ON THE HOMOLOGICAL CONSTRUCTION OF CASIMIR ALGEBRAS

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ABSTRACT

We propose a BRST (homological) construction of the Casimir extended conformal algebras by quantizing a classical observation of Drinfeld and Sokolov. We give the explicit expression for the Virasoro generators and compute the discrete series of the Casimir algebras. The unitary subseries agrees with that of the coset construction for the case of simply laced algebras. With the help of a free field realization of the affine algebra, we compute the BRST cohomology of this system with coefficients in a Fock module. This allows us to generalize and prove a conjecture of Bershadsky and Ooguri.

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§1 Introduction

Extended conformal algebras are associative operator product algebras which contain the Virasoro algebra as a subalgebra and are finitely generated by, in addition to the Virasoro generator, holomorphic Virasoro primaries in such a way that the following closure property is satisfied: that in the singular part of the operator product expansion (OPE) of these fields there appear only Virasoro descendents of the identity and of these primary fields themselves as well as normal ordered products thereof. Clearly if we dropped the requirement of these algebras being finitely generated we could always satisfy the closure property by augmenting the number of generators by whichever new primaries appear in the right hand side of the OPE, but then we would be dealing with objects far too general to offer any hopes of classification. The study of extended conformal algebras is interesting because they are the only hope of classifying rational conformal field theories with $c \geq 1$, this being the central poblem in conformal field theory.

The systematic study of extended conformal algebras was initiated by Zamolodchikov in **[1]** , where he analysed the possible associative operator algebras generated by a stress tensor (generating a Virasoro subalgebra) and one or more holomorphic primary fields of half-integral conformal weight $s \leq 3$ with the above closure property. He found a lot of already existing conformal field theories: free fermions ($s = \frac{1}{2}$), affine Lie algebras ($s = 1$), superVirasoro algebras ($s = \frac{3}{2}$), direct product of Virasoro algebras ($s = 2$); as well as two new algebras ($s = \frac{5}{2}$ and $s = 3$) which, unlike the others, are not Lie (super)algebras. The case $s = \frac{5}{2}$ satisfies the associativity condition for a specific value of the central charge ($c = -\frac{13}{14}$); whereas the case $s = 3$ yields an associative operator algebra—called W_3—for all values of the central charge. This extended algebra has been the focus of a lot of recent work and, in particular, it has been shown to be the symmetry algebra of the 3-state Potts model at criticality[2].

In **[3]** , Bouwknegt continued the approach of Zamolodchikov by investigating the possible extensions of the Virasoro algebra by a holomorphic primary field of weight s. He argued that, if one demands associativity of the resulting operator algebra for generic values of the Virasoro central charge, the only solutions with integer s can occur for $s = 1, 2, 3, 4, 6$. Other extended algebras have been constructed following this approach: $\mathfrak{o}_N$-extended superconformal algebras[4],[5], and other algebras with an assortment of fields with low spins[6].

However this approach suffers from the drawback that one must know in advance the Virasoro primaries that occur in the algebra due to the fact that the closure criterion mentioned above allows for normal ordered products of primaries to appear among the singular terms in the OPE. Abstractly (as Virasoro primaries) there is no way to test whether the field which appears can in fact be written as normal ordered products of the "primitive" primaries. In fact, the existence of an extended conformal algebra as defined above can only be probed via its explicit construction in terms an underlying associative conformal field theory like the one corresponding to, say, free fields or affine algebras.

One such approach to extended conformal algebras is the construction of the so-called Casimir algebras. Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra of rank ℓ. Then the center of the universal enveloping algebra of $\mathfrak{g}$ is ℓ-dimensional and is spanned by the casimirs of $\mathfrak{g}$. Associated to each casimir we can define an operator (also referred to as the casimir with a little abuse of notation) in the universal enveloping algebra of the affinization $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$. This is a generalisation of the Sugawara construction which from the quadratic casimir obtains the Sugawara stress tensor. It turns out that these operators (except for the Sugawara tensor itself) are primary of weight equal to the order of the casimir with respect to the Virasoro algebra generated by the Sugawara tensor. However, and except for specific values of the central charge, the operator algebra generated by the casimirs does not close in the sense described above. For example[7], in the case of the affine algebra $A_2^{(1)}$, in the OPE of the cubic casimir with itself there appears a new primary field which only decouples for $c = 2$. The way to make a closed operator algebra from the casimirs relies in a coset construction analogous to that of Goddard-Kent-Olive[8] for $A_1^{(1)}$. For example in the case of $A_2^{(1)}$, the authors of **[9]** obtained the W_3 operator algebra by constructing a weight three primary operator in the universal enveloping algebra of $A_2^{(1)} \times A_2^{(1)}$ which commutes with the diagonal $A_2^{(1)}$ subalgebra: hence a weight three primary field in the coset theory $(A_2^{(1)} \times A_2^{(1)})/A_2^{(1)}$. This operator turns out to form a closed operator algebra with the coset Virasoro generator if and only if one of the $A_2^{(1)}$ factors is at level 1. The weight three primary is constructed out of the cubic casimirs of the two factors but also contains mixed terms which are rather mysterious. In fact, the explicit construction of the operators generating the casimir algebras associated to other affine algebras is still lacking and the very existence of the higher Casimir algebras has not been proven except for $A_2^{(1)}$, although there is certainly some evidence of it coming from character formulas for affine algebras[3]. In fact the conjecture stands that all extended conformal algebras (generated by primaries of integer weights) are indeed Casimir algebras. So far, the only supporting evidence for this claim comes from the results of **[3]** , where it is suggested that the extended conformal algebras obtained by adding just one extra primary field correspond to the casimir algebras resulting from $A_1 \times D_1$, $A_1 \times A_1$, A_2, B_2, and G_2 respectively. The coset construction (or any other construction, for that matter) of these casimir algebras is known explicitly only for the first three cases. The explicit form of the algebra (as an abstract associative algebra) is nevertheless known in all cases: **[10]** for the case of spin 4 and **[11]** for the case of spin 6. However the identification of these two last algebras as casimir algebras is still lacking.

