Hyperbolic monopoles and supersymmetry

José Miguel Figueroa O’Farrill

hep-th seminar
Tōkyō, 12 December 2013
Supersymmetry

- Supersymmetry relates
Supersymmetry

- Supersymmetry relates
  - fermions: (usually) satisfy 1st order field equations
Supersymmetry

- Supersymmetry relates
  - **fermions**: (usually) satisfy 1st order field equations
  - **bosons**: (usually) satisfy 2nd order field equations
Supersymmetry

- Supersymmetry relates
  - fermions: (usually) satisfy 1st order field equations
  - bosons: (usually) satisfy 2nd order field equations

- It should not come as a surprise to find supersymmetry whenever 1st-order equations imply 2nd-order equations
Supersymmetry

- Supersymmetry relates
  - fermions: (usually) satisfy 1st order field equations
  - bosons: (usually) satisfy 2nd order field equations
- It should not come as a surprise to find supersymmetry whenever 1st-order equations imply 2nd-order equations

A metamathematical principle?
Supersymmetry underlies any situation where
Supersymmetry

- Supersymmetry relates
  - **fermions**: (usually) satisfy 1st order field equations
  - **bosons**: (usually) satisfy 2nd order field equations

- It should **not** come as a surprise to find supersymmetry whenever 1st-order equations imply 2nd-order equations

**A metamathematical principle?**

Supersymmetry underlies any situation where

- 1st order PDE implies 2nd order PDE
Supersymmetry

- Supersymmetry relates
  - **fermions**: (usually) satisfy 1st order field equations
  - **bosons**: (usually) satisfy 2nd order field equations
- It should **not** come as a surprise to find supersymmetry whenever 1st-order equations imply 2nd-order equations

A metamathematical principle?

Supersymmetry underlies any situation where
- 1st order PDE implies 2nd order PDE
- solutions of the 1st order PDE are **optimal** among all solutions of the 2nd order PDE
Well-known examples

- **calibrated geometry**

  calibrated $\implies$ minimal

  and calibrated submanifolds are **volume-minimizing** in their homology class
Well-known examples

- calibrated geometry

  calibrated $\implies$ minimal

  and calibrated submanifolds are **volume-minimizing** in their homology class

- instantons

  (anti)self-dual $\implies$ Yang–Mills

  and (A)SD gauge fields saturate the topological bound
Well-known examples

- **calibrated geometry**
  
  calibrated $\implies$ minimal

  and calibrated submanifolds are **volume-minimizing** in their homology class

- **instantons**
  
  (anti)self-dual $\implies$ Yang–Mills

  and (A)SD gauge fields saturate the topological bound

- **monopoles**
  
  Bogomol’nyi $\implies$ Yang–Mills–Higgs

  and Bogomol’nyi monopoles saturate the topological bound
In this talk

- Bogomol’nyi monopoles in hyperbolic space
In this talk

- Bogomol’nyi monopoles in hyperbolic space
- Reformulation as BPS configurations in a supersymmetric Yang–Mills–Higgs theory on hyperbolic space
In this talk

- Bogomol’nyi monopoles in hyperbolic space
- Reformulation as BPS configurations in a supersymmetric Yang–Mills–Higgs theory on hyperbolic space
- Determination of the geometry of the monopole moduli space

Based on joint work ([1311.3588]) with Moustafa Gharamti
In this talk

- Bogomol’nyi monopoles in hyperbolic space
- Reformulation as BPS configurations in a supersymmetric Yang–Mills–Higgs theory on hyperbolic space
- Determination of the geometry of the monopole moduli space
- Based on joint work (1311.3588) with Moustafa Gharamti
Outline of talk

1. Hyperbolic monopoles
Outline of talk

1. Hyperbolic monopoles
2. Supersymmetric Yang–Mills–Higgs in hyperbolic space
Outline of talk

1. Hyperbolic monopoles
2. Supersymmetric Yang–Mills–Higgs in hyperbolic space
3. The geometry of the monopole moduli space
Outline of talk

1. Hyperbolic monopoles
2. Supersymmetric Yang–Mills–Higgs in hyperbolic space
3. The geometry of the monopole moduli space
4. Conclusions and future directions
1 Hyperbolic monopoles

2 Supersymmetric Yang–Mills–Higgs in hyperbolic space

3 The geometry of the monopole moduli space

4 Conclusions and future directions
Monopoles

- The Bogomol’nyi equation in $\mathbb{R}^3$ is

$$d_A \phi = - \star F_A$$

where
Monopoles

- The Bogomol’nyi equation in $\mathbb{R}^3$ is

$$d_A \phi = - \star F_A$$

where

- $A$ is a connection on a principal $G$ bundle $P$ over $\mathbb{R}^3$ and
- $F_A = dA + \frac{1}{2} [A, A]$ its curvature
Monopoles

- The Bogomol’nyi equation in $\mathbb{R}^3$ is

$$\nabla_A \phi = - \ast F_A$$

where

- $A$ is a connection on a principal $G$ bundle $P$ over $\mathbb{R}^3$ and $F_A = dA + \frac{1}{2} [A, A]$ its curvature
- The Higgs field $\phi$ is a section of the adjoint bundle $adP$ over $\mathbb{R}^3$ satisfying suitable boundary conditions which ensure that the $L^2$ norm of $F_A$ is finite
Monopoles

- The Bogomol’nyi equation in \( \mathbb{R}^3 \) is

\[
d_A \phi = - \star F_A
\]

where

- \( A \) is a connection on a principal \( G \) bundle \( P \) over \( \mathbb{R}^3 \) and \( F_A = dA + \frac{1}{2} [A, A] \) its curvature
- The Higgs field \( \phi \) is a section of the adjoint bundle \( \text{ad} P \) over \( \mathbb{R}^3 \) satisfying suitable boundary conditions which ensure that the \( L^2 \) norm of \( F_A \) is finite
- \( d_A \phi = d\phi + [A, \phi] \)
Monopoles

- The Bogomol’nyi equation in $\mathbb{R}^3$ is

$$d_A \phi = - \star F_A$$

where

- $A$ is a connection on a principal $G$ bundle $P$ over $\mathbb{R}^3$ and $F_A = dA + \frac{1}{2} [A, A]$ its curvature
- The Higgs field $\phi$ is a section of the adjoint bundle $ad P$ over $\mathbb{R}^3$ satisfying suitable boundary conditions which ensure that the $L^2$ norm of $F_A$ is finite
- $d_A \phi = d\phi + [A, \phi]$
- $\star$ is the Hodge star operator
Monopoles

- The Bogomol’nyi equation in $\mathbb{R}^3$ is

$$d_A \phi = - \star F_A$$

where

- $A$ is a connection on a principal $G$ bundle $P$ over $\mathbb{R}^3$ and

$$F_A = dA + \frac{1}{2} [A, A]$$

its curvature

- The Higgs field $\phi$ is a section of the adjoint bundle $ad P$ over $\mathbb{R}^3$ satisfying suitable boundary conditions which ensure that the $L^2$ norm of $F_A$ is finite

- $d_A \phi = d\phi + [A, \phi]$

- $\star$ is the Hodge star operator

- A pair $(A, \phi)$ satisfying the Bogomol’nyi equation is called an **euclidean monopole**
Translationally invariant instantons

- Interpret $\phi$ as the fourth component of a connection $A$ in $\mathbb{R}^4$
Translationally invariant instantons

- Interpret $\phi$ as the fourth component of a connection $A$ in $\mathbb{R}^4$.
- If $A$ is now independent of the 4th coordinate, the Bogomol’nyi equation becomes the self-duality equation on $\mathbb{R}^4$:

$$F_A = \star F_A$$

where $\star$ is now the Hodge star in $\mathbb{R}^4$. 
Translationally invariant instantons

- Interpret $\phi$ as the fourth component of a connection $A$ in $\mathbb{R}^4$
- If $A$ is now independent of the 4th coordinate, the Bogomol'nyi equation becomes the self-duality equation on $\mathbb{R}^4$

$$F_A = \star F_A$$

where $\star$ is now the Hodge star in $\mathbb{R}^4$
- In other words, euclidean monopoles are translationally invariant instantons
Hyperbolic monopoles

Hyperbolic monopoles are solutions to the Bogomol’nyi equation in hyperbolic space $H^3$
Hyperbolic monopoles

- Hyperbolic monopoles are solutions to the Bogomol’nyi equation in hyperbolic space $H^3$
- They can be constructed from rotationally invariant instantons

Atiyah (1984)
Hyperbolic monopoles

- Hyperbolic monopoles are solutions to the Bogomol’nyi equation in hyperbolic space $H^3$
- They can be constructed from rotationally invariant instantons
- Write the euclidean metric in $\mathbb{R}^4$

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

using polar coordinates in the $(x_3, x_4)$ plane:

$$ds^2 = dx_1^2 + dx_2^2 + dr^2 + r^2 d\theta^2$$
Hyperbolic monopoles

- Hyperbolic monopoles are solutions to the Bogomol’nyi equation in hyperbolic space $\mathbb{H}^3$
- They can be constructed from rotationally invariant instantons
- Write the euclidean metric in $\mathbb{R}^4$

