ON THE CASIMIR ALGEBRA OF B_2

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Abstract

Using the conformal bootstrap we explicitly construct WB_2, the Casimir algebra of B_2. This algebra contains, besides an extra dimension 4 field, also a fermionic dimension 5/2 field. We then construct the most general free field realization of this algebra using two free bosons and one free fermion. Finally, we derive the screening operators and show that they are related to the long and short root of B_2. Our construction hence proves several conjectures of the recent literature.

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1 Introduction

The classification of all two-dimensional Conformal Field Theories is one of the major goals in string theory and 2d critical phenomena. Since this turns out to be a tremendously difficult problem, one restricts oneself to study those CFTs with only a finite number of primary fields, *i.e.* the so-called Rational CFTs. However, as shown by Cardy, a CFT which is rational relative to the Virasoro algebra, necessarily has central charge $c < 1$ [1]. Therefore, in order to construct RCFTs with $c \geq 1$, one is led to extended conformal algebras.

There exists by now a wealth of examples of extended conformal algebras. Among the best known are the affine Lie algebras and the superconformal algebras. Another class of extended conformal algebras consists of W-algebras. These algebras are special in that they only close in their enveloping algebra: the (anti)commutators of the modes of the generators of the chiral algebra contain normal ordered products. Some examples of such W-algebras have already been constructed: the Bershadsky-Knizhnik algebras [2, 3], which are superconformal as well; W_3 [4], the spin 4 algebra [5, 6], the spin 6 algebra [7],..., and more recently, also supersymmetric W-algebras [8, 9, 10].

A general class of W-algebras have been conjectured to exist in [11, 12, 15]. They are related to the simple Lie algebras through the independent Casimirs of the latter. If the simple Lie algebra is simply-laced, the number of the primary generators of the corresponding "Casimir" algebra is equal to the number of the independent Casimirs of the former, *i.e.* its rank. The primary fields of the Casimir algebra have dimension equal to the order of these independent Casimirs.

In [12, 15] these algebras have been defined by their realization in terms of free fields, via a generalized Miura transformation. This leads directly to the construction of their minimal models and hence of the RCFTs with these algebras as a chiral algebra. As an example, the WA_n algebras contain besides the energy-momentum tensor, $n-1$ primary fields of dimension $3, 4, \ldots, n+1$. The central charge of the corresponding unitary minimal models is given by following discrete series [12]

$$c_{A_n} = n\left(1 - \frac{(n+1)(n+2)}{p(p+1)}\right), \tag{1}$$

where $p = n+2, n+3, \ldots$.

When the simple roots of the underlying algebra do not all have the same length, one has, according to [15], to introduce extra primary fields corresponding to the short roots. In the case of WB_n one has, besides the energy-momentum tensor and Virasoro primaries of dimension $4, 6, \ldots, 2n$ also a fermionic field of dimension $n + \frac{1}{2}$. The central charge of the unitary minimal models for these Casimir algebras[1] is given, for $n \geq 2$, by [15]

$$c_{B_n} = (n + \frac{1}{2}) \left(1 - \frac{(2n-1)2n}{p(p+1)} \right), \tag{2}$$

$p = 2n, 2n+1, \ldots$, and describes again RCFTs with $c \geq 1$. Since $B_1 \cong A_1$ is simply-laced, (2) is not valid for $n = 1$. Indeed, to obtain (2), one explicitly uses the fact that the algebra contains simple roots of different length.

It is important to realize that, although there is certainly some evidence for it [16], it has not yet been proven that these Casimir algebras indeed exist, *i.e.* that they are associative for generic values of the central charge. Evenmore, only very few of these algebras have already been constructed: the Virasoro and super Virasoro algebra correspond to WA_1 and WB_1, Zamolodchikov's W_3 algebra is WA_2 and more recently, WA_4 has been explicitly constructed in [17, 18, 19].

In this paper we will prove explicitly the existence of WB_2. Using the perturbative conformal bootstrap [20, 5, 9], we will prove that the algebra is associative for generic values of the central charge. The algebra will be written down in terms of quasiprimary families. We will construct the most general realization of this algebra with two free bosons and one free fermion and show that it is equivalent to the realization proposed by Fateev and Lukyanov [15]. Finally, using this realization, we check that the screening operators are related to the long and short root of B_2 leading to formulae for the degenerate representations.