In this paper we propose a different construction of Casimir algebras than the coset one. This is a generalization of the one in **[12]** , where an explicit construction of W_3 from $A_2^{(1)}$ was provided. The method is founded on the result of Drinfeld and Sokolov[13] (after an observation by Reiman and Semenov-Tyan-Shanskii) which showed that a classical version of W_3 (as fundamental Poisson brackets of a Poisson manifold) arises from the symplectic reduction of the infinite dimensional Poisson manifold defined by the dual of the affine algebra $A_2^{(1)}$ relative to the action of a unipotent subgroup. The rôle of the casimirs in this construction has recently been clarified in **[14]** . The construction of **[12]** can be thought of as a quantization of the Drinfeld-Sokolov construction for the case of A_2. There the cubic casimir of A_2 also played a fundamental rôle. In fact, the operators which generate the W_3 algebra are induced from the BRST completions of deformations of the Sugawara stress tensor and of the cubic casimir by terms of lower order in the $A_2^{(1)}$ currents.

Let us briefly describe the Drinfeld-Sokolov construction for an arbitrary affine algebra $\widehat{\mathfrak{g}}$. The dual space M of $\widehat{\mathfrak{g}}$ has a canonical Poisson structure which is invariant under the action of the corresponding loop group. Let $\mathfrak{n}_+$ denote the nilpotent subalgebra of $\mathfrak{g}$ generated by the elements associated to the positive roots and $\widehat{\mathfrak{n}}_+$ the corresponding affine algebra. The action of the corresponding loop group is Poisson and gives rise to a moment mapping $J : M \to \widehat{\mathfrak{n}}_+^*$. Drinfeld and Sokolov consider the level set M_o of J coresponding to the following constraints:

$$E_{-\alpha}(z) = \begin{cases} 1, & \text{if } \alpha \in \Delta, \\ 0, & \text{otherwise;} \end{cases} \tag{1.1}$$

where α is a positive root, $E_{-\alpha}$ the generator of $\mathfrak{g}$ corresponding to the negative root $-\alpha$, and Δ is a choice of simple roots. Equivariance of the moment mapping implies that M_o is stabilized by the loop group corresponding to $\widehat{\mathfrak{n}}_+$ which allows us to quotient by its action and obtain a manifold $\widetilde{M}$ which inherits a natural Poisson structure from that on M. In order to write the fundamental Poisson brackets of $\widetilde{M}$ explicitly we need to coordinatize $\widetilde{M}$. Since $\widetilde{M}$ is defined as a quotient there are no preferred coordinates. In order for $\widetilde{M}$ to inherit coordinates from M_o it is necessary to exhibit it as a submanifold of M_o, *i.e.* , to fix a gauge. Different choices of a gauge slice will give different coordinates and, hence, different fundamental Poisson brackets. For the case of $\mathfrak{g} = A_n$, Drinfeld and Sokolov show there exists a choice of gauge slice in which the fundamental Poisson brackets are those of n free bosons; whereas there exists a different gauge slice in which the fundamental Poisson brackets represent the Gelfand-Dickey algebras which, for A_n, is a classical version of W_{n+1}, the generalization of W_3 due to Fateev and Lykyanov[**15**]. Moreover the change of coordinates from one gauge slice to another (*i.e.* , the corresponding gauge transformation) is nothing but the Miura transformation of the KdV theory. This was exploited by Fateev and Lykyanov to construct the Coulomb gas representation for W_n.

This paper is organized as follows. Section 2 contains the general construction. In it we impose the Drinfeld-Sokolov constraints in the BRST formalism. We construct a BRST invariant stress tensor showing that the quantal space of the reduced theory inherits the structure of a Virasoro module and determine its central charge. We do this in detail for all simple Lie algebras. In the simply-laced case, and for suitable choices of the level, we recover the discrete series of the Casimir algebras. The unitary subseries agrees with the results obtained in the coset construction of **[7]** . In Section 3 we analyze the cohomology of the BRST operator in some more detail with the help of a free field realization for the affine algebra. We compute the BRST cohomology of a Fock module and show that it is (as a Virasoro module) isomorphic to the Fock space of ℓ free bosons with background charge. This allows us to prove for general $\mathfrak{g}$ a conjecture of Bershadsky and Ooguri[16] for the case A_n. The proof of the main theorem in section 3 requires a technical result that is discussed in the appendix. Finally section 4 contains some concluding remarks.

§2 The General Construction

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra of rank ℓ and $\mathfrak{h}$ a Cartan subalgebra which will remain fixed throughout. Let Φ_+ denote the positive roots and $\Delta = \{\alpha_i\}$ a fixed choice of simple roots. Any given $\alpha \in \Phi_+$ can be written as a unique linear combination $\sum_{i=1}^{\ell} n_i\alpha_i$ where the n_i are non-negative integers. We call $\mathrm{ht}\,\alpha = \sum_i n_i$ the height of α. Obviously simple roots have height 1 and all other positive roots have higher heights. Associated to each positive root α there are generators $E_{\pm\alpha} \in \mathfrak{g}$.

We now consider the affine algebra $\widehat{\mathfrak{g}}$ of $\mathfrak{g}$. Its Lie bracket is encoded in the operator product expansion (OPE) of the currents $X(z)$ for $X \in \mathfrak{g}$:

$$X(z)\,Y(w) = \frac{k(X,Y)}{(z-w)^2} + \frac{[X\,,\,Y](w)}{z-w} + \mathrm{reg}\ , \tag{2.1}$$

where $(\,,\,)$ is a fixed symmetric invariant bilinear form on $\mathfrak{g}$—*i.e.* , a multiple of the Killing form. In particular the currents $\{E_\alpha(z)\}_{\alpha\in\Phi_+}$ (respectively $\{E_{-\alpha}(z)\}_{\alpha\in\Phi_+}$) form a subalgebra of $\widehat{\mathfrak{g}}$ without central extension:

$$E_\alpha(z)\, E_\beta(w) = \frac{m_{\alpha\beta} E_{\alpha+\beta}(w)}{z-w} + \text{reg} \ , \tag{2.2}$$

where the coefficients $m_{\alpha\beta}$ are defined by $[E_\alpha\, ,\, E_\beta] = m_{\alpha\beta} E_{\alpha+\beta}$.