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

using polar coordinates in the $(x_3, x_4)$ plane:

$$ds^2 = dx_1^2 + dx_2^2 + dr^2 + r^2 d\theta^2$$

Atiyah (1984)
Hyperbolic monopoles

- Hyperbolic monopoles are solutions to the Bogomol’nyi equation in hyperbolic space $\mathbb{H}^3$
- They can be constructed from rotationally invariant instantons
- Write the euclidean metric in $\mathbb{R}^4$

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

using polar coordinates in the $(x_3, x_4)$ plane:

$$ds^2 = dx_1^2 + dx_2^2 + dr^2 + r^2 d\theta^2 = r^2 \left( \frac{dx_1^2 + dx_2^2 + dr^2}{r^2} + \frac{d\theta^2}{s^1} \right)$$
Rotationally invariant instantons

- This description is valid in the complement of the 2-plane
  \[ x_3 = x_4 = 0 \]
Rotationally invariant instantons

- This description is valid in the complement of the 2-plane $x_3 = x_4 = 0$
- We see that $\mathbb{R}^4 \setminus \mathbb{R}^2$ is conformal to $H^3 \times S^1$
Rotationally invariant instantons

- This description is valid in the complement of the 2-plane $x_3 = x_4 = 0$
- We see that $\mathbb{R}^4 \setminus \mathbb{R}^2$ is conformal to $\mathbb{H}^3 \times S^1$
- The self-duality equation is conformally invariant, so instantons on $\mathbb{R}^4 \setminus \mathbb{R}^2$ are in one-to-one correspondence with instantons on $\mathbb{H}^3 \times S^1$
Rotationally invariant instantons

- This description is valid in the complement of the 2-plane $x_3 = x_4 = 0$
- We see that $\mathbb{R}^4 \setminus \mathbb{R}^2$ is conformal to $H^3 \times S^1$
- The self-duality equation is conformally invariant, so instantons on $\mathbb{R}^4 \setminus \mathbb{R}^2$ are in one-to-one correspondence with instantons on $H^3 \times S^1$
- Instantons on $\mathbb{R}^4 \setminus \mathbb{R}^2$ invariant under rotations in the $(x_3, x_4)$ plane give solutions of the Bogomol’nyi equation in $H^3$

\[ d_A \phi = - \star F_A \]

where $\phi$ is the $\theta$-component of $A$
Mass and charge

- Hyperbolic monopoles are determined by their mass:
  $$ m \in \mathbb{R}^+ $$

  $$ m = \lim_{r \to \infty} |\phi(r)| $$
Mass and charge

- Hyperbolic monopoles are determined by their **mass**
  \[ m \in \mathbb{R}^+ \]
  \[ m = \lim_{r \to \infty} |\phi(r)| \]

- and their **charge** \( k \in \mathbb{Z}^+ \)
  \[ k = \lim_{r \to \infty} \frac{1}{4\pi m} \int_{H^3} \text{tr}(F_A \wedge d_A \phi) \]
Mass and charge

- Hyperbolic monopoles are determined by their mass
  \[ m \in \mathbb{R}^+ \]
  \[ m = \lim_{r \to \infty} |\phi(r)| \]

- and their charge \( k \in \mathbb{Z}^+ \)
  \[ k = \lim_{r \to \infty} \frac{1}{4\pi m} \int_{H^3} \text{tr}(F_A \wedge d_A \phi) \]

- They exist for all values of \( m \) and \( k \) \textbf{Sibner+Sibner (2012)}
Mass and charge

- Hyperbolic monopoles are determined by their **mass**
  \[ m \in \mathbb{R}^+ \]
  \[ m = \lim_{r \to \infty} |\phi(r)| \]

- and their **charge** \( k \in \mathbb{Z}^+ \)
  \[ k = \lim_{r \to \infty} \frac{1}{4\pi m} \int_{H^3} \text{tr}(F_A \wedge d_A \phi) \]

- They exist for all values of \( m \) and \( k \) \textbf{Sibner+Sibner (2012)}

- We can rescale the mass to unity, but this changes the curvature of \( H^3 \) from \(-1\) to \(-1/m^2\)
Mass and charge

- Hyperbolic monopoles are determined by their mass
  \[ m \in \mathbb{R}^+ \]
  \[ m = \lim_{r \to \infty} |\phi(r)| \]

- and their charge \( k \in \mathbb{Z}^+ \)
  \[ k = \lim_{r \to \infty} \frac{1}{4\pi m} \int_{H^3} \text{tr}(F_A \wedge d_A \phi) \]

- They exist for all values of \( m \) and \( k \) \( \text{Sibner+Sibner (2012)} \)
- We can rescale the mass to unity, but this changes the curvature of \( H^3 \) from \(-1\) to \(-1/m^2\)
- A hyperbolic monopole extends to a rotationally invariant instanton on all of \( \mathbb{R}^4 \) if and only if \( m \in \mathbb{Z} \)
Moduli space of euclidean monopoles

- Low-energy dynamics of euclidean monopoles = geodesic motion on the moduli space $\mathcal{M}$

Manton (1982)
Moduli space of euclidean monopoles

- Low-energy dynamics of euclidean monopoles = geodesic motion on the moduli space $\mathcal{M}$
- $\mathcal{M}$ is the space of solutions modulo gauge equivalence

Manton (1982)
Moduli space of euclidean monopoles

- Low-energy dynamics of euclidean monopoles = geodesic motion on the moduli space $\mathcal{M}$
  - $\mathcal{M}$ is the space of solutions modulo gauge equivalence
  - $\mathcal{M}$ inherits a metric from the $L^2$ metric on the space of solutions of the linearised Bogomol’nyi equation

Manton (1982)
Moduli space of euclidean monopoles

- Low-energy dynamics of euclidean monopoles = geodesic motion on the moduli space $M$
  - $M$ is the space of solutions modulo gauge equivalence
  - $M$ inherits a metric from the $L^2$ metric on the space of solutions of the linearised Bogomol’nyi equation
  - this metric is hyperkähler

Atiyah+Hitchin (1984)
Moduli space of euclidean monopoles

- Low-energy dynamics of euclidean monopoles = geodesic motion on the moduli space $\mathcal{M}$  
  \[ Manton (1982) \]
- $\mathcal{M}$ is the space of solutions modulo gauge equivalence
- $\mathcal{M}$ inherits a metric from the $L^2$ metric on the space of solutions of the linearised Bogomol’nyi equation
- this metric is hyperkähler  
  \[ Atiyah+Hitchin (1984) \]
- the metric for $k = 2$ is known explicitly, as is the metric for well-separated monopoles  
  \[ Manton (1985) \]
Moduli space of hyperbolic monopoles (I)

Let $\mathcal{M}_{k,m}$ be the moduli space of monopoles on $H^3$ with charge $k$ and mass $m$. 
Moduli space of hyperbolic monopoles (I)

- Let $\mathcal{M}_{k,m}$ be the moduli space of monopoles on $H^3$ with charge $k$ and mass $m$.
- The moduli space $\mathcal{M}_{k,m}$ is diffeomorphic to the space of rational maps (for $k \geq 1$)

$$
\frac{a_1 z^{k-1} + a_2 z^{k-2} + \cdots + a_k}{z^k + b_1 z^{k-1} + \cdots + b_k}
$$

where numerator and denominator polynomials are coprime.

Atiyah (1984)
Moduli space of hyperbolic monopoles (I)

- Let $\mathcal{M}_{k,m}$ be the moduli space of monopoles on $H^3$ with charge $k$ and mass $m$

- The moduli space $\mathcal{M}_{k,m}$ is diffeomorphic to the space of rational maps (for $k \geq 1$)

$$\frac{a_1 z^{k-1} + a_2 z^{k-2} + \cdots + a_k}{z^k + b_1 z^{k-1} + \cdots + b_k}$$

where numerator and denominator polynomials are coprime

- Since $a_1, \ldots, a_k, b_1, \ldots, b_k$ are complex numbers, $\mathcal{M}_{k,m}$ is a real $4k$-dimensional manifold

Atiyah (1984)
Moduli space of hyperbolic monopoles (II)

- The $L^2$ metric for linearised monopoles does not converge in $H^3$  

Braam+Austin (1990)
Moduli space of hyperbolic monopoles (II)

- The $L^2$ metric for linearised monopoles does not converge in $H^3$.
- Therefore $\mathcal{M}_{k,m}$ does not (seem to) inherit a metric from the gauge theory.
Moduli space of hyperbolic monopoles (II)

- The $L^2$ metric for linearised monopoles does not converge in $H^3$ \[ \text{Braam+Austin (1990)} \]
- Therefore $\mathcal{M}_{k,m}$ does not (seem to) inherit a metric from the gauge theory
- This suggests that the geometry of $\mathcal{M}_{k,m}$ is not riemannian
Moduli space of hyperbolic monopoles (II)

- The $L^2$ metric for linearised monopoles does not converge in $H^3$. 
  [Braam+Austin (1990)]

- Therefore $\mathcal{M}_{k,m}$ does not (seem to) inherit a metric from the gauge theory.