2 WB_2 from the Conformal Bootstrap

In this section we will construct WB_2 explicitly using the conformal bootstrap approach [20], which we briefly discuss.

Because the local fields of a Conformal Field Theory assemble themselves into representations of the Virasoro algebra, the Operator Product Expansion of two Virasoro primary fields decomposes into Virasoro conformal families

$$\phi_m \times \phi_n \longrightarrow \sum_p C_{mn}{}^p [\phi_p]. \tag{3}$$

The conformal bootstrap is a powerful method to check associativity of this Operator Product Algebra. First, one determines the conformal family $[\phi_p]$ of every primary field by requiring conformal covariance, *i.e.* associativity with the energy-momentum tensor $T(z)$. The result can be written in terms of the inverse of the Šapovalov form on the Verma module generated by the action of this primary field on the $SL_2(\mathbb{C})$ invariant vacuum [7]. This fixes the operator algebra up to the couplings $C_{mn}{}^p$ between the primary fields. These couplings are then determined by requiring crossing symmetry of the four-point functions $G^{lk}_{nm}(x) = \langle k|\phi_l(1)\phi_n(x)|m\rangle$, which can be computed perturbatively around

[1] A classical version of WB_n arises from a Hamiltonian reduction of the superalgebra $B(0,n)$ rather than B_n [13, 14]. WB_n is, however, related to B_n through the root system of the latter (see later), and therefore, we will continue to refer to it as a Casimir algebra.

$x = 0$. The crossing symmetry constraints can be implemented using a group theoretical method due to Bouwknegt [5], that was generalized in [9]. It is this method we will use to construct WB_2 explicitly.

Since B_2 has two independent Casimirs, one of order two and one of order four, WB_2 contains, besides an energy-momentum tensor $T(z)$ corresponding to the second order Casimir, also a Virasoro primary field of dimension four, $W(z)$. Although there exists an operator algebra that is associative for generic values of the central charge with only this field content [5, 6], this is—in our terminology—not yet the Casimir algebra of B_2. To get WB_2, one has to introduce a primary weight 5/2 field $Q(z)$, which, in fact, generates the entire algebra [15]. This is reminescent of the fact that the Frenkel-Kač level one realization of the non-simply laced affine Lie algebras $\widehat{B}_n$, requires, due to the short root, the introduction of an extra fermion [21].

The OPA of WB_2 can be written schematically as

$$\begin{aligned} Q \times Q &\longrightarrow \frac{2c}{5}[0] + C_{\frac{5}{2}\frac{5}{2}}{}^{4}[W], \\ W \times W &\longrightarrow \frac{c}{4}[0] + C_{44}{}^{4}[W] + C_{44}{}^{6}[\Phi], \\ Q \times W &\longrightarrow C_{\frac{5}{2}4}{}^{\frac{5}{2}}[Q], \end{aligned} \tag{4}$$

with [0] the conformal family of the identity. Notice the symmetry property $C_{\frac{5}{2}\frac{5}{2}}{}^{4} = \frac{8}{5}C_{\frac{5}{2}4}{}^{\frac{5}{2}} = \frac{8}{5}C_{4\frac{5}{2}}{}^{\frac{5}{2}}$. Crucial for associativity of the OPA is the appearance of the new dimension 6 Virasoro primary $\Phi(z) = Q\partial Q(z) +$ corrections. Let us briefly discuss the solutions of the crossing symmetry constraints. Due to the appearance of $W(w)$ in the OPE $Q(z)Q(w)$, the crossing symmetry constraints for $\langle QQQQ \rangle$ don't fix the central charge—as in [4]—, but rather the coupling $C_{\frac{5}{2}\frac{5}{2}}{}^{4}$. Similarly $\langle WWWW \rangle$ leaves the central charge and one of the couplings $C_{44}{}^{4}$ or $C_{44}{}^{6}$ free. This is not surprising since the algebra generated by $\{T(z), W(z)\}$, *i.e.* $C_{44}{}^{6} = 0$, is well known to be associative for all values of c [5]. Finally, the mixed correlators fix the remaining coupling, leaving the central charge as a free parameter. Explicitly, one finds that the algebra is associative for all values of the central charge[2] provided the couplings are given by

$$\begin{aligned} C_{\frac{5}{2}\frac{5}{2}}{}^{4} &= \sqrt{\frac{6(14c+13)}{5c+22}}\,\epsilon_1, \\ C_{44}{}^{4} &= \frac{3\sqrt{6}(2c^2+83c-490)}{\sqrt{(14c+13)(5c+22)}(2c+25)}\,\epsilon_1, \\ C_{44}{}^{6} &= \frac{12\sqrt{5(6c+49)(4c+115)(c-1)}(5c+22)}{\sqrt{(14c+13)(7c+68)(2c-1)(c+24)}(2c+25)}\,\epsilon_2, \end{aligned} \tag{5}$$

where ϵ_1 and ϵ_2 are some arbitrary signs.