In the universal enveloping algebra of $\widehat{\mathfrak{g}}$ we find a Virasoro subalgebra generated by the Sugawara stress tensor:

$$T^{\mathfrak{g}}(z) = \frac{1}{2k + c_{\mathfrak{g}}} g^{ab} \left(X_a X_b\right)(z) \ , \tag{2.3}$$

where g^{ab} is the inverse of $g_{ab} \equiv (X_a, X_b)$ with $\{X_a\}$ any basis of $\mathfrak{g}$; $c_{\mathfrak{g}}$ is the eigenvalue of the casimir element $g^{ab} X_a \otimes X_b$ in the adjoint representation; and where by the normal ordered product (AB) of any two operators we mean the following:

$$(AB)\,(z) = \oint_{C_z} \frac{dw}{2\pi i} \frac{A(w)B(z)}{w-z} \ . \tag{2.4}$$

The central charge of this representation of the Virasoro algebra is given by

$$c(\mathfrak{g}, k) \equiv \frac{2k d_{\mathfrak{g}}}{2k + c_{\mathfrak{g}}} \ , \tag{2.5}$$

where $d_{\mathfrak{g}}$ is the dimension of $\mathfrak{g}$.

As explained in the intoduction and following Drinfeld-Sokolov we now impose the following constraints:

$$E_{-\alpha}(z) = \begin{cases} 1, & \text{if } \alpha \in \Delta, \\ 0, & \text{if } \alpha \in \Phi_+ \backslash \Delta \end{cases} \tag{2.6}$$

These constraints close under commutators since the Lie bracket of two positive roots can never be a simple root. Therefore we can quantize the system à la BRST.

To this effect we introduce an antighost–ghost pair (b_α, c_α) for each positive root α with the following OPE:

$$b_\alpha(z)\, c_\beta(w) = \frac{\delta_{\alpha\beta}}{z-w} + \text{reg} \ ; \tag{2.7}$$

and we define the BRST operator d by

$$d \equiv \oint_{C_0} \frac{dz}{2\pi i}\, j_{\text{BRST}}(z) \ , \tag{2.8}$$

where

$$j_{\text{BRST}} = -\sum_{\alpha\in\Delta} c_\alpha + \sum_{\alpha\in\Phi_+} (c_\alpha E_{-\alpha}) + \frac{1}{2} \sum_{\alpha,\beta\in\Phi_+} m_{\alpha\beta}\left(c_\alpha c_\beta b_{\alpha+\beta}\right) \ . \tag{2.9}$$

The closure of the constraint algebra implies that the BRST operator is square-zero: $d^2 = 0$, whence its cohomology is defined. The operators in the reduced theory will be the BRST invariant operators in the algebra generated by the $\widehat{\mathfrak{g}}$ currents and the (anti)ghost fields; two such operators inducing the same operator in the reduced theory if they are BRST cohomologous, *i.e.* , if their difference can be written as $[d\,,\text{something}]$.

Since we want the reduced theory to be a conformal field theory we want to induce in BRST cohomology the structure of a Virasoro module. In other words, we want to exhibit a BRST invariant field which obeys (at least up to BRST coboundaries) the OPE of the Virasoro algebra. This requires the constraints to be conformally covariant with respect to this Virasoro algebra. For example, under the Virasoro algebra generated by the Sugawara tensor, the currents $E_{-\alpha}(z)$ are all primary of weight one. However the constraints set $E_\alpha(z)$ for α a simple root equal to a constant (1 in our conventions) which has zero conformal weight. Therefore the constraints corresponding to the simple roots are not conformally covariant with respect to the Sugawara stress tensor.

In order to obviate this problem we can try to deform the Sugawara stress tensor in such a way that the currents corresponding to the negative simple roots are primary but of zero conformal weight. Let us define a deformed stress tensor $T_{\rm def}$ by adding to it the derivative of a current

$$T_{\rm def}(z) \equiv T^{\mathfrak{g}}(z) - \partial h(z) \ . \tag{2.10}$$

It is clear that demanding that the currents $E_{-\alpha}(z)$ be primary forces h to lie in the Cartan subalgebra. Computing the OPE of $T_{\rm def}$ with $E_{-\alpha}$ we find

$$T_{\rm def}(z)\, E_{-\alpha}(w) = \frac{(1-\alpha(h))E_{-\alpha}(w)}{(z-w)^2} + \frac{\partial E_{-\alpha}(w)}{z-w} + \text{reg} \ , \tag{2.11}$$

where we have used that $[h\,,\, E_{-\alpha}] = -\alpha(h)E_{-\alpha}$. Demanding that for α a simple root, $E_{-\alpha}$ have zero conformal weight we get the following conditions on h: $\alpha(h) = 1$ for all simple roots α. Since the simple roots span the dual $\mathfrak{h}^*$ to the Cartan subalgebra, these conditions fix h uniquely.

We now determine h. The restriction of the invariant symmetric bilinear form $(,)$ to the Cartan subalgebra $\mathfrak{h}$ is non-degenerate. Under the induced isomorphism $\mathfrak{h} \to \mathfrak{h}^*$, h gets sent to $h^\flat$. We shall determine $h^\flat$. Since the fundamental weights $\{\lambda_i\}$ span $\mathfrak{h}^*$ we can expand $h^\flat$ as $h^\flat = \sum_i \mu_i \lambda_i$. Using the fact that $(\lambda_i, \alpha_j) = \delta_{ij}\frac{1}{2}(\alpha_j, \alpha_j)$ and also that $(h^\flat, \alpha_j) = \alpha_j(h) = 1$, we can solve for the coefficients μ_i. Therefore $h^\flat = \sum_i (2/(\alpha_i, \alpha_i))\lambda_i$. Notice that if $\mathfrak{g}$ is simply laced then $h^\flat$ is simply a multiple of δ, half the sum of all the positive roots. We will compute the norm of h (or equivalently that of $h^\flat$) for all $\mathfrak{g}$ later on when we compute the central charges of the Casimir algebras.

The upshot of all this is that $T_{\rm def}$ still satisfies a Virasoro algebra with central charge given by $c_{\rm def} \equiv c(\mathfrak{g}, k) - 12k(h,h)$ and that the currents $E_{\pm\alpha}$ for α a positive root are still primary with respect to $T_{\rm def}$ with weights $\Delta_{\pm\alpha} = 1 \pm \text{ht}\,\alpha$.