- This suggests that the geometry of $\mathcal{M}_{k,m}$ is not riemannian.

- Nevertheless $\mathcal{M}_{2,m}$ admits a self-dual Einstein metric (for $m \in \mathbb{Z}$) whose $m \to \infty$ limit is the Atiyah–Hitchin metric for euclidean monopoles. 
  [Hitchin (1996)]
Moduli space of hyperbolic monopoles (II)

- The $L^2$ metric for linearised monopoles does not converge in $H^3$ \[ \text{Braam+Austin (1990)} \]
- Therefore $\mathcal{M}_{k,m}$ does not (seem to) inherit a metric from the gauge theory
- This suggests that the geometry of $\mathcal{M}_{k,m}$ is not riemannian
- Nevertheless $\mathcal{M}_{2,m}$ admits a self-dual Einstein metric (for $m \in \mathbb{Z}$) whose $m \to \infty$ limit is the Atiyah–Hitchin metric for euclidean monopoles \[ \text{Hitchin (1996)} \]
- It is still an open problem to relate the Hitchin family of metrics to the gauge theory
Moduli space of hyperbolic monopoles (III)

- The geometry of $\mathcal{M}_{k,m}$ has been investigated using twistor methods.

  \textit{Nash (2007)}
Moduli space of hyperbolic monopoles (III)

- The geometry of $\mathcal{M}_{k,m}$ has been investigated using twistor methods.
- Using this, it was recently identified as a pluricomplex geometry.

Nash (2007)

Bielawski + Schwachhöfer (2011)
Moduli space of hyperbolic monopoles (III)

- The geometry of $\mathcal{M}_{k,m}$ has been investigated using twistor methods. 
  \textit{Nash (2007)}

- Using this, it was recently identified as a \textbf{pluricomplex geometry}. 
  \textit{Bielawski+Schwachhöfer (2011)}

- Pluricomplex manifolds have a 2-sphere worth of integrable complex structures, but no compatible metric.
Moduli space of hyperbolic monopoles (III)

- The geometry of $\mathcal{M}_{k,m}$ has been investigated using twistor methods
  - \textit{Nash (2007)}

- Using this, it was recently identified as a pluricomplex geometry
  - \textit{Bielawski+Schwachhöfer (2011)}

- Pluricomplex manifolds have a 2-sphere worth of integrable complex structures, but no compatible metric

- Neither are they hypercomplex; although they can be characterised as admitting a complex-linear hypercomplex structure on the complexification of the tangent bundle
Moduli space of hyperbolic monopoles (III)

- The geometry of $\mathcal{M}_{k,m}$ has been investigated using twistor methods \textbf{Nash (2007)}
- Using this, it was recently identified as a pluricomplex geometry \textbf{Bielawski+Schwachhöfer (2011)}
- Pluricomplex manifolds have a 2-sphere worth of integrable complex structures, but no compatible metric
- Neither are they hypercomplex; although they can be characterised as admitting a complex-linear hypercomplex structure on the complexification of the tangent bundle
- In the euclidean limit, the pluricomplex structure gives rise to a hyperkähler structure \textbf{Bielawski+Schwachhöfer (2012)}
Supersymmetry

In this talk we will show how the pluricomplex structure arises naturally from supersymmetry
Supersymmetry

- In this talk we will show how the pluricomplex structure arises naturally from supersymmetry.
- We will construct a supersymmetric gauge theory on hyperbolic space, whose BPS configurations are precisely the hyperbolic monopoles.
In this talk we will show how the pluricomplex structure arises naturally from supersymmetry.

We will construct a supersymmetric gauge theory on hyperbolic space, whose BPS configurations are precisely the hyperbolic monopoles.

The lack of $L^2$ metric means that there is no effective action for the moduli.
Supersymmetry

- In this talk we will show how the pluricomplex structure arises naturally from supersymmetry.
- We will construct a supersymmetric gauge theory on hyperbolic space, whose BPS configurations are precisely the hyperbolic monopoles.
- The lack of $L^2$ metric means that there is no effective action for the moduli.
- But we can constrain the geometry by demanding the closure of the supersymmetry algebra.
Supersymmetry

- In this talk we will show how the pluricomplex structure arises naturally from supersymmetry.
- We will construct a supersymmetric gauge theory on hyperbolic space, whose BPS configurations are precisely the hyperbolic monopoles.
- The lack of $L^2$ metric means that there is no effective action for the moduli.
- But we can constrain the geometry by demanding the closure of the supersymmetry algebra.
- This is reminiscent of 4d Wess–Zumino sigma models without actions, in which case the target space geometry need not be Kähler.

*Stelle+Van Proeyen (2003)*
1. Hyperbolic monopoles

2. Supersymmetric Yang–Mills–Higgs in hyperbolic space

3. The geometry of the monopole moduli space

4. Conclusions and future directions
Supersymmetric Yang–Mills theories

We construct a supersymmetric Yang–Mills–Higgs theory in hyperbolic space as follows:

- Start with $d = 4$ $N = 1$ supersymmetric Yang–Mills theory
Supersymmetric Yang–Mills theories

We construct a supersymmetric Yang–Mills–Higgs theory in hyperbolic space as follows:

- Start with $d = 4$ $N = 1$ supersymmetric Yang–Mills theory
- Euclideanise it using the approach of Van Nieuwenhuizen+Waldron (1996)
Supersymmetric Yang–Mills theories

We construct a supersymmetric Yang–Mills–Higgs theory in hyperbolic space as follows:

- Start with $d = 4 \ N = 1$ supersymmetric Yang–Mills theory
- Euclideanise it using the approach of Van Nieuwenhuizen+Waldron (1996)
- This complexifies the fields: spinors in $\mathbb{R}^4$ are not real
Supersymmetric Yang–Mills theories

We construct a supersymmetric Yang–Mills–Higgs theory in hyperbolic space as follows:

- Start with $d = 4$ $N = 1$ supersymmetric Yang–Mills theory
- Euclideanise it using the approach of Van Nieuwenhuizen+Waldron (1996)
- This complexifies the fields: spinors in $\mathbb{R}^4$ are not real
- Dimensionally reduce to $\mathbb{R}^3$
Supersymmetric Yang–Mills theories

We construct a supersymmetric Yang–Mills–Higgs theory in hyperbolic space as follows:

- Start with $d = 4 \ N = 1$ supersymmetric Yang–Mills theory
- Euclideanise it using the approach of Van Nieuwenhuizen+Waldron (1996)
- This complexifies the fields: spinors in $\mathbb{R}^4$ are not real
- Dimensionally reduce to $\mathbb{R}^3$
- Deform the theory from $\mathbb{R}^3$ to $\mathbb{H}^3$
The lagrangian

The lagrangian density is given by

\[ \mathcal{L} = -i \chi^\dagger \slashed{D} \psi - \chi^\dagger [\phi, \psi] - i \lambda \chi^\dagger \psi - \frac{1}{4} F^2 - \frac{1}{2} |\slashed{D} \phi|^2 - \frac{1}{2} D^2 \]

where all fields are Lie algebra valued (\text{Tr} is implicit)
The lagrangian

The lagrangian density is given by

\[ \mathcal{L} = -i\chi^\dagger \mathcal{D}\psi - \chi^\dagger [\phi, \psi] - i\lambda \chi^\dagger \psi - \frac{1}{4} F^2 - \frac{1}{2} |\mathcal{D}\phi|^2 - \frac{1}{2} D^2 \]

where all fields are Lie algebra valued (Tr is implicit) and

- \( \chi, \psi \) are two-component complex spinor fields on \( H^3 \)
The lagrangian

The lagrangian density is given by

\[ \mathcal{L} = -i\chi^\dagger \slashed{D}\psi - \chi^\dagger [\phi, \psi] - i\lambda\chi^\dagger \psi - \frac{1}{4} F^2 - \frac{1}{2} |\slashed{D}\phi|^2 - \frac{1}{2} D^2 \]

where all fields are Lie algebra valued (Tr is implicit) and
- \( \chi, \psi \) are two-component complex spinor fields on \( H^3 \)
- \( \phi \) is a complexified Higgs
The lagrangian

The lagrangian density is given by

$$\mathcal{L} = -i\chi^\dagger \slashed{D} \psi - \chi^\dagger [\phi, \psi] - i\lambda \chi^\dagger \psi - \frac{1}{4} F^2 - \frac{1}{2} |\slashed{D} \phi|^2 - \frac{1}{2} D^2$$

where all fields are Lie algebra valued (Tr is implicit) and

- \(\chi, \psi\) are two-component complex spinor fields on \(H^3\)
- \(\phi\) is a complexified Higgs
- \(F\) is the curvature of the complexified gauge field \(A\)
The lagrangian

