
[P_0\,,\,P_i] &= K_i
\end{aligned}
$$

For $t_3 < 0$ after some rescaling and rotating P_i and K_i we obtain the algebra N_- defined by

$$
[M_{ij}\,,\,M_{kl}] = g_{jk}\,M_{il} + g_{il}\,M_{jk} - g_{ik}\,M_{jl} - g_{jl}\,M_{ik}
$$

$$
\begin{aligned}
[M_{ij}\,,\,K_k] &= g_{jk}\,K_i - g_{ik}\,K_j\\
[M_{ij}\,,\,P_k] &= g_{jk}\,P_i - g_{ik}\,P_j\\
[P_0\,,\,K_i] &= K_i\\
[P_0\,,\,P_i] &= -P_i
\end{aligned}
$$

These results are summarized graphically in Figure 1 which represents the $t_1 = t_4 = 0$ plane in the parameter space of infinitesimal deformations.

In the second case, $t_2 = 0$, we have that J_2 is a coboundary; but the obstruction cocycle at level 3 to which this second-order deformation leads, is not integrable unless $t_1 = 0$ or $t_3 = 0$. In the first case, $t_1 = 0$, we see that J_2 is automatically zero so that this is already a deformation. The deformed algebra is given by

$$
\begin{aligned}
[M_{ij}\,,\,M_{kl}] &= g_{jk}\,M_{il} + g_{il}\,M_{jk} - g_{ik}\,M_{jl} - g_{jl}\,M_{ik}\\
[M_{ij}\,,\,K_k] &= g_{jk}\,K_i - g_{ik}\,K_j\\
[M_{ij}\,,\,P_k] &= g_{jk}\,P_i - g_{ik}\,P_j\\
[P_0\,,\,K_i] &= -P_i + t_4 K_i\\
[P_0\,,\,P_i] &= t_4 P_i + t_3 K_i
\end{aligned}
$$

This is a non-semisimple Lie algebra which does not seem particularly interesting. In the second case, $t_3 = 0$, a long calculation yields an obstruction cocycle at level 5 which is not a coboundary unless $t_1 = 0$ or $t_4 = 0$. In any of these cases $J_2 = 0$ to begin with and we already have deformations. The case $t_1 = 0$ yields (after some rescaling) the algebra

$$
\begin{aligned}
[M_{ij}\,,\,M_{kl}] &= g_{jk}\,M_{il} + g_{il}\,M_{jk} - g_{ik}\,M_{jl} - g_{jl}\,M_{ik}\\
[M_{ij}\,,\,K_k] &= g_{jk}\,K_i - g_{ik}\,K_j\\
[M_{ij}\,,\,P_k] &= g_{jk}\,P_i - g_{ik}\,P_j\\
[P_0\,,\,K_i] &= -P_i + K_i\\
[P_0\,,\,P_i] &= P_i
\end{aligned}
$$

which is a contraction of the previously found algebra; whereas in the second case, $t_4 = 0$, we find, after some rescaling, the deformed algebra

$$
\begin{aligned}
[M_{ij}\,,\,M_{kl}] &= g_{jk}\,M_{il} + g_{il}\,M_{jk} - g_{ik}\,M_{jl} - g_{jl}\,M_{ik}\\
[M_{ij}\,,\,K_k] &= g_{jk}\,K_i - g_{ik}\,K_j\\
[M_{ij}\,,\,P_k] &= g_{jk}\,P_i - g_{ik}\,P_j\\
[K_i\,,\,K_j] &= \epsilon_{ijk} g^{kl} P_l\\
[P_0\,,\,K_i] &= -P_i
\end{aligned}
$$

In brief, we have a four parameter family $\{t_a\}$ of non-trivial infinitesimal deformations. The "domain of integrability" of these infinitesimal deformations, *i.e.* the subset of $\mathbb{R}^4$ corresponding to those values of $\{t_a\}$ for which the infinitesimal deformations are integrable, is given by the (3,4)-plane, the (2,3)-plane and the 1-axis. The interesting deformations seem to be in the (2,3)-plane as Figure 1 shows.

Centrally Extended Galilean Algebra

The summary of the calculations for the centrally extended Galilean algebra is as follows:

Space	Dimension
$C^0(\mathfrak{h};\mathfrak{g})^\mathfrak{s}$	2
$Z^0(\mathfrak{h};\mathfrak{g})^\mathfrak{s}$	1
$H^0(\mathfrak{h};\mathfrak{g})^\mathfrak{s}$	1
$B^1(\mathfrak{h};\mathfrak{g})^\mathfrak{s}$	1
$C^1(\mathfrak{h};\mathfrak{g})^\mathfrak{s}$	10
$Z^1(\mathfrak{h};\mathfrak{g})^\mathfrak{s}$	4
$H^1(\mathfrak{h};\mathfrak{g})^\mathfrak{s}$	3
$B^2(\mathfrak{h};\mathfrak{g})^\mathfrak{s}$	6
$C^2(\mathfrak{h};\mathfrak{g})^\mathfrak{s}$	25
$Z^2(\mathfrak{h};\mathfrak{g})^\mathfrak{s}$	9
$H^2(\mathfrak{h};\mathfrak{g})^\mathfrak{s}$	3
$B^3(\mathfrak{h};\mathfrak{g})^\mathfrak{s}$	16