Since the BRST operator is a closed contour integral of a current, BRST invariance of the Virasoro stress tensor is equivalent to the current being primary of weight 1. This is clearly the case if we assign to the (anti)ghost pair (b_α, c_α) the conformal weights $(1 - \mathrm{ht}\,\alpha, \mathrm{ht}\,\alpha)$, which corresponds to the stress tensor

$$T^{\mathrm{gh}} = \sum_{\alpha\in\Phi_+} \left[(\mathrm{ht}\,\alpha - 1)\,(b_\alpha \partial c_\alpha) + \mathrm{ht}\,\alpha\,(\partial b_\alpha c_\alpha)\right] \ , \tag{2.12}$$

which obeys the Virasoro algebra with central charge

$$c_{\mathrm{gh}} = -2 \sum_{\alpha\in\Phi_+} \left(6(\mathrm{ht}\,\alpha)^2 - 6\mathrm{ht}\,\alpha + 1\right) \ . \tag{2.13}$$

Since the total stress tensor $T_{\mathrm{tot}} \equiv T_{\mathrm{def}} + T^{\mathrm{gh}}$ is BRST invariant and obeys a Virasoro algebra with central charge $c = c_{\mathrm{def}} + c_{\mathrm{gh}}$, the BRST cohomology $H_d(\mathfrak{M})$ of any $\widehat{\mathfrak{g}}$-module $\mathfrak{M}$ inherits the structure of a Virasoro module with central charge given by c.

We now come to the calculation of the central charge for the Casimir algebra of an arbitrary finite dimensional simple Lie algebra $\mathfrak{g}$. If we let θ stand for the maximal root, the level x of the representation of the affine algebra $\widehat{\mathfrak{g}}$ is given by $x = 2k/|\theta|^2$. Moreover the eigenvalue of the quadratic Casimir in the adjoint representation is given by $c_{\mathfrak{g}} = |\theta|^2 g^*$, where g^* is the dual Coxeter number. The central charge $c = c_{\mathrm{def}} + c_{\mathrm{gh}}$ is then given by

$$c = \ell - \frac{d_{\mathfrak{g}} g^*}{x + g^*} - 6x|\theta|^2|h|^2 - 12 \sum_{\alpha\in\Phi_+} \left((\mathrm{ht}\,\alpha)^2 - \mathrm{ht}\,\alpha\right) \ . \tag{2.14}$$

This expression is manifestly independent of the choice of invariant bilinear form $(\,,)$, since the multiplicative factor in $\mathfrak{h}$ is compensated by the inverse factor in $\mathfrak{h}^*$. Therefore we choose to compute things with the Killing form κ. The eigenvalue of the quadratic Casimir on the adjoint representation is identically 1 with this choice and this means that $|\theta|^2 = 1/g^*$. Also we notice that since h acts diagonally on $\mathfrak{g}$ in such a way that $[h\,,\,E_\alpha] = \mathrm{ht}\,\alpha\,E_\alpha$, its norm relative to the Killing form is just

$|h|^2 = 2\sum_{\alpha\in\Phi_+}(\mathrm{ht}\,\alpha)^2$. These remarks allow us to rewrite (2.14) as

$$c = \ell - \frac{d_{\mathfrak{g}} g^*}{x+g^*} - \frac{12}{g^*}\sum_{\alpha\in\Phi_+}(\mathrm{ht}\,\alpha)^2(x+g^*) + 12\sum_{\alpha\in\Phi_+}\mathrm{ht}\,\alpha \ . \tag{2.15}$$

Therefore the central charge of the Casimir algebra is expressed in terms of the rank, the dimension, and the dual Coxeter number of the algebra—all of which are tabulated in standard references on the subject; as well as the sums over the positive roots of the heights and their squares. These are straightforward to compute from the explicit realizations for the root systems in terms of linear combinations of the standard orthonormal basis of euclidean n-space. Such realizations are standard and can be found, for example, in **[17]** . The following table summarizes the information relevant to the calculation of the central charges.

$\mathfrak{g}$	$d_{\mathfrak{g}}$	g^*	$\sum_{\alpha\in\Phi_+}\mathrm{ht}\,\alpha$	$\sum_{\alpha\in\Phi_+}(\mathrm{ht}\,\alpha)^2$
A_ℓ	$\ell(\ell+2)$	$\ell+1$	$\frac{1}{6}\ell(\ell+1)(\ell+2)$	$\frac{1}{12}\ell(\ell+1)^2(\ell+2)$
B_ℓ	$\ell(2\ell+1)$	$2\ell-1$	$\frac{1}{6}\ell(\ell+1)(4\ell-1)$	$\frac{1}{6}\ell(\ell+1)(2\ell-1)(2\ell+1)$
C_ℓ	$\ell(2\ell+1)$	$\ell+1$	$\frac{1}{6}\ell(\ell+1)(4\ell-1)$	$\frac{1}{6}\ell(\ell+1)(2\ell-1)(2\ell+1)$
D_ℓ	$\ell(2\ell-1)$	$2(\ell-1)$	$\frac{1}{3}\ell(\ell-1)(2\ell-1)$	$\frac{1}{3}\ell(\ell-1)^2(2\ell-1)$
E_6	78	12	156	936
E_7	133	18	399	3591
E_8	248	30	1240	18600
F_4	52	9	110	702
G_2	14	4	16	56

With this information it is then straightforward to compute the central charge of the Casimir algebra according to equation (2.15). For the (A, D, E) series—*i.e.* , the simply-laced algebras—the choice $x + g^* = p/q \in \mathbb{Q}$ where p and q are coprime natural numbers corresponds to the discrete series of the Casimir algebras:

$$c_{A_\ell} = \ell\left(1-(\ell+1)(\ell+2)\frac{(p-q)^2}{pq}\right) \ , \tag{2.16}$$

$$c_{D_\ell} = \ell\left(1 - 2(\ell-1)(2\ell-1)\frac{(p-q)^2}{pq}\right) \ , \tag{2.17}$$

$$c_{E_6} = 6\left(1 - 156\frac{(p-q)^2}{pq}\right) \ , \tag{2.18}$$

$$c_{E_7} = 7\left(1 - 342\frac{(p-q)^2}{pq}\right) \ , \tag{2.19}$$

$$c_{E_8} = 8\left(1 - 930\frac{(p-q)^2}{pq}\right) \ . \tag{2.20}$$

In particular notice that the A_ℓ discrete series (2.16) agrees with that of the $W_{\ell+1}$ algebras, the Virasoro algebra being W_2. The unitary subseries are obtained setting $p = m$ and $q = m + 1$. Then for $m \geq g^*$ we obtain the unitary discrete series of the coset construction **[7]** .