The lagrangian density is given by

$$\mathcal{L} = -i\chi^\dagger \not{D}\psi - \chi^\dagger [\phi, \psi] - i\lambda\chi^\dagger \psi - \frac{1}{4} F^2 - \frac{1}{2} |\not{D}\phi|^2 - \frac{1}{2} D^2$$

where all fields are Lie algebra valued (Tr is implicit) and

- $\chi, \psi$ are two-component complex spinor fields on $H^3$
- $\phi$ is a complexified Higgs
- $F$ is the curvature of the complexified gauge field $A$
- $D$ is an auxiliary field for off-shell closure of supersymmetry
The lagrangian

The lagrangian density is given by

\[ \mathcal{L} = -i\chi^\dagger \mathcal{D}\psi - \chi^\dagger [\phi, \psi] - i\lambda\chi^\dagger \psi - \frac{1}{4} F^2 - \frac{1}{2} |\mathcal{D}\phi|^2 - \frac{1}{2} D^2 \]

where all fields are Lie algebra valued (Tr is implicit) and

- \( \chi, \psi \) are two-component complex spinor fields on \( H^3 \)
- \( \phi \) is a complexified Higgs
- \( F \) is the curvature of the complexified gauge field \( A \)
- \( D \) is an auxiliary field for off-shell closure of supersymmetry
- \( \mathcal{D} \) is the fully covariant derivative: \( \mathcal{D}_i = \nabla_i + [A_i, -] \)
The lagrangian

The lagrangian density is given by

\[ \mathcal{L} = -i\chi^\dagger \mathcal{D}\psi - \chi^\dagger [\phi, \psi] - i\lambda \chi^\dagger \psi - \frac{1}{4} F^2 - \frac{1}{2} |\mathcal{D}\phi|^2 - \frac{1}{2} D^2 \]

where all fields are Lie algebra valued (Tr is implicit) and
- \( \chi, \psi \) are two-component complex spinor fields on \( \mathbb{H}^3 \)
- \( \phi \) is a complexified Higgs
- \( F \) is the curvature of the complexified gauge field \( A \)
- \( D \) is an auxiliary field for off-shell closure of supersymmetry
- \( \mathcal{D} \) is the fully covariant derivative: \( \mathcal{D}_i = \nabla_i + [A_i, -] \)
- and \(-\lambda^2\) is proportional to the scalar curvature of \( \mathbb{H}^3 \)


Supersymmetry transformations (I)

\( \mathcal{L} \) transforms as

\[
\delta_L \mathcal{L} = \nabla_i \left( -i\chi^\dagger (\sigma^i D + \sigma_j F^{ij} - i\mathcal{D}^i \phi) \epsilon_L \right)
\]

under

\[
\delta_L A_i = i\chi^\dagger \sigma_i \epsilon_L \\
\delta_L \phi = \chi^\dagger \epsilon_L \\
\delta_L \chi^\dagger = 0 \\
\delta_L \psi = D\epsilon_L + i(\frac{1}{2} \varepsilon_{ijk} F^{ij} - \mathcal{D}_k \phi) \sigma^k \epsilon_L \\
\delta_L D = i\chi^\dagger \mathcal{D} \epsilon_L + [\phi, \chi^\dagger] \epsilon_L - i\lambda \chi^\dagger \epsilon_L
\]

provided that

\[
\nabla_i \epsilon_L = \lambda \sigma_i \epsilon_L
\]
Supersymmetry transformations (II)

\( \mathcal{L} \) also transforms as

\[
\delta_R \mathcal{L} = \nabla_i \left( \epsilon^{ijk} \epsilon^\dagger_R \left( -\frac{1}{2} F_{jk} + i \mathcal{D}_j \phi \sigma_k \right) \psi \right)
\]

under

\[
\delta_R A_i = -i \epsilon^\dagger_R \sigma_i \psi
\]

\[
\delta_R \phi = -\epsilon^\dagger_R \psi
\]

\[
\delta_R \chi^\dagger = -D \epsilon^\dagger_R - i \left( \frac{1}{2} \epsilon^{ijk} F_{ij} + \mathcal{D}_k \phi \right) \epsilon^\dagger_R \sigma^k
\]

\[
\delta_R \psi = 0
\]

\[
\delta_R D = i \epsilon^\dagger_R \mathcal{D} \psi + \epsilon^\dagger_R [\phi, \psi] + i \lambda \epsilon^\dagger_R \psi
\]

provided that

\[
\nabla_i \epsilon^\dagger_R = -\lambda \epsilon^\dagger_R \sigma_i
\]
Closure

The above supersymmetry transformations obey the following superalgebra:

\[
[\delta_L, \delta'_L] = 0 = [\delta_R, \delta'_R] \\
[\delta_L, \delta_R] = \mathcal{L}_\xi + \delta^\text{gauge}_\Lambda + \delta^R_\omega
\]
Closure

The above supersymmetry transformations obey the following superalgebra:

\[[\delta_L, \delta'_L] = 0 = [\delta_R, \delta'_R]\]
\[[\delta_L, \delta_R] = \mathcal{L}\xi + \delta^\text{gauge}_\Lambda + \delta^\text{R}_\omega\]

where

- \(\xi^i = 2i\epsilon^\dagger_R \sigma^i \epsilon_L\) is a Killing vector field: \(\nabla_i \xi_j = -2i\lambda \epsilon_{ijk} \xi^k\)
Closure

The above supersymmetry transformations obey the following superalgebra:

\[ [\delta_L, \delta'_L] = 0 = [\delta_R, \delta'_R] \]
\[ [\delta_L, \delta_R] = \mathcal{L}_\xi + \delta_{\Lambda}^{\text{gauge}} + \delta_{\varpi}^{R} \]

where

- \( \xi^i = 2i \epsilon_R^\dagger \sigma^i \epsilon_L \) is a Killing vector field: \( \nabla_i \xi_j = -2i \lambda \epsilon_{ijk} \xi^k \)
- \( \Lambda = \xi^i A_i + 2 \epsilon_R^\dagger \epsilon_L \phi \)
Closure

The above supersymmetry transformations obey the following superalgebra:

\[
\begin{align*}
[\delta_L, \delta_L'] &= 0 = [\delta_R, \delta_R'] \quad [\delta_L, \delta_R] = \mathcal{L}_\xi + \delta_{\Lambda}^{\text{gauge}} + \delta_{\omega}^{\text{R}}
\end{align*}
\]

where

- \( \xi^i = 2i \epsilon_R^\dagger \sigma^i \epsilon_L \) is a Killing vector field: \( \nabla_i \xi_j = -2i \lambda \epsilon_{ijk} \xi^k \)
- \( \Lambda = \xi^i A_i + 2 \epsilon_R^\dagger \epsilon_L \phi \)
- \( \delta_{\omega}^{\text{R}} \) is an R-symmetry transformation:

\[
\delta_{\omega}^{\text{R}} \psi = i \omega \psi \quad \text{and} \quad \delta_{\omega}^{\text{R}} \chi^\dagger = -i \omega \chi^\dagger
\]

with \( \omega = -4\lambda \epsilon_R^\dagger \epsilon_L \), which is indeed constant.
Some remarks

- All fields are **complex** and the lagrangian as written is not real.
- The theory has **8 real supercharges**, because $\epsilon_{L,R}$ are Killing spinors on $H^3$, which admits the maximum number of Killing spinors with either sign of the Killing constant.
- Similar (but not identical) to supersymmetric theories in “Family A” in work of **Blau (2000)**.
BPS configurations

The bosonic BPS configurations are precisely the hyperbolic monopoles with $D = 0$
BPS configurations

- The bosonic BPS configurations are precisely the hyperbolic monopoles with $D = 0$
- Write $\delta_L \psi = (D + i(\frac{1}{2} \varepsilon_{ijk} F^{ij} - \mathcal{D}_k \phi) \sigma^k)\epsilon_L$
BPS configurations

- The bosonic BPS configurations are precisely the hyperbolic monopoles with $D = 0$
- Write $\delta_L \psi = (D + i(\frac{1}{2} \varepsilon_{ijk} F^{ij} - \mathcal{D}_k \phi) \sigma^k) \epsilon_L$
- Then $\det(D + i(\frac{1}{2} \varepsilon_{ijk} F^{ij} - \mathcal{D}_k \phi) \sigma^k) = 0$ if and only if $D = 0$ and $\frac{1}{2} \varepsilon_{ijk} F^{ij} - \mathcal{D}_k \phi = 0$
BPS configurations

- The bosonic BPS configurations are precisely the hyperbolic monopoles with $D = 0$
- Write $\delta_L \psi = (D + i(\frac{1}{2}\epsilon_{ijk}F^{ij} - D_k \phi)\sigma^k)\epsilon_L$
- Then $\det(D + i(\frac{1}{2}\epsilon_{ijk}F^{ij} - D_k \phi)\sigma^k) = 0$ if and only if $D = 0$ and $\frac{1}{2}\epsilon_{ijk}F^{ij} - D_k \phi = 0$
- Similarly, bosonic configurations with $D_k \phi = -\frac{1}{2}\epsilon_{ijk}F^{ij}$ and $D = 0$ are precisely the ones which preserve the $\delta_R$ supersymmetries
BPS configurations