[2]Except, of course, for these c-values for which $C_{\frac{5}{2}\frac{5}{2}}{}^{4}$ and $C_{44}{}^{4}$ have poles; the extra poles in $C_{44}{}^{6}$ are cancelled by the zeros in the normalization constant $\mathcal{N}$ (13). Notice also that the coupling constants are imaginary for $-22/5 < c < -13/14$ [22].

Notice that for $c=-13/14$ the subalgebra generated by $\{T(z),Q(z)\}$ corresponds to the spin 5/2 algebra of Zamolodchikov [4].

For completeness, we now present the more complicated OPEs:

$$\begin{aligned} Q(z)Q(w) &= \frac{2c/5}{(z-w)^5}+\frac{2T(w)}{(z-w)^3}+\frac{\partial T(w)}{(z-w)^2} \\ &+\frac{\frac{3}{10}\partial^2 T(w)+\frac{27}{5c+22}\Lambda(w)+C_{\frac{5}{2}\frac{5}{2}}{}^4 W(w)}{z-w}+\text{regular} \end{aligned} \tag{6}$$

with, as usual[3],

$$\Lambda(z)=TT(z)-\frac{3}{10}\partial^2 T(z). \tag{7}$$

It is convenient to write $W(z)W(w)$ in the following form

$$W(z)W(w)=\frac{c/4}{(z-w)^8}+\sum_i \{\phi^{(i)}_{\Delta_i}\}(z|w), \tag{8}$$

where $\{\phi^{(i)}_{\Delta_i}\}(z|w)$ denotes the projective family of the quasiprimary fields $\phi^{(i)}_{\Delta_i}(w)$ and is given by

$$\{\phi^{(i)}_{\Delta_i}\}(z|w)=(z-w)^{-8+\Delta_i}\sum_{n\geq 0}(z-w)^n\alpha^i_n\partial^n\phi^{(i)}_{\Delta_i}(w), \tag{9}$$

with

$$\alpha^i_0=1, \qquad \alpha^i_n=\prod_{j=1}^{n}\frac{j+\Delta_i-1}{j^2+j(2\Delta_i-1)} \tag{10}$$

and Δ_i the weight ($\equiv L_0$ eigenvalue) of $\phi^{(i)}_{\Delta_i}(w)$. The quasiprimary fields $\phi^{(i)}_{\Delta_i}(w)$ appearing in the singular part of the Operator Product Expansion $W(z)W(w)$ are given by

$$\begin{aligned} \phi^{(1)}_2 &= 2T, \qquad \phi^{(2)}_4=\frac{42}{5c+22}\Lambda, \\ \phi^{(3)}_6 &= -\frac{95c^2+1254c-10904}{6(7c+68)(5c+22)(2c-1)}\left[\partial T\partial T-\frac{4}{5}\partial^2 TT-\frac{1}{42}\partial^4 T\right], \\ \phi^{(4)}_6 &= \frac{24(72c+13)}{(7c+68)(5c+22)(2c-1)}\left[(T(TT))-\frac{9}{10}\partial^2 TT-\frac{1}{28}\partial^4 T\right], \\ \phi^{(5)}_4 &= C_{44}{}^4 W, \quad \phi^{(6)}_6=-\frac{14}{9(c+24)}C_{44}{}^4\left[\partial^2 W-6TW\right], \\ \phi^{(7)}_6 &= C_{44}{}^6\Phi. \end{aligned} \tag{11}$$

Finally, $\Phi(z)$ is the unique (up to a normalization) dimension 6 Virasoro primary given by

$$\begin{aligned} \Phi(z) &= \mathcal{N}\left[Q\partial Q(z)+\alpha_1\partial^2 W(z)+\alpha_2 TW(z)+\alpha_3\partial T\partial T(z)\right. \\ &\left.+\alpha_4 T\partial^2 T(z)+\alpha_5\partial^4 T(z)+\alpha_6(T(TT))(z)\right], \end{aligned} \tag{12}$$