§3 The Cohomology of the BRST Operator

In this section we will analyze the cohomology of the BRST operator in some detail. A naive counting of degrees of freedom suggests that the reduced quantum theory (*i.e.* , the one whose quantal space is given by the BRST cohomology at zero ghost number) has ℓ degrees of freedom in "units" where a free boson would constitute one degree of freedom. We will make this observation precise by computing the BRST cohomology of a highly reducible module, namely the Fock space of a free field representation for the affine algebra $\widehat{\mathfrak{g}}$. Under some assumptions we will see at the end of the section what this implies for the BRST cohomology of an irreducible $\widehat{\mathfrak{g}}$-module $\mathfrak{M}$. Fortunately we will not need the explicit form of the $\widehat{\mathfrak{g}}$ currents in terms of free fields but only some qualitative properties of it and, of course, its existence. This last assertion is guaranteed by the work of **[18]** who, although unable to give a general formula for the $\widehat{\mathfrak{g}}$ currents, give us a method to compute it. It seems clear that there is no obstruction to their construction and that, in fact, a free field realization exists for any $\widehat{\mathfrak{g}}$ in terms of ℓ free bosons with background charges and dim Φ_+ bosonic $(\beta\gamma)$ systems.

In this realization the Sugawara tensor decomposes into a sum of free field tensors:

$$T^{\mathfrak{g}} = T'_{\varphi} + T'_{\beta,\gamma} \ , \tag{3.1}$$

where

$$T'_{\varphi} = \sum_{i=1}^{\ell} \left(-\frac{1}{2}(\partial\varphi_i)^2 + Q_i \partial^2 \varphi_i \right) \tag{3.2}$$

$$T'_{\beta,\gamma} = \sum_{\alpha\in\Phi_+} (\beta_\alpha \partial\gamma_\alpha) \tag{3.3}$$

where the Q_i are background charges. The deformed energy momentum tensor is obtained from $T^{\mathfrak{g}}$ by adding to it $-\partial h$. The general form of h in the free field representation is

$$h = \vec{\alpha}\cdot\partial\vec{\varphi} + \sum_{\alpha\in\Phi_+} \mathrm{ht}\,\alpha\,(\beta_\alpha\gamma_\alpha) \ . \tag{3.4}$$

To see this notice that in the free field representation $E_{-\alpha} = \beta_\alpha + \cdots$, where the dots correspond to terms of the form $\gamma_{\alpha_1}\gamma_{\alpha_2}\cdots\gamma_{\alpha_n}\beta_{\alpha+\alpha_1+\cdots+\alpha_n}$. Since $[h\,,\,E_{-\alpha}] = -\mathrm{ht}\,\alpha\,E_{-\alpha}$ this fixes the coefficient in front of $(\beta_\alpha\gamma_\alpha)$ in the free field expansion of h. Therefore the deformed stress tensor in the free field representation takes the form

$$T_{\mathrm{def}} = T_{\varphi} - \sum_{\alpha\in\Phi_+} \left[(\mathrm{ht}\,\alpha - 1)\,(\beta_\alpha\partial\gamma_\alpha) + \mathrm{ht}\,\alpha\ (\partial\beta_\alpha\gamma_\alpha) \right] \ , \tag{3.5}$$

where T_φ has the same form as T'_φ except for different background charges. The full BRST invariant stress tensor is thus the sum of two terms: T_φ and T given by

$$T = \sum_{\alpha\in\Phi_+} \left[(\mathrm{ht}\,\alpha - 1)\,((b_\alpha\partial c_\alpha) - (\beta_\alpha\partial\gamma_\alpha)) + \mathrm{ht}\,\alpha\,((\partial b_\alpha c_\alpha) - (\partial\beta_\alpha\gamma_\alpha)) \right] \ . \tag{3.6}$$

The general form of the BRST current can be read off from (2.9):

$$j_{\mathrm{BRST}} = -\sum_{\alpha\in\Delta} c_\alpha + \sum_{\alpha\in\Phi_+} (\beta_\alpha c_\alpha) + \frac{1}{2}\sum_{\alpha,\beta\in\Phi_+} m_{\alpha\beta} c_\alpha c_\beta b_{\alpha+\beta} + \cdots \tag{3.7}$$

where dots stand for terms of the form $\gamma\beta c$ coming from the $\gamma\beta$ terms in $E_{-\alpha}$. In particular we notice that the BRST operator is independent of φ.

We now consider a Fock space representation of the free field algebra:

$$\mathcal{F} = \mathcal{H}_\varphi \otimes \mathcal{H}_{\beta\gamma} \otimes \mathcal{H}_{\rm gh} \ .$$

The BRST operator acts as the identity on $\mathcal{H}_\varphi$ and hence its cohomology is

$$H_d(\mathcal{F}) \cong \mathcal{H}_\varphi \otimes H_d(\mathcal{H}_{\beta\gamma} \otimes \mathcal{H}_{\rm gh}) \ . \tag{3.8}$$

Letting $C \equiv \mathcal{H}_{\beta\gamma} \otimes \mathcal{H}_{\rm gh}$ we will prove in this section that $H_d(C) \cong \mathbb{C}$. The proof is a straightforward extension to the general case of the one given for $\mathfrak{g} = A_2$ in **[12]** as part of the explicit construction of W_3 from $A_2^{(1)}$.

The crux of the proof relies on the observation that the BRST operator d contains a piece d_0 given by the contour integral around 0 of $\sum_{\alpha\in\Phi_+} (\beta_\alpha c_\alpha)$ obeying $d_0^2 = 0$ whose cohomology is almost trivial. In fact we have

Lemma. $H_{d_0}(C) \cong \mathbb{C}$.

Proof: We need to introduce some notation first. C is generated by the creation operators of the fields $\{b_\alpha, c_\alpha, \beta_\alpha, \gamma_\alpha\}$ acting on the $\mathfrak{sl}_2$ invariant vacuum Ω_0, which is annihilated by the following modes:$(b_\alpha)_p$, $(\beta_\alpha)_p$ for $p \geq \mathrm{ht}\,\alpha$; and $(c_\alpha)_p$, $(\gamma_\alpha)_p$ for $p \geq (1 - \mathrm{ht}\,\alpha)$. For every kind of field let us now define the following creation ($a^\dagger$) and annihilation (a) operators:

$$\left.\begin{aligned} a^\dagger(\beta_\alpha)_p &\equiv (\beta_\alpha)_{-p} \\ a(\beta_\alpha)_p &\equiv -(\gamma_\alpha)_p \end{aligned}\right\} \forall p \geq (1 - \mathrm{ht}\,\alpha)$$
$$\left.\begin{aligned} a^\dagger(\gamma_\alpha)_p &\equiv (\gamma_\alpha)_{-p} \\ a(\gamma_\alpha)_p &\equiv (\beta_\alpha)_p \end{aligned}\right\} \forall p \geq \mathrm{ht}\,\alpha \tag{3.9}$$

and the same for b and c except for the minus signs in the definitions of $a(b_\alpha)$. With this notation and if X is a free field

$$[a(X)_p\,,\,a^\dagger(X)_q]_\pm=\delta_{pq}\ , \tag{3.10}$$

for allowed p,q. A short calculation further shows that

$$d_0=\sum_{\substack{\alpha,p\\ \text{allowed}}}\Big(a^\dagger(\beta_\alpha)_p a(b_\alpha)_p+a^\dagger(c_\alpha)_p a(\gamma_\alpha)_p\Big)\ . \tag{3.11}$$