- The bosonic BPS configurations are precisely the hyperbolic monopoles with $D = 0$.
- Write $\delta_L \psi = (D + i(\frac{1}{2} \varepsilon_{ijk} F^{ij} - \mathcal{D}_k \phi) \sigma^k) \epsilon_L$
- Then $\det(D + i(\frac{1}{2} \varepsilon_{ijk} F^{ij} - \mathcal{D}_k \phi) \sigma^k) = 0$ if and only if $D = 0$ and $\frac{1}{2} \varepsilon_{ijk} F^{ij} - \mathcal{D}_k \phi = 0$
- Similarly, bosonic configurations with $\mathcal{D}_k \phi = -\frac{1}{2} \varepsilon_{ijk} F^{ij}$ and $D = 0$ are precisely the ones which preserve the $\delta_R$ supersymmetries.
- We will study the moduli space $\mathcal{M}$ of bosonic configurations preserving the $\delta_R$ supersymmetries.
1. Hyperbolic monopoles

2. Supersymmetric Yang–Mills–Higgs in hyperbolic space

3. The geometry of the monopole moduli space

4. Conclusions and future directions
Bosonic zero modes

Let \((A(t), \phi(t))\) be a family of bosonic BPS configurations:

\[ \mathcal{D}_i(t) \phi(t) + \varepsilon_{ijk} F^j_k(t) = 0 \]
Bosonic zero modes

- Let \((A(t), \phi(t))\) be a family of bosonic BPS configurations:

\[
\mathcal{D}_i(t)\phi(t) + \varepsilon_{ijk} F^{jk}(t) = 0
\]

- Differentiating w.r.t. \(t\) at \(t = 0\) we obtain the **linearised Bogomol’nyi equation**:

\[
\mathcal{D}_i(0)\dot{\phi} - [\phi(0), \dot{A}_i] + \varepsilon_{ijk} \mathcal{D}^j(0)\dot{A}^k = 0
\]

where
Bosonic zero modes

Let \((A(t), \phi(t))\) be a family of bosonic BPS configurations:

\[
\mathcal{D}_i(t)\phi(t) + \epsilon_{ijk} F^{jk}(t) = 0
\]

Differentiating w.r.t. \(t\) at \(t = 0\) we obtain the linearised Bogomol’nyi equation:

\[
\mathcal{D}_i(0)\dot{\phi} - [\phi(0), \dot{A}_i] + \epsilon_{ijk} \mathcal{D}^j(0)\dot{A}^k = 0
\]

where

\[
\dot{A}_i = \left. \frac{\partial A_i}{\partial t} \right|_{t=0}
\]
Bosonic zero modes

- Let \((A(t), \phi(t))\) be a family of bosonic BPS configurations:
  \[\mathcal{D}_i(t)\phi(t) + \varepsilon_{ijk} F^{jk}(t) = 0\]

- Differentiating w.r.t. \(t\) at \(t = 0\) we obtain the linearised Bogomol’nyi equation:
  \[\mathcal{D}_i(0)\dot{\phi} - [\phi(0), \dot{A}_i] + \varepsilon_{ijk} \mathcal{D}^j(0)\dot{A}^k = 0\]

where
  - \(\dot{A}_i = \frac{\partial A_i}{\partial t} \bigg|_{t=0}\)
  - \(\dot{\phi} = \frac{\partial \phi}{\partial t} \bigg|_{t=0}\)
Bosonic zero modes

Let \((A(t), \phi(t))\) be a family of bosonic BPS configurations:

\[
\mathcal{D}_i(t)\phi(t) + \varepsilon_{ijk}F^{jk}(t) = 0
\]

Differentiating w.r.t. \(t\) at \(t = 0\) we obtain the linearised Bogomol’nyi equation:

\[
\mathcal{D}_i(0)\dot{\phi} - [\phi(0), \dot{A}_i] + \varepsilon_{ijk}\mathcal{D}^j(0)\dot{A}^k = 0
\]

where

- \(\dot{A}_i = \frac{\partial A_i}{\partial t} \bigg|_{t=0}\)
- \(\dot{\phi} = \frac{\partial \phi}{\partial t} \bigg|_{t=0}\)
- \(\mathcal{D}_i(0) = \nabla_i + [A_i(0), -]\)
Gauge orbits

Some $(\dot{A}, \dot{\phi})$ are tangent to the orbit $\mathcal{O}$ of $\mathcal{A}_0 = (A(0), \phi(0))$ under the group of gauge transformations.
Gauge orbits

- Some \((\dot{A}, \dot{\phi})\) are tangent to the orbit \(\mathcal{O}\) of \(A_0 = (A(0), \phi(0))\) under the group of gauge transformations.
- We identify \(T_{[A_0]} \mathcal{M}\) with a suitable complement to \(T_{A(0)} \mathcal{O}\).
Gauge orbits

- Some $(\dot{A}, \dot{\phi})$ are tangent to the orbit $O$ of $A_0 = (A(0), \phi(0))$ under the group of gauge transformations.
- We identify $T_{[A_0]} M$ with a suitable complement to $T_{A(0)} O$.
- For euclidean monopoles, there is a riemannian metric on the space of solutions of the linearised Bogomol’nyi equation, so $T_{[A_0]} M \cong (T_{A(0)} O)^\perp$ (i.e., Gauss’s Law).
Gauge orbits

- Some $(\dot{A}, \dot{\phi})$ are tangent to the orbit $\mathcal{O}$ of $A_0 = (A(0), \phi(0))$ under the group of gauge transformations.
- We identify $T_{A_0} M$ with a suitable complement to $T_{A(0)} \mathcal{O}$.
- For euclidean monopoles, there is a riemannian metric on the space of solutions of the linearised Bogomol’nyi equation, so $T_{A_0} M \cong (T_{A(0)} \mathcal{O})^\perp$ (i.e., Gauss’s Law).
- For hyperbolic monopoles there is no natural riemannian metric, so we will employ supersymmetry to define this complement.
Fermionic zero modes

- A **fermionic zero mode** $\psi$ is a solution of the (already linear) Dirac equation in the presence of the monopole $A_0 = (A(0), \phi(0))$:

  $\mathcal{D}(0)\psi - i[\phi(0), \psi] + \lambda\psi = 0$

  (Notice that the equation has a mass term which goes to zero in the euclidean limit.)
Fermionic zero modes

- A **fermionic zero mode** $\psi$ is a solution of the (already linear) Dirac equation in the presence of the monopole $\mathcal{A}_0 = (A(0), \phi(0))$:

$$\mathcal{D}(0)\psi - i[\phi(0), \psi] + \lambda \psi = 0$$

(Notice that the equation has a mass term which goes to zero in the euclidean limit.)

- We could determine the number of fermionic zero modes by an index theory calculation. **Callias (1978), Råde (1994)**
Fermionic zero modes

A **fermionic zero mode** $\psi$ is a solution of the (already linear) Dirac equation in the presence of the monopole $A_0 = (A(0), \phi(0))$:

$$D(0)\psi - i[\phi(0), \psi] + \lambda\psi = 0$$

(Notice that the equation has a mass term which goes to zero in the euclidean limit.)

- We could determine the number of fermionic zero modes by an index theory calculation: **Callias (1978), Råde (1994)**
- But we will instead use supersymmetry: **Zumino (1977)**
Supersymmetry between zero modes (I)

- Let $\eta$ be a Killing spinor on $H^3$ satisfying $\nabla_i \eta = \lambda \sigma_i \eta$
Supersymmetry between zero modes (I)

- Let $\eta$ be a Killing spinor on $H^3$ satisfying $\nabla_i \eta = \lambda \sigma_i \eta$
- Let $(\dot{A}, \dot{\phi})$ obey the linearised Bogomol’nyi equation

The last term might be surprising...
Supersymmetry between zero modes (I)

- Let \( \eta \) be a Killing spinor on \( H^3 \) satisfying \( \nabla_i \eta = \lambda \sigma_i \eta \)
- Let \( (\dot{A}, \dot{\phi}) \) obey the linearised Bogomol’nyi equation
- Define \( \dot{\psi} = i\dot{A}_i \sigma^i \eta - \dot{\phi} \eta \)
Supersymmetry between zero modes (I)

- Let $\eta$ be a Killing spinor on $H^3$ satisfying $\nabla_i \eta = \lambda \sigma_i \eta$
- Let $(\dot{A}, \dot{\phi})$ obey the linearised Bogomol’nyi equation
- Define $\dot{\psi} = i\dot{A}_i \sigma^i \eta - \dot{\phi} \eta$
- Then $\dot{\psi}$ is a fermionic zero mode if and only if $(\dot{A}, \dot{\phi})$ obey in addition the \textbf{generalised Gauss Law}

$$D^i(0)\dot{A}_i + [\phi(0), \dot{\phi}] + 4i\lambda \dot{\phi} = 0$$
Supersymmetry between zero modes (I)