[3]The normal ordering used is the usual point splitting regularization.

where the normalization factor

$$\mathcal{N} = \sqrt{\frac{(2c-1)(c+24)(7c+68)}{5(c-1)(4c+115)(6c+49)(14c+13)}} \tag{13}$$

is fixed such that $\langle\Phi(z)\Phi(w)\rangle = (c/6)(z-w)^{-12}$ and the coefficients $\alpha_1, \ldots, \alpha_6$ are

$$\begin{array}{lcllcl}
\alpha_1 & = & -\frac{13c+350}{36(c+24)} C_{\frac{5}{2}\frac{5}{2}}{}^4, & \alpha_2 & = & \frac{19}{3(c+24)} C_{\frac{5}{2}\frac{5}{2}}{}^4, \\
\alpha_3 & = & -6(566c^2 + 5295c - 3998)/N, & \alpha_4 & = & -6(530c^2 + 6141c - 1487)/N, \\
\alpha_5 & = & -(46c^3 - 125c^2 - 6235c + 1778)/N, & \alpha_6 & = & 12(734c + 49)/N,
\end{array}$$

with $N = 12(7c+68)(5c+22)(2c-1)$.

Notice that the fermionic primary $Q(z)$ can be viewed as some kind of "generalized supersymmetry" generator. The appearance of the dimension four field is then similar to *e.g.* the appearance of the affine $\widehat{\mathrm{u}}(1)$ current in the $N = 2$ super Virasoro algebra. Similarly, one can view the WB_n algebras as some "generalized supersymmetry" algebras, the rôle of the higher spin fields being to ensure associativity of the OPA for generic values of the central charge.

3 Coulomb Gas Realization

The next step in the analysis of the WB_2 algebra consists in constructing a Coulomb gas realization, *i.e.* a realization of the primary fields of the OPA in terms of free fields. In order to find this realization, one needs two free bosons and one free fermion. The propagators for the free fields are defined to be[4]

$$\begin{aligned}
\langle\varphi_i(z)\varphi_j(w)\rangle &= \delta_{ij}\ln(z-w), \\
\langle\psi(z)\psi(w)\rangle &= \frac{1}{z-w}.
\end{aligned} \tag{14}$$

The energy-momentum tensor is simply the free energy-momentum tensor of the two free bosons with background charge and the free fermion. Due to rotational invariance, one can always transform the background charge into one direction and hence we take

$$T(z) = \frac{1}{2}\partial\varphi_1\partial\varphi_1(z) + \frac{1}{2}\partial\varphi_2\partial\varphi_2(z) + \alpha_0\partial^2\varphi_1(z) + \frac{1}{2}\partial\psi\psi(z), \tag{15}$$

which satisfies a Virasoro algebra with central charge

$$c = \frac{5}{2} - 12\alpha_0^2. \tag{16}$$

It is a long and boring task to find the explicit form of the dimension 4 and dimension 5/2 primaries. Therefore we developed a program in *Mathematica*^{TM},

[4] We have chosen the positive sign for the boson propagator in order to avoid factors of i.

that is able to compute, given OPEs of a set of "basic" fields, the OPE of any two normal ordered products of these fields [23]. Using this program, one can try to construct the most general primary dimension four field and require the $W(z)W(w)$ operator product expansion to be satisfied. This, however, leads to a system of quadratic equations which is very difficult to solve. A somewhat easier way is to construct the most general primary dimension 5/2 field. This leads to a four parameter family of such primaries. The $Q(z)Q(w)$ operator product expansion gives (up to some discrete automorphisms) three possible solutions for these parameters, and hence also three candidate dimension four primary fields. Finally, matching the $W(z)W(w)$ OPE eliminates two of the solutions, and yields a unique construction. We wish to stress that we have checked all these statements explicitly, including the appearance of the dimension 6 primary mentioned earlier.