Let us define the operator K as follows

$$K\equiv\sum_{\substack{\alpha,p\\ \text{allowed}}}\Big(a^\dagger(b_\alpha)_p a(\beta_\alpha)_p+a^\dagger(\gamma_\alpha)_p a(c_\alpha)_p\Big)\ . \tag{3.12}$$

This operator obeys $\{d_0\,,\,K\}=N$, where N is the full number operator diagonalised by the basis states

$$\prod_{\substack{\alpha,p\\ \text{allowed}}}\Big[a^\dagger(\beta_\alpha)_p\Big]^{k_{\alpha,p}}\Big[a^\dagger(b_\alpha)_p\Big]^{k'_{\alpha,p}}\Big[a^\dagger(\gamma_\alpha)_p\Big]^{l_{\alpha,p}}\Big[a^\dagger(c_\alpha)_p\Big]^{l'_{\alpha,p}}\cdot\Omega_0 \tag{3.13}$$

with eigenvalue

$$\sum_{\substack{\alpha,p\\ \text{allowed}}}\big(k_{\alpha,p}+k'_{\alpha,p}+l_{\alpha,p}+l'_{\alpha,p}\big)<\infty\ , \tag{3.14}$$

where $k_{\alpha,p}$ and $l_{\alpha,p}$ take integer non-negative values and $k'_{\alpha,p}$ and $l'_{\alpha,p}$ are either 0 or 1. Since N commutes with d_0, the cohomology of d_0 splits into a direct sum

$$H_{d_0}(C)=\bigoplus_{n\geq 0}H_{d_0}(C^{(n)})\ , \tag{3.15}$$

where $C^{(n)}$ is the eigenspace of N with eigenvalue n. But since N is d_0-exact, the cohomology resides in $C^{(0)}$; for any d_0-cocycle $\omega\in C^{(n\neq 0)}$ is also a coboundary: $\omega=d_0\frac{1}{n}K\,\omega$. But $C^{(0)}$ is spanned by the vacuum which is thus a non-trivial d_0-cocycle. ■

The construction now consists in breaking up d in a natural way such that the piece d_0 is isolated allowing us to compute the cohomology of d from the our knowledge of the one of d_0. One way to achieve this is to grade C in such a way that under the grading d breaks up as $\sum_{i\geq 0} d_i$ with the degree of d_0 being zero and the degrees of the other homogeneous terms in d being positive. Towards this end let us define a degree deg on C as follows. We first define $\deg \Omega_0 = 0$. Since C is generated by the action of the modes of the fundamental fields on Ω_0, we need only define the degree of the fundamental fields to define it on all of C. Demanding that the operator algebra respect the grading, we conclude that $\deg \beta_\alpha = -\deg \gamma_\alpha$ and $\deg b_\alpha = -\deg c_\alpha$. Moreover demanding that d_0 have zero degree forces $\deg \beta_\alpha = -\deg c_\alpha$ and, thus, $\deg c_\alpha = \deg \gamma_\alpha = -\deg b_\alpha = -\deg \beta_\alpha = \lambda_\alpha$. However the $\{\lambda_\alpha\}$ must satisfy two extra conditions: first, that the homogeneous terms in $d - d_0$ have positive degrees; and, second, that the grading be bounded above in each of the L_0-eigenspaces of C.

This last condition requires further clarification. Since $T(z)$ is BRST invariant so is L_0. Moreover we can split C into L_0 eigenspaces $C = \bigoplus_h C^h$, where $C^h = \{\phi \in C \mid L_0 \psi = h\,\psi\}$, each of which is preserved by the action of d. Consequently the cohomology of d breaks up as $H_d(C) \cong \bigoplus_h H_d(C^h)$. We will prove that $H_d(C^{h\neq 0}) \cong 0$ whereas $H_d(C^0) \cong \mathbb{C}$. But in the proof it will be crucial that the grading be bounded from above in each of the C^h.

The two aforementioned conditions on the grading translate into inequalities that the λ_α must satisfy. It can be proven that such a grading exists, but the the proof is rather technical and not very illuminating and hence it is best relegated to the appendix. Hence from now on we will assume that we have graded C in such a way that the two conditions mentioned above are satisfied and, moreover, that the $\{\lambda_\alpha\}$ are all positive integers.

Under this grading the BRST operator decomposes naturally into a finite sum $d = \sum_i d_i$, where $\deg d_i = i$ and where some of the d_i may be zero. Notice that since $\deg L_0 = 0$, $[d_i\,,\,L_0] = 0$ for all i. In particular, $[d_0\,,\,L_0] = 0$ and so

$H_{d_0}(C) \cong \bigoplus_h H_{d_0}(C^h)$. From the lemma, however, and since the $\mathfrak{sl}_2$ invariant vacuum is annihilated by L_0, $H_{d_0}(C^{h\neq 0}) \cong 0$ and $H_{d_0}(C^0) \cong \mathbb{C}$. In fact, the same is true for the full BRST operator, as we now prove.

Theorem. $H_d(C^{h\neq 0}) \cong 0$, $H_d(C^0) \cong \mathbb{C}$.

Proof: Let $\psi \in C^h$. We say that ψ has *order* p if ψ has no piece of deg $< p$. Let deg be bounded above in C^h by μh, where μ is some integer that could depend on h. Then the order of ψ is clearly bounded above but not below; although any element of C^h has finite (albeit arbitrarily negative) order. Let ψ be a d-cocycle in C^h of order p. Under the decomposition $C^h = \bigoplus_{n=-\infty}^{N} C_n^h$ induced by the grading, ψ breaks up as $\psi = \psi_p \oplus \cdots \oplus \psi_N$.