- Let $\eta$ be a Killing spinor on $H^3$ satisfying $\nabla_i \eta = \lambda \sigma_i \eta$
- Let $(\dot{A}, \dot{\phi})$ obey the linearised Bogomol'nyi equation
- Define $\dot{\psi} = i \dot{A}_i \sigma^i \eta - \dot{\phi} \eta$
- Then $\dot{\psi}$ is a fermionic zero mode if and only if $(\dot{A}, \dot{\phi})$ obey in addition the **generalised Gauss Law**

$$D^i(0)\dot{A}_i + [\phi(0), \dot{\phi}] + 4i\lambda \dot{\phi} = 0$$

- The last term might be surprising...
Supersymmetry between zero modes (I)

- Let $\eta$ be a Killing spinor on $H^3$ satisfying $\nabla_i \eta = \lambda \sigma_i \eta$
- Let $(\dot{A}, \dot{\phi})$ obey the linearised Bogomol’nyi equation
- Define $\dot{\psi} = i \dot{A}_i \sigma^i \eta - \dot{\phi} \eta$
- Then $\dot{\psi}$ is a fermionic zero mode if and only if $(\dot{A}, \dot{\phi})$ obey in addition the **generalised Gauss Law**

$$\mathcal{D}^i(0) \dot{A}_i + [\phi(0), \dot{\phi}] + 4i\lambda \dot{\phi} = 0$$

- The last term might be surprising...
- The generalised Gauss Law is invariant under $G$ and defines a complement to the tangent space to the gauge orbits
Conversely, let $\zeta$ be a Killing spinor in $H^3$ obeying
\[ \nabla_i \zeta^\dagger = -\lambda \zeta^\dagger \sigma_i \]
Supersymmetry between zero modes (II)

- Conversely, let $\zeta$ be a Killing spinor in $H^3$ obeying
  $\nabla_i \zeta^\dagger = -\lambda \zeta^\dagger \sigma_i$

- Let $\psi$ be a fermionic zero mode
Supersymmetry between zero modes (II)

Conversely, let $\zeta$ be a Killing spinor in $H^3$ obeying
$$\nabla_i \zeta^\dagger = -\lambda \zeta^\dagger \sigma_i$$

Let $\psi$ be a fermionic zero mode

Then $\dot{A}_i = -i \zeta^\dagger \sigma_i \psi$ and $\dot{\phi} = -\zeta^\dagger \psi$ obey the linearised Bogomol’nyi equation and the generalised Gauss Law.
Conversely, let $\zeta$ be a Killing spinor in $H^3$ obeying
\[ \nabla_i \zeta^\dagger = -\lambda \zeta^\dagger \sigma_i \]
Let $\psi$ be a fermionic zero mode
Then $\dot{A}_i = -i \zeta^\dagger \sigma_i \psi$ and $\dot{\phi} = -\zeta^\dagger \psi$ obey the linearised Bogomol’nyi equation and the generalised Gauss Law
In summary, there are linear maps (parametrised by Killing spinors on $H^3$) mapping between bosonic and fermionic zero modes
Conversely, let $\zeta$ be a Killing spinor in $H^3$ obeying
$$\nabla_i \zeta^\dagger = -\lambda \zeta^\dagger \sigma_i$$

Let $\psi$ be a fermionic zero mode.

Then $\dot{A}_i = -i \zeta^\dagger \sigma_i \psi$ and $\dot{\phi} = -\zeta^\dagger \psi$ obey the linearised Bogomol'nyi equation and the generalised Gauss Law.

In summary, there are linear maps (parametrised by Killing spinors on $H^3$) mapping between bosonic and fermionic zero modes.

We will see these maps are isomorphisms, so that there are $4k$ fermionic zero modes as well.
Supersymmetry between zero modes (II)

- Conversely, let $\zeta$ be a Killing spinor in $H^3$ obeying
  $\nabla_i \zeta^\dagger = -\lambda \zeta^\dagger \sigma_i$

- Let $\psi$ be a fermionic zero mode

- Then $\dot{A}_i = -i\zeta^\dagger \sigma_i \psi$ and $\dot{\phi} = -\zeta^\dagger \psi$ obey the linearised
  Bogomol’nyi equation and the generalised Gauss Law

- In summary, there are linear maps (parametrised by Killing
  spinors on $H^3$) mapping between bosonic and fermionic
  zero modes

- We will see these maps are isomorphisms, so that there
  are $4k$ fermionic zero modes as well

- But it is easier to see this in a four-dimensional formalism
A four-dimensional formalism

- We work formally in $H^3 \times S^1$ but fields are $S^1$-invariant
A four-dimensional formalism

- We work formally in $\mathbb{H}^3 \times S^1$ but fields are $S^1$-invariant
- $\Gamma_\mu$ are complex $4 \times 4$ matrices representing $\mathbb{C}\ell(0, 4)$
A four-dimensional formalism

- We work formally in $\mathbb{H}^3 \times S^1$ but fields are $S^1$-invariant
- $\Gamma_\mu$ are complex $4 \times 4$ matrices representing $\mathbb{C}\ell(0,4)$
- Spinors $\eta$ and $\zeta^\dagger$ in $\mathbb{H}^3$ lift to chiral spinors in $\mathbb{H}^3 \times S^1$:

$$\eta_R = \begin{pmatrix} 0 \\ \eta \end{pmatrix} \quad \zeta^\dagger_R = (0 \  \zeta^\dagger)$$
A four-dimensional formalism

- We work formally in $H^3 \times S^1$ but fields are $S^1$-invariant
- $\Gamma_\mu$ are complex $4 \times 4$ matrices representing $C\ell(0, 4)$
- Spinors $\eta$ and $\zeta^\dagger$ in $H^3$ lift to chiral spinors in $H^3 \times S^1$:

$$\eta_R = \begin{pmatrix} 0 \\ \eta \end{pmatrix} \quad \zeta_R^\dagger = \begin{pmatrix} 0 & \zeta \end{pmatrix}$$

- The Killing spinor equations in $H^3$ become

$$\nabla_i \eta_R = -i\lambda \Gamma_i \Gamma_4 \eta_R \quad \nabla_i \zeta_R^\dagger = -i\lambda \zeta_R^\dagger \Gamma_4 \Gamma_i$$

in addition to $\nabla_4 \eta_R = 0$ and $\nabla_4 \zeta_R^\dagger = 0$
Zero modes in four-dimensional formalism

- In this formalism, a **fermionic zero mode** $\Psi_L = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$ obeys

$$\mathcal{D}\Psi_L = -i\lambda \Gamma_4 \Psi_L$$
Zero modes in four-dimensional formalism

- In this formalism, a fermionic zero mode $\Psi_L = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$ obeys
  \[ \mathcal{D} \Psi_L = -i\lambda \Gamma_4 \Psi_L \]

- and a bosonic zero mode $\dot{A}_\mu = (\dot{A}_i, \dot{\phi})$ obeys
  \[ \mathcal{D}_{[\mu} \dot{A}_{\nu]} = -\frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} \mathcal{D}^\rho \dot{A}^\sigma \]
  \[ \mathcal{D}^\mu \dot{A}_\mu = -4i\lambda \dot{A}_4 \]
Zero modes in four-dimensional formalism

- In this formalism, a **fermionic zero mode** $\Psi_L = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$ obeys

$$\mathcal{D}\Psi_L = -i\lambda \Gamma_4 \Psi_L$$

- and a **bosonic zero mode** $\dot{A}_\mu = (\dot{A}_i, \dot{\phi})$ obeys

$$\mathcal{D}[\mu \dot{A}_\nu] = -\frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} \mathcal{D}^\rho \dot{A}^\sigma$$

$$\mathcal{D}^\mu \dot{A}_\mu = -4i\lambda \dot{A}_4$$

- Of course, $\nabla_4 \dot{\Psi}_L = 0$ and $\nabla_4 \dot{A}_\mu = 0$
Supersymmetry between zero modes (III)

Let $Z_0$ and $Z_1$ denote the vector space of bosonic and fermionic zero modes.
Supersymmetry between zero modes (III)

- Let $Z_0$ and $Z_1$ denote the vector space of bosonic and fermionic zero modes.
- Let $K^\pm$ denote the vector space of Killing spinors (on $H^3$).

$$K^\pm = \{ \xi_R | \nabla_i \xi_R = \mp i \lambda \Gamma_i \Gamma_4 \xi_R \quad \text{and} \quad \nabla_4 \xi_R = 0 \}$$
Supersymmetry between zero modes (III)

- Let $Z_0$ and $Z_1$ denote the vector space of bosonic and fermionic zero modes.
- Let $\mathbf{K}^\pm$ denote the vector space of Killing spinors (on $\mathbb{H}^3$)

\[
\mathbf{K}^\pm = \{ \xi_R | \nabla_i \xi_R = \mp i\lambda \Gamma_i \Gamma_4 \xi_R \quad \text{and} \quad \nabla_4 \xi_R = 0 \}
\]