In particular there is a three parameter family of non-trivial infinitesimal deformations induced by the following cocycles:

$$\begin{aligned}
C_1^1 &= P^{0*} \wedge P^{i*} \otimes K_i \\
C_1^2 &= P^{0*} \wedge P^{i*} \otimes P_i + P^{0*} \wedge K^{i*} \otimes K_i - 2c^* \wedge P^{0*} \otimes c \\
C_1^3 &= \frac{1}{2} K^{i*} \wedge K^{j*} \otimes M_{ij} + c^* \wedge K^{i*} \otimes P_i \ .
\end{aligned}$$

Again the most general non-trivial infinitesimal deformation is $\sum_{a=1}^3 t_a \, C_1^a$. The first obstruction cocycle is given by

$$J_2 = t_1 t_3 \left(P^{0*} \wedge c^* \wedge K^{i*} \otimes K_i - P^{0*} \wedge c^* \wedge P^{i*} \otimes P_i \right) + 2 t_2 t_3 P^{0*} \wedge c^* \wedge K^{i*} \otimes P_i \ . \tag{4.4}$$

This is a coboundary if and only if it vanishes identically. That is, $t_1 = t_2 = 0$ or $t_3 = 0$. In this latter case the deformed algebra looks like

$$\begin{aligned}
[M_{ij} \, , \, M_{kl}] &= g_{jk} \, M_{il} + g_{il} \, M_{jk} - g_{ik} \, M_{jl} - g_{jl} \, M_{ik} \\
[M_{ij} \, , \, K_k] &= g_{jk} \, K_i - g_{ik} \, K_j
\end{aligned}$$

$$
\begin{aligned}
[M_{ij}\,,\,P_k] &= g_{jk}\,P_i - g_{ik}\,P_j \\
[P_0\,,\,K_i] &= -P_i + t_2 K_i \\
[P_0\,,\,P_i] &= t_1 K_i + t_2 P_i \\
[P_i\,,\,K_j] &= g_{ij} c \\
[c\,,\,P_0] &= -2t_2 c\ .
\end{aligned}
$$

If $t_2 = 0$ and depending on the sign of t_1 we get centrally extended versions of the Newton algebras N_+ and N_- considered in the previous subsection. If $t_1 = 0$ we get, after some rescaling,

$$
\begin{aligned}
[M_{ij}\,,\,M_{kl}] &= g_{jk}\,M_{il} + g_{il}\,M_{jk} - g_{ik}\,M_{jl} - g_{jl}\,M_{ik} \\
[M_{ij}\,,\,K_k] &= g_{jk}\,K_i - g_{ik}\,K_j \\
[M_{ij}\,,\,P_k] &= g_{jk}\,P_i - g_{ik}\,P_j \\
[P_0\,,\,K_i] &= -P_i + K_i \\
[P_0\,,\,P_i] &= P_i \\
[P_i\,,\,K_j] &= g_{ij} c \\
[c\,,\,P_0] &= -2c\ .
\end{aligned}
$$

In the case that $t_1 = t_2 = 0$ we get the following deformed algebra

$$
\begin{aligned}
[M_{ij}\,,\,M_{kl}] &= g_{jk}\,M_{il} + g_{il}\,M_{jk} - g_{ik}\,M_{jl} - g_{jl}\,M_{ik} \\
[M_{ij}\,,\,K_k] &= g_{jk}\,K_i - g_{ik}\,K_j \\
[M_{ij}\,,\,P_k] &= g_{jk}\,P_i - g_{ik}\,P_j \\
[K_i\,,\,P_0] &= P_i \\
[K_i\,,\,K_j] &= t_3 M_{ij} \\
[c\,,\,K_i] &= t_3 P_i \\
[P_i\,,\,K_j] &= g_{ij} c\ ,
\end{aligned}
$$

Notice that depending on the sign of t_3 this algebra has a Poincaré or $E(4)$ subalgebra. However this seems an accident since the rôle of the fourth momentum generator is played, not by P_0, but by the central extension c. In fact we get either Poincaré or $E(4)$ with an extra generator which we call D obeying $[D\,,\, M_{0i}] = -P_i$.

In brief, we have a three parameter family of non-trivial infinitesimal deformations, whose integrability domain is the subset of $\mathbb{R}^3$ consisting of the (1,2)-plane and the 3-axis.

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