We first show that if $p > 0$ then ψ is a d-coboundary. The proof will proceed by backwards induction. Suppose $p = N$ so that $\psi = \psi_N$. Then $d\psi = 0 \Rightarrow d_0\psi = 0$. By the lemma, $\psi = d_0\xi_N$ for some $\xi_N \in C_N^h$. But then $d_i\xi_N = 0$ for $i > 0$ since $d_i\xi_N \in C_{N+i}^h = 0$. Therefore $\psi = d\xi_N$ and hence a d-coboundary. Suppose that this is the case for all positive orders $> n$. Then if ψ has order n, again $d_0\psi_n = 0$ and by the lemma $\psi_n = d_0\xi_n$ for some $\xi_n \in C_n^h$. But then $\psi - d\xi_n$ is a d-cocycle of order $n+1$, which by the induction hypothesis is a d-coboundary. Hence by induction we find that all d-cocycles of positive order are d-coboundaries. Exactly the same argument shows that for $h \neq 0$ all d-cocycles are d-coboundaries since d_0 has no cohomology here. This proves the first of the two isomorphisms of the theorem.

Suppose now that $h = 0$. In this case deg is bounded above by zero. If ψ is a d-cocycle of order zero then $\psi = \psi_0$ and $d_0\psi = 0$. By the lemma it is d_0-cohomologous to some multiple of the vacuum, that is, $\psi = \alpha\Omega_0 + d_0\xi_0$ for some $\xi_0 \in C_0^0$. But $d_i\xi_0 \in C_i^0 = 0$ so that ψ is actually d-cohomologous to some multiple of the vacuum. Suppose now that this is the case for all d-cocycles of order $> p$ ($p < 0$). Then if ψ has order p, so that $\psi = \psi_p \oplus \cdots \oplus \psi_0$, we have again that $d\psi = 0 \Rightarrow d_0\psi_p = 0$. By the lemma, $\psi_p = d_0\xi_p$ for some $\xi_p \in C_p^0$. Therefore $\psi - d\xi_n$

is a d-cocycle of order $p+1$ and hence, by the induction hypothesis, cohomologous to some multiple of the vacuum. Hence we conclude that all d-cocycles in C^0 are cohomologous to a multiple of the vacuum, proving the second isomorphism of the theorem. ■

Notice that the fact that the grading is bounded from above was instrumental in the proof. If this had not been the case there would be no guarantee that the cochains ξ obeying $\psi = d\xi$ would be in C, which, by definition, consists of finite linear combinations of basis elements of the type (3.13). Note also that the grading is not bounded from below. Had this been the case then each C^h would have been a bounded complex and the above theorem would have followed trivially from the lemma by a standard spectral sequence argument.

As an immediate corollary of this theorem we conclude that under the isomorphism $H_d(\mathcal{F}) \cong \mathcal{H}_\varphi$ and given that $T(z)$ induces the zero operator in cohomology, the operator induced by $T_{\rm tot}$ is precisely T_φ. Therefore, as a Virasoro module, $H_d(\mathcal{F})$ is precisely the one defined by the Fock space of ℓ free bosons with background charges. Moreover, by abstract nonsense, it follows that $T(z) = \{d\,,\,t(z)\}$, which was proven by Bershadsky and Ooguri[**16**] for the case of A_1 and A_2 and conjectured for the rest of the A_n series.

Moreover if $\mathfrak{M}$ is a $\widehat{\mathfrak{g}}$-module embedding equivariantly in a Fock space $\mathcal{H}_\varphi \otimes \mathcal{H}_{\beta\gamma}$ in such a way that the embedding splits, *i.e.* , that there is a $\widehat{\mathfrak{g}}$-module $\mathfrak{N}$ such that $\mathcal{H}_\varphi \otimes \mathcal{H}_{\beta\gamma} \cong \mathfrak{M} \oplus \mathfrak{N}$, then the above theorem implies that the BRST cohomology for non-zero ghost number vanishes:

$$H_d^{g\neq 0}(\mathfrak{M} \otimes \mathcal{H}_{\rm gh}) \cong 0\ , \tag{3.16}$$

which means that the complex $(\mathfrak{M} \otimes \mathcal{H}_{\rm gh}, d)$ provides a resolution of $H_d^0(\mathfrak{M} \otimes \mathcal{H}_{\rm gh})$ and thus allows the use of Lefschetz-trace-type formulas in order to compute the traces of operators on $H_d^0(\mathfrak{M} \otimes \mathcal{H}_{\rm gh})$ in terms of the alternating traces of their lifts to BRST invariant operators in $\mathfrak{M} \otimes \mathcal{H}_{\rm gh}$. It is not clear whether there are any interesting $\widehat{\mathfrak{g}}$-modules satisfying these properties. But even if the embedding does not split one can still take advantage of the vanishing of the BRST cohomology for the Fock module by first constructing a Fock resolution of the relevant $\widehat{\mathfrak{g}}$-module $\mathfrak{M}$. In the case of $A_1^{(1)}$ and $A_2^{(1)}$ (which are the only cases for which this construction has been explicitly worked out) the BRST operator intertwines between the Fock resolutions of the affine algebra and of the Casimir algebra. The resulting double complex is easy to analyze using standard techniques in homological algebra[16].

§4 Conclusions

In this paper we have outlined a quantum homological construction of the Casimir (extended conformal) algebra associated to a finite dimensional simple Lie algebra $\mathfrak{g}$ based on the quantization of the constrained system introduced by Drinfeld and Sokolov. We have found the stress tensor in the general case and have proven that, in the framework of a free field representation for the currents of the affine algebra $\widehat{\mathfrak{g}}$, it agrees with the one of $\ell \equiv \operatorname{rank} \mathfrak{g}$ free bosons with background charges.

We have computed the central charge of the Casimir algebras as a function of the level for arbitrary Lie algebra $\mathfrak{g}$ and have found, in the simply laced case and for certain values of the level, their discrete series. The unitary subseries agrees with the one suggested by the proposed coset construction. For the non simply-laced algebras we don't seem to recover the coset discrete series in the same fashion. Understanding why the construction is different in this case is, in our opinion, a very interesting and important open problem.

ACKNOWLEDGEMENTS

It is a pleasure to thank Peter Bouwknegt, Alvaro Díaz, Stany Schrans, Walter Troost, Toine Van Proeyen, and Dirk Verstegen for many useful conversations. I am particularly grateful to Dirk Verstegen for his ideas on how to compute the discrete series.