- We have real bilinear maps

\[
\begin{align*}
\mathbf{K}^+ \times Z_0 & \to Z_1 \\
(\eta_R, \dot{A}_\mu) & \mapsto i\dot{A}_\mu \Gamma^\mu \eta_R \\
\mathbf{K}^- \times Z_1 & \to Z_0 \\
(\zeta_R, \dot{\Psi}_L) & \mapsto -i\zeta_R^\dagger \Gamma_\mu \dot{\Psi}_L
\end{align*}
\]
Supersymmetry between zero modes (IV)

We can compose them:

\[ K^+ \times K^- \times Z_1 \rightarrow Z_1 \]

\[ (\eta_R, \zeta_R, \psi_L) \mapsto 2\zeta_R^{\dagger} \eta_R \psi_L \]
Supersymmetry between zero modes (IV)

- We can compose them:

\[ K^+ \times K^- \times Z_1 \rightarrow Z_1 \]

\[(\eta_R, \zeta_R, \Psi_L) \mapsto 2\zeta^+_R\eta_R\Psi_L\]

- Normalising so that \(2\zeta^+_R\eta_R = 1\), we see that this composition is the identity.
Supersymmetry between zero modes (IV)

- We can compose them:
  \[ K^+ \times K^- \times Z_1 \rightarrow Z_1 \]
  \[ (\eta_R, \zeta_R, \Psi_L) \mapsto 2\zeta_R^\dagger \eta_R \Psi_L \]

- Normalising so that \( 2\zeta_R^\dagger \eta_R = 1 \), we see that this composition is the identity.

- In particular, both maps are isomorphisms and hence \( \dim Z_0 = \dim Z_1 \).
Complex structures from Killing spinors (I)

- Let $\eta_R \in K^+$ and $\zeta_R \in K^-$
Complex structures from Killing spinors (I)

- Let $\eta_R \in K^+$ and $\zeta_R \in K^-$
- They define a complex-linear endomorphism of $T_C(H^3 \times S^1)$ by

$$E^\gamma_\mu = -i \zeta_R^\dagger \Gamma^\gamma_\mu \eta_R$$
Complex structures from Killing spinors (I)

- Let $\eta_R \in K^+$ and $\zeta_R \in K^-$.
- They define a complex-linear endomorphism of $T_C(H^3 \times S^1)$ by

$$E_{\mu}{}^{\nu} = -i\zeta_R^+ \Gamma_{\mu}{}^{\nu} \eta_R$$

- It follows from the chirality of $\eta_R$ and $\zeta_R$ that $E$ is self-dual:

$$\frac{1}{2} \varepsilon_{\mu \nu \rho \sigma} E^{\rho \sigma} = E_{\mu \nu}$$
Complex structures from Killing spinors (I)

- Let $\eta_R \in K^+$ and $\zeta_R \in K^-$
- They define a complex-linear endomorphism of $T_C(H^3 \times S^1)$ by
  \[ E_\mu{}^\nu = -i\zeta_R^\dagger \Gamma_\mu{}^\nu \eta_R \]

- It follows from the chirality of $\eta_R$ and $\zeta_R$ that $E$ is self-dual:
  \[ \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} E^{\rho\sigma} = E_{\mu\nu} \]

- Also it follows from Fierz identities that
  \[ E_\mu{}^\rho E_\rho{}^\nu = -(\zeta_R^\dagger \eta_R)^2 \delta_\mu{}^\nu \]
Complex structures from Killing spinors (I)

- Let $\eta_R \in K^+$ and $\zeta_R \in K^-$.
- They define a complex-linear endomorphism of $T_C(H^3 \times S^1)$ by
  \[ E_{\mu}{}^{\nu} = -i\zeta_R^\dagger \Gamma_{\mu}{}^{\nu} \eta_R \]

- It follows from the chirality of $\eta_R$ and $\zeta_R$ that $E$ is self-dual:
  \[ \frac{1}{2} \epsilon_{\mu \nu \rho \sigma} E^{\rho \sigma} = E_{\mu}{}^{\nu} \]

- Also it follows from Fierz identities that
  \[ E_{\mu}{}^{\rho} E_{\rho}{}^{\nu} = - (\zeta_R^\dagger \eta_R)^2 \delta_{\mu}{}^{\nu} \]

- If we normalise $\zeta_R^\dagger \eta_R = 1$, then $E$ is a complex structure.
Complex structures from Killing spinors (II)

Since $\eta_R$ and $\zeta_R$ are Killing spinors, $\nabla_4 E_{\mu\nu} = 0$ and

\[ \nabla_i E_{4j} = 2i\lambda E_{ij} \quad \nabla_i E_{jk} = -2i\lambda (\delta_{ij} E_{4k} - \delta_{ik} E_{4j}) \]
Complex structures from Killing spinors (II)

- Since $\eta_R$ and $\zeta_R$ are Killing spinors, $\nabla_4 E_{\mu \nu} = 0$ and
  
  $$\nabla_i E_{4j} = 2i\lambda E_{ij}, \quad \nabla_i E_{jk} = -2i\lambda (\delta_{ij} E_{4k} - \delta_{ik} E_{4j})$$

- This implies that if $\dot{A}_\mu$ is a bosonic zero mode, so is $E_{\mu}^{\nu} \dot{A}_\nu$
Complex structures from Killing spinors (II)

- Since $\eta_R$ and $\zeta_R$ are Killing spinors, $\nabla_4 E_{\mu\nu} = 0$ and
  \[
  \nabla_i E_{4j} = 2i\lambda E_{ij} \quad \nabla_i E_{jk} = -2i\lambda (\delta_{ij} E_{4k} - \delta_{ik} E_{4j})
  \]

- This implies that if $\hat{A}_\mu$ is a bosonic zero mode, so is $E_\mu \gamma \hat{A}_\nu$
- If $\hat{A}_{\alpha\mu}$ denotes a basis for $Z_0$, then
  \[
  E_\mu \gamma \hat{A}_{\alpha\nu} = \mathcal{E}_{\alpha}^b \hat{A}_{b\mu}
  \]
  defines an almost complex structure $\mathcal{E}$ on $T_{\mathbb{C}M}$
Complex structures from Killing spinors (II)

- Since $\eta_R$ and $\zeta_R$ are Killing spinors, $\nabla_4 E_{\mu\nu} = 0$ and
  \[
  \nabla_i E_{4j} = 2i\lambda E_{ij} \quad \nabla_i E_{jk} = -2i\lambda (\delta_{ij} E_{4k} - \delta_{ik} E_{4j})
  \]

- This implies that if $\dot{A}_\mu$ is a bosonic zero mode, so is $E_{\mu} \gamma \dot{A}_\nu$
- If $\dot{A}_{a\mu}$ denotes a basis for $Z_0$, then
  \[
  E_{\mu} \gamma \dot{A}_{a\nu} = \epsilon^{ab}_{\mu} \dot{A}_{b\mu}
  \]
  defines an almost complex structure $\mathcal{E}$ on $\mathcal{T}_C \mathcal{M}$

- Varying $\eta_R$ and $\zeta_R$ subject to $\zeta_R^\dagger \eta_R = 1$, we find a 2-sphere worth of almost complex structures
Complex structures from Killing spinors (II)

- Since $\eta_R$ and $\zeta_R$ are Killing spinors, $\nabla_4 E_{\mu\nu} = 0$ and

$$\nabla_i E_{4j} = 2i\lambda E_{ij} \quad \nabla_i E_{jk} = -2i\lambda (\delta_{ij} E_{4k} - \delta_{ik} E_{4j})$$

- This implies that if $\dot{A}_\mu$ is a bosonic zero mode, so is $E_{\mu} \gamma \dot{A}_\nu$

- If $\dot{A}_{a\mu}$ denotes a basis for $Z_0$, then

$$E_{\mu} \gamma \dot{A}_{a\nu} = \mathcal{E}_a^b \dot{A}_{b\mu}$$

defines an almost complex structure $\mathcal{E}$ on $T_{\mathcal{C}M}$

- Varying $\eta_R$ and $\zeta_R$ subject to $\zeta_R^\dagger \eta_R = 1$, we find a 2-sphere worth of almost complex structures

- Supersymmetry $\implies$ they are integrable
Linearising the supersymmetry transformations (I)

- In 4d-language, the supersymmetry transformation of $A_\mu$ is

$$\delta_\epsilon A_\mu = -i\epsilon^\dagger_R \Gamma_\mu \Psi_L$$

$$\quad\Rightarrow\quad$$

$$\delta_\epsilon \dot{A}_\mu = -i\epsilon^\dagger_R \Gamma_\mu \dot{\Psi}_L$$
Linearising the supersymmetry transformations (I)

- In 4d-language, the supersymmetry transformation of $A_\mu$ is
  \[
  \delta_\epsilon A_\mu = -i\epsilon^\dagger R_{\Gamma_\mu} \Psi_L \implies \delta_\epsilon \dot{A}_\mu = -i\epsilon^\dagger R_{\Gamma_\mu} \dot{\Psi}_L
  \]