Let us now present the explicit solution[5]. The dimension 5/2 field is given by

$$\begin{aligned} Q(z) &= \xi \Big[\frac{3}{2}\partial\varphi_1\partial\varphi_1\psi(z) - \frac{3}{2}\partial\varphi_2\partial\varphi_2\psi(z) + 4\partial\varphi_1\partial\varphi_2\psi(z) + \alpha_0\partial^2\varphi_1\psi(z) \\ &\quad +3\alpha_0\partial^2\varphi_2\psi(z) + 4\alpha_0\partial\varphi_1\partial\psi(z) + 2\alpha_0\partial\varphi_2\partial\psi(z) + 2\alpha_0^2\partial^2\psi(z)\Big], \end{aligned} \tag{17}$$

where $\xi = 1/\sqrt{5(5-4\alpha_0^2)}$. The dimension 4 field is somewhat more complicated

$$\begin{aligned} W(z) &= \sigma\big[N^{ijkl}\partial\varphi_i\partial\varphi_j\partial\varphi_k\partial\varphi_l(z) + N^{ijk}\partial^2\varphi_i\partial\varphi_j\partial\varphi_k(z) + N^{ij}\partial^3\varphi_i\partial\varphi_j(z) \\ &\quad +\tilde{N}^{ij}\partial^2\varphi_i\partial^2\varphi_j(z) + N^i\partial^4\varphi_i(z) + M^{ij}\partial\varphi_i\partial\varphi_j\partial\psi\psi(z) + M^i\partial^2\varphi_i\partial\psi\psi(z) \\ &\quad +\tilde{M}^i\partial\varphi_i\partial^2\psi\psi(z) + K_1\partial^3\psi\psi(z) + K_2\partial^2\psi\partial\psi(z)\Big], \end{aligned} \tag{18}$$

with N^{ijkl}, $\tilde{N}^{ij}$ and M^{ij} completely symmetric and N^{ijk} symmetric in the last two indices. The coefficients appearing in (18) are given explicitly by

$$\begin{array}{lcllcl}
N^{1111} &=& N^{2222} = 81/80, & N^{1112} &=& -N^{1222} = -\mu/20, \\
N^{1122} &=& (560\alpha_0^2-79)/720, & & & \\
N^{111} &=& 81\alpha_0/20, & N^{112} &=& -7\alpha_0\mu/60, \\
N^{122} &=& \alpha_0(80\alpha_0^2+197)/60, & N^{211} &=& -11\alpha_0\mu/30, \\
N^{212} &=& -\alpha_0\mu/10, & N^{222} &=& \alpha_0\mu/5, \\
N^{11} &=& (128\alpha_0^2-25)/60, & N^{12} &=& -\alpha_0^2\mu/30, \\
N^{21} &=& -\alpha_0^2\mu/5, & N^{22} &=& (80\alpha_0^4+82\alpha_0^2-25)/60, \\
\tilde{N}^{11} &=& (34\alpha_0^2+25)/40, & \tilde{N}^{12} &=& -\alpha_0^2\mu/15, \\
\tilde{N}^{22} &=& (80\alpha_0^4-174\alpha_0^2+25)/40, & & & \\
N^1 &=& \alpha_0(128\alpha_0^2-25)/360, & N^2 &=& -\alpha_0^3\mu/30, \\
M^{11} &=& M^{22} = (82\alpha_0^2-35)/6, & M^{12} &=& 0, \\
M^1 &=& \alpha_0(2\alpha_0^2+11)/3, & M^2 &=& \alpha_0\mu/3, \\
\tilde{M}^1 &=& -\alpha_0\mu/3, & \tilde{M}^2 &=& -\alpha_0\mu/6, \\
K_1 &=& (36\alpha_0^4-22\alpha_0^2+5)/18, & K_2 &=& (52\alpha_0^4-26\alpha_0^2-15)/6,
\end{array}$$

[5]By convention, normal ordering is always from the right to the left, *e.g.* $\partial\varphi_i\partial\varphi_j\partial\varphi_k = (\partial\varphi_i(\partial\varphi_j\partial\varphi_k))$.

where

$$\sigma = \sqrt{\frac{3}{2(2-7\alpha_0^2)(23-40\alpha_0^2)}\frac{1}{4\alpha_0^2-5}}, \quad \mu = 23 - 40\alpha_0^2. \tag{19}$$

This solution corresponds to the sign choices $\epsilon_1 = \epsilon_2 = +1$ in (5).

Some remarks are in order.