Appendix A The Construction of the Grading

In this appendix we prove the existence of the grading used in the proof of the theorem in section 3 by its explicit construction. Recall that the grading was defined on $C \equiv \mathcal{H}_{\beta\gamma} \otimes \mathcal{H}_{\rm gh}$ by setting $\deg \Omega_0 = 0$ and $\deg c_\alpha = \deg \gamma_\alpha = -\deg b_\alpha = -\deg \beta_\alpha = \lambda_\alpha$. We further demand that under this grading d breaks up as $d = \sum_i d_i$ with $\deg d_i = i$, some of which could be zero. This implies certain conditions on the $\{\lambda_\alpha\}$ which we now discuss.

From the expression (3.7) for the BRST current we see that apart from the term yielding d_0 upon integration, it contains terms of three different forms:

$$
\begin{aligned}
&(1) \quad c_\alpha, \quad \forall \alpha \in \Delta \\
&(2) \quad c_\alpha c_\beta b_{\alpha+\beta}, \quad \forall \alpha, \beta, \alpha+\beta \in \Phi_+ \\
&(3) \quad c_\alpha \gamma_{\alpha_i} \cdots \gamma_{\alpha_n} \beta_{\alpha+\alpha_1+\cdots+\alpha_n}, \quad \forall \alpha, \alpha_i, \alpha + \sum_i \alpha_i \in \Phi_+
\end{aligned}
$$

Demanding that the degree of the terms of type (1) and (2) have positive degree we find the following inequalities on the $\{\lambda_\alpha\}$:

$$\lambda_\alpha > 0 \qquad \forall\, \alpha \in \Delta \tag{A.1}$$

$$\lambda_\alpha + \lambda_\beta > \lambda_{\alpha+\beta} \qquad \forall\, \alpha, \beta, \alpha+\beta \in \Phi_+ \tag{A.2}$$

The inequalities obtained from type (3) terms are already contained in (A.2) since they are of the form $\sum_i \lambda_{\alpha_i} > \lambda_{\alpha_1+\cdots+\alpha_n}$.

These inequalities are to be supplemented by the requirement that deg be bounded above on C^h, the eigenspace of L_0 with eigenvalue h. C^h is spanned by monomials of the form (3.13) satisfying

$$\sum_{\substack{\alpha,p \\ \text{allowed}}} p\left(k_{\alpha,p} + k'_{\alpha,p} + l_{\alpha,p} + l'_{\alpha,p}\right) = h \ , \tag{A.3}$$

where we have used that for a primary field $\phi(z) = \sum_{p\in\mathbb{Z}} \phi_p z^{-p-\Delta}$ of weight Δ,

$[L_0\,,\,\phi_p] = -p\phi_p$. The degree of such a basis element is given by

$$\sum_{\substack{\alpha,p\\ \text{allowed}}} \lambda_\alpha \left(-k_{\alpha,p} - k'_{\alpha,p} + l_{\alpha,p} + l'_{\alpha,p}\right) \ . \tag{A.4}$$

Demanding that this be bounded on each C^h is clearly equivalent to the following inequality:

$$\sum_{\substack{\alpha,p\\ \text{allowed}}} \left[(-\lambda_\alpha - \mu p)\left(k_{\alpha,p} + k'_{\alpha,p}\right) + (\lambda_\alpha - \mu p)\left(l_{\alpha,p} + l'_{\alpha,p}\right)\right] \leq 0 \ , \tag{A.5}$$

for some μ. Since $k_{\alpha,p} + k'_{\alpha,p}$ and $l_{\alpha,p} + l'_{\alpha,p}$ can grow arbitrarily large, (A.5) is equivalent to the following two inequalities on the $\{\lambda_\alpha\}$:

$$-\lambda_\alpha \leq \mu p \qquad \forall\, p \geq 1 - \mathrm{ht}\,\alpha \tag{A.6}$$

$$\lambda_\alpha \leq \mu p \qquad \forall\, p \geq \mathrm{ht}\,\alpha \ , \tag{A.7}$$

which can only be true if μ is positive. Now, (A.6) is equivalent to $-\lambda_\alpha \leq \mu\,(1 - \mathrm{ht}\,\alpha)$ and (A.7) is equivalent to $\lambda_\alpha \leq \mu\,\mathrm{ht}\,\alpha$ which combine together into

$$\mu\,(\mathrm{ht}\,\alpha - 1) \leq \lambda_\alpha \leq \mu\,\mathrm{ht}\,\alpha \ . \tag{A.8}$$

For convenience let us define $\widetilde{\lambda}_\alpha = \lambda_\alpha/\mu$. In terms of these new variables the system of inequalities (A.1), (A.2), and (A.8) becomes:

$$\widetilde{\lambda}_\alpha > 0 \quad \forall\, \alpha \in \Delta \ , \tag{A.9}$$

$$\widetilde{\lambda}_\alpha + \widetilde{\lambda}_\beta > \widetilde{\lambda}_{\alpha+\beta} \quad \forall\, \alpha, \beta, \alpha+\beta \in \Phi_+ \ , \tag{A.10}$$

$$\mathrm{ht}\,\alpha - 1 \leq \widetilde{\lambda}_\alpha \leq \mathrm{ht}\,\alpha \quad \forall\, \alpha \in \Phi_+ \ . \tag{A.11}$$

Evidently $\widetilde{\lambda}_\alpha = \mathrm{ht}\,\alpha$ almost satisfies the second inequality. Thus it behooves us to choose the $\{\widetilde{\lambda}_\alpha\}$ just below $\mathrm{ht}\,\alpha$ at a distance which decreases as the height increases so that the second inequality can be satisfied. It is then obvious that there is no problem in satisfying everything. Basically it all comes down to the fact that there is a lot of room in the interval. A possible choice of the $\{\widetilde{\lambda}_\alpha\}$ is the following:

$$\widetilde{\lambda}_\alpha = \mathrm{ht}\,\alpha - 4^{\mathrm{ht}\,\alpha}\,\varepsilon \ , \tag{A.12}$$

where $\varepsilon = 4^{-(M+1)}$, M being the height of the maximal root. Choosing $\mu = 4^{M+1}$ we have that

$$\lambda_\alpha = 4^{M+1}\,\mathrm{ht}\,\alpha - 4^{\mathrm{ht}\,\alpha} \ , \tag{A.13}$$

which is an integer satisfying the system of inequalities (A.1), (A.2), (A.8). This proves the claim in section 3.

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