The primary dimension 5/2 field $Q(z)$ can be rewritten in a more suggestive form. Indeed, rotating

$$\begin{aligned}\bar{\varphi}_1(z) &= \frac{1}{\sqrt{10}}(3\varphi_1(z) - \varphi_2(z)),\\ \bar{\varphi}_2(z) &= \frac{1}{\sqrt{10}}(\varphi_1(z) + 3\varphi_2(z)),\end{aligned} \tag{20}$$

one can rewrite

$$Q(z) = 5\left[\sqrt{\frac{2}{5}}\alpha_0\partial + \partial\bar{\varphi}_1(z)\right]\left[\sqrt{\frac{2}{5}}\alpha_0\partial + \partial\bar{\varphi}_2(z)\right]\psi(z), \tag{21}$$

which is exactly the starting point of the analysis of Fateev and Lukyanov [15]. Our construction hence proves that the algebra generated by (21) is indeed associative for all values of the central charge, as was conjectured in [15].

A second remark concerns the fact that there is (up to some discrete automorphisms) only one free field realization of WB_2 with two free bosons and one free fermion. In the case of $W_3 \equiv WA_2$, Fateev and Zamolodchikov found two inequivalent free field realizations with two free bosons [24]. The simplest one was related to $\mathfrak{su}(3)$ and led Fateev and Lukyanov to generalize this to the WA_n algebras using n free bosons [12], while the more complicated one has been shown to be related to parafermions (at least for a specific value of the central charge) [25]. In fact it was argued in [25] that such a realization exists (for fixed c) for all WA_n algebras, and that in a limit $n \to \infty$ it corresponds to the $c = 2$ free field realization of W_∞ by Bakas and Kiritsis [26]. For WB_2, there does not seem to be a similar construction.

4 Degenerate Representations

Degenerate representations are highest weight representations of the WB_2 algebra that contain at least two independent null vectors in their Verma modules [15]. These representations are not necessarily completely degenerate. In Virasoro CFT a highest weight vector is created by the action of a Virasoro primary field on the projective invariant vacuum.

There is, however, no unique definition of what a W-primary field is. In the case of Virasoro CFT, the first order pole of the OPE of a primary field with the energy-momentum tensor is required to be the derivative of the primary field. This extra requirement is not needed for constructing highest weight representations; it arises because conformal transformations have a geometrical interpretation, *viz.* as diffeomorphisms of S^1. There is, however, no known geometrical interpretation for the transformations of the currents generating a nonlinear algebra, and, therefore, one cannot—at present—give a consistent definition of primary fields w.r.t. a W-algebra[6]. Nevertheless, in the Neveu-Schwarz representation, for a field $\phi_\lambda(z)$ to create a WB_2 highest-weight state when acting on the projective invariant vacuum, the following operator product expansions must be valid.

$$\begin{aligned} T(z)\phi_\lambda(w) &= \frac{\Delta_\lambda \phi_\lambda(w)}{(z-w)^2} + \cdots , \\ W(z)\phi_\lambda(w) &= \frac{\omega_\lambda \phi_\lambda(w)}{(z-w)^4} + \cdots , \\ Q(z)\phi_\lambda(w) &= \frac{\psi_\lambda(w)}{(z-w)^2} + \cdots . \end{aligned} \tag{22}$$

Here the dots mean lower order poles + regular terms. We will call, for convenience, such a field $\phi_\lambda(z)$ a WB_2-preprimary field.

Examples of WB_2-preprimary fields are easily constructed in the Coulomb gas realization. They are given by exponentials of the free bosons

$$V_{\vec\beta}(z) = e^{\vec\beta.\vec\varphi}(z), \tag{23}$$

with $\vec\beta \equiv (\beta_1, \beta_2)$ and $\vec\varphi(z) \equiv (\varphi_1(z), \varphi_2(z))$. $V_{\vec\beta}(z)$ has Virasoro dimension

$$\Delta_{\vec\beta} = \frac{1}{2}\beta_1^2 + \frac{1}{2}\beta_2^2 - \beta_1\alpha_0, \tag{24}$$

and its W-weight can be written as

$$w_{\vec\beta} = \frac{\sigma}{480}\left[(40\alpha_0^2 - 23)\left(\prod_{\mathfrak{w}\in\mathcal{W}} \mathfrak{w}(\vec\beta - \vec\alpha_0).\vec\rho\right)^{1/2} + 8(128\alpha_0^2 - 25)\Delta_{\vec\beta} + 1944\Delta_{\vec\beta}^2\right]. \tag{25}$$

Here we have introduced the root system of B_2: $\vec\rho = \frac{1}{2}(3\vec e_L + 4\vec e_S)$ is half the sum of the positive roots, taking as simple roots $\vec e_L \equiv \sqrt{\frac{2}{5}}(1, -2)$ and $\vec e_S \equiv \sqrt{\frac{1}{10}}(1, 3)$, which are related to the canonical choice of the long and short root of B_2 by a Weyl reflection, followed by a rotation. $\mathcal{W}$ denotes the Weyl group of B_2 and $\vec{\alpha_0} \equiv (\alpha_0, 0)$ is parallel to $\vec\rho$.

Of course, this realization provides us with an explicit example of what should be called a W-primary field. In the case at hand, one finds at the third order pole, besides the Virasoro descendants of $V_{\vec\beta}(z)$ a new Virasoro primary field, proportional to $(\beta_2\partial\varphi_1 + (2\alpha_0 - \beta_1)\partial\varphi_2))V_{\vec\beta}(z)$. As mentioned before, we have no intrinsically geometric way to express this new Virasoro primary as well as the ones appearing in the second and first order poles.

[6] This lack of geometrical interpretation also seems to jeopardize the possibility of constructing manifestly W-covariant CFTs.

The weights $\Delta_{\vec{\beta}}$ and $w_{\vec{\beta}}$ clearly enjoy the following symmetry properties: $\Delta_i[\vec{\beta}] = \Delta_i[\mathfrak{w}(\vec{\beta} - \vec{\alpha_0}) + \vec{\alpha_0}]$ with $\mathfrak{w}$ an arbitrary element of $\mathcal{W}$ and $\Delta_1[\vec{\beta}] = \Delta_{\vec{\beta}}$ and $\Delta_2[\vec{\beta}] = w_{\vec{\beta}}$ [15].

Screening operators are characterized by the fact that the singular part of their OPE with any of the basic currents $X(z) \in \{T(z), W(z), Q(z)\}$ can be written as a total derivative

$$X(z)\phi(w) = \sum_n \frac{[X\phi]_n(w)}{(z-w)^n} = \frac{d}{dw}[\text{something}] + \text{regular}\,. \tag{26}$$

This is equivalent to $[\phi X]_1 = 0$. Using the realization of the previous section, one can construct two different kinds of screening operators. They can be written in such a way that their relation with the root system of B_2 is manifest and are given by $V^L_\pm(z) = \exp(\beta_\pm \vec{e}_L.\vec{\varphi})(z)$ and $V^S_\pm(z) = \psi \exp(\beta_\pm \vec{e}_S.\vec{\varphi})(z)$ with $\beta_+ + \beta_- = \sqrt{\frac{2}{5}}\alpha_0$ and $\beta_+\beta_- = -1$. They are thus seen to be equal to the screening charges presented in [15].

Given the screening operators, it is a standard construction to derive the degenerate representations, see [27, 24, 15, 28] to which we refer for details.

5 Conclusion

In this paper we have proven the existence, for generic c, of the Casimir algebra of B_2 by explicitly constructing it using the conformal bootstrap. Using a Coulomb gas realization in terms of two free bosons and one free fermion, we have been able to show the equivalence with the results conjectured by Fateev and Lukyanov [15] and, finally, using the explicit form of the realization we derived the screening charges, which lead, in turn, to the degenerate representations of WB_2.

It is clear from the explicit form of the algebra, that to prove the existence of general WB_n algebras one will have to recur to other methods.

Let us finally remark that the Casimir algebras provide, compared to other W-algebras, very tractable CFTs. Indeed, not only are they defined by a free field realization, but it was also conjectured that they arise as certain coset models from affine Lie algebras [11]. These two constructions provide powerful means to analyse their representation theory. For more general W-algebras, on the contrary, no such constructions exist, at present. In fact, we have checked explicitly, that the spin four algebra generated by the energy-momentum tensor and a dimension four field [5], does not have a free field realization in terms of two free bosons only for generic values of the central charge. Similarly, one should not be surprised if the spin 6 algebra [7] does not have a free field realization in terms of only two free bosons, at least for generic values of the central charge. It is still an open question whether these algebras can be realized by free fields at all.

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