- Choose a basis $\dot{A}_{a\mu}$ for $Z_0$ and let $\dot{\Psi}_{La} = i\dot{A}_{a\mu} \Gamma^\mu \eta_R$ be the corresponding basis for $Z_1$
Linearising the supersymmetry transformations (I)

In 4d-language, the supersymmetry transformation of $A_\mu$ is

$$\delta_\epsilon A_\mu = -i\epsilon^\dagger_R \Gamma_\mu \Psi_L \implies \delta_\epsilon \dot{A}_\mu = -i\epsilon^\dagger_R \Gamma_\mu \dot{\Psi}_L$$

Choose a basis $\dot{A}_{a\mu}$ for $Z_0$ and let $\dot{\Psi}_{La} = i\dot{A}_{a\mu} \Gamma^\mu \eta_R$ be the corresponding basis for $Z_1$

Expand $\dot{A}_\mu = \dot{A}_{a\mu} X^a$ and $\dot{\Psi}_L = \dot{\Psi}_{La} \theta^a$
Linearising the supersymmetry transformations (I)

In 4d-language, the supersymmetry transformation of $\mathcal{A}_\mu$ is

$$\delta_\epsilon \mathcal{A}_\mu = -i\epsilon^\dagger R \Gamma_\mu \Psi_L \implies \delta_\epsilon \dot{\mathcal{A}}_\mu = -i\epsilon^\dagger R \Gamma_\mu \dot{\Psi}_L$$

Choose a basis $\dot{\mathcal{A}}_{a\mu}$ for $Z_0$ and let $\dot{\Psi}_{La} = i\dot{\mathcal{A}}_{a\mu} \Gamma^\mu \eta_R$ be the corresponding basis for $Z_1$

Expand $\dot{\mathcal{A}}_\mu = \dot{\mathcal{A}}_{a\mu} X^a$ and $\dot{\Psi}_L = \dot{\Psi}_{La} \theta^a$

On the one hand, $\delta_\epsilon \dot{\mathcal{A}}_\mu = \dot{\mathcal{A}}_{a\mu} \delta_\epsilon X^a$
Linearising the supersymmetry transformations (I)

- In 4d-language, the supersymmetry transformation of $\dot{A}_\mu$ is
  \[
  \delta_\epsilon \dot{A}_\mu = -i\epsilon_R^{\dagger} \Gamma_\mu \Psi_L \quad \Rightarrow \quad \delta_\epsilon \dot{A}_\mu = -i\epsilon_R^{\dagger} \Gamma_\mu \Psi_L
  \]

- Choose a basis $\dot{A}_{a\mu}$ for $Z_0$ and let $\dot{\Psi}_{La} = i\dot{A}_{a\mu} \Gamma^\mu \eta_R$ be the corresponding basis for $Z_1$
- Expand $\dot{A}_\mu = \dot{A}_{a\mu} X^a$ and $\dot{\Psi}_L = \dot{\Psi}_{La} \theta^a$
- On the one hand, $\delta_\epsilon \dot{A}_\mu = \dot{A}_{a\mu} \delta_\epsilon X^a$
- but also $\delta_\epsilon \dot{A}_\mu = -i\epsilon_R^{\dagger} \Gamma_\mu \Psi_{La} \theta^a = \dot{A}_{a\nu} \epsilon_R^{\dagger} \Gamma_\mu \Gamma_\nu \eta_R \theta^a$
Linearising the supersymmetry transformations (II)

- Putting both together and using the Clifford relations

\[ \hat{A}_{\alpha\mu} \delta \epsilon X^a = \hat{A}_{\alpha\mu} \epsilon_R^\dagger \eta_R \theta^a + \epsilon_R^\dagger \Gamma_\mu \gamma \eta_R \hat{A}_{\alpha\nu} \theta^a \]
Linearising the supersymmetry transformations (II)

Putting both together and using the Clifford relations

\[ \dot{A}_{a\mu}\delta_{\epsilon} X^a = \dot{A}_{a\mu} \epsilon^\dagger_R \eta_R \theta^a + \epsilon^\dagger_R \Gamma_{\mu} \gamma \eta_R \dot{A}_{a\nu} \theta^a \]

Let \( \epsilon^\dagger_R \eta_R = \epsilon^1 \) and \( \epsilon^\dagger_R \Gamma_{\mu} \gamma \eta_R = \epsilon^2 E_{\mu} \gamma \), so that

\[ \dot{A}_{a\mu}\delta_{\epsilon} X^a = \dot{A}_{a\mu} \epsilon^1 \theta^a + \epsilon^2 \mathcal{E}_a^b \dot{A}_{b\mu} \theta^a \]

where we have used \( E_{\mu} \gamma \dot{A}_{a\nu} = \mathcal{E}_a^b \dot{A}_{b\mu} \).
Linearising the supersymmetry transformations (II)

Putting both together and using the Clifford relations

\[ \dot{A}_{a \mu} \delta \epsilon X^a = \dot{A}_{a \mu} \epsilon^\dagger_R \eta_R \theta^a + \epsilon^\dagger_R \Gamma_{\mu} \eta_R \dot{A}_{a \nu} \theta^a \]

Let \( \epsilon^\dagger_R \eta_R = \epsilon^1 \) and \( \epsilon^\dagger_R \Gamma_{\mu} \eta_R = \epsilon^2 E_{\mu} \gamma \), so that

\[ \dot{A}_{a \mu} \delta \epsilon X^a = \dot{A}_{a \mu} \epsilon^1 \theta^a + \epsilon^2 E_{\mu} \gamma \dot{A}_{b \mu} \theta^a \]

where we have used \( E_{\mu} \gamma \dot{A}_{a \nu} = \epsilon_a^b \dot{A}_{b \mu} \)

Since the \( \dot{A}_{a \mu} \) are a basis,

\[ \delta \epsilon X^a = \epsilon^1 \theta^a + \epsilon^2 \epsilon_b^a \theta^b \]
A one-dimensional supersymmetric sigma model

By analogy with the case of euclidean monopoles, we will explore the geometry of $\mathcal{M}$ by considering a one-dimensional sigma model with fields $X^a$ and $\theta^a$. In contrast with the case of euclidean monopoles, there is no action for this sigma model due to the lack of natural Riemannian metric on $\mathcal{M}$. Since hyperbolic monopoles are $1/2$-BPS, we expect that this sigma model should have 4 real supercharges, although (in this talk) I work with two supercharges at a time.
A one-dimensional supersymmetric sigma model

- By analogy with the case of euclidean monopoles, we will explore the geometry of $\mathcal{M}$ by considering a one-dimensional sigma model with fields $X^a$ and $\theta^a$.
- In contrast with the case of euclidean monopoles, there is no action for this sigma model due to the lack of natural riemannian metric on $\mathcal{M}$.
A one-dimensional supersymmetric sigma model

- By analogy with the case of euclidean monopoles, we will explore the geometry of $\mathcal{M}$ by considering a one-dimensional sigma model with fields $X^a$ and $\theta^a$.
- In contrast with the case of euclidean monopoles, there is no action for this sigma model due to the lack of natural riemannian metric on $\mathcal{M}$.
- Since hyperbolic monopoles are $\frac{1}{2}$-BPS, we expect that this sigma model should have 4 real supercharges, although (in this talk) I work with two supercharges at a time.
Closing the supersymmetry algebra (I)

- Introduce odd derivations $\delta_1$ and $\delta_2$ by

$$\delta_{\epsilon} X^a = \epsilon^1 \delta_1 X^a + \epsilon^2 \delta_2 X^a$$
Closing the supersymmetry algebra (I)

- Introduce odd derivations $\delta_1$ and $\delta_2$ by
  \[ \delta_\epsilon X^a = \epsilon^1 \delta_1 X^a + \epsilon^2 \delta_2 X^a \]

- Explicitly,
  \[ \delta_1 X^a = \theta^a \quad \delta_2 X^a = \mathcal{E}^a_b \theta^b \]
Closing the supersymmetry algebra (I)

- Introduce odd derivations $\delta_1$ and $\delta_2$ by

$$\delta_\epsilon X^a = \epsilon^1 \delta_1 X^a + \epsilon^2 \delta_2 X^a$$

- Explicitly,

$$\delta_1 X^a = \theta^a \quad \text{and} \quad \delta_2 X^a = \mathcal{E}^a_b \theta^b$$

- We demand that they obey the supersymmetry algebra

$$\delta_A \delta_B + \delta_B \delta_A = 2i \delta_{AB} \frac{d}{dt}$$
Closing the supersymmetry algebra (I)

- Introduce odd derivations $\delta_1$ and $\delta_2$ by
  \[
  \delta_\epsilon X^a = \epsilon_1 \delta_1 X^a + \epsilon_2 \delta_2 X^a
  \]

- Explicitly,
  \[
  \delta_1 X^a = \theta^a \quad \quad \delta_2 X^a = \epsilon_b a \theta^b
  \]

- We demand that they obey the supersymmetry algebra
  \[
  \delta_A \delta_B + \delta_B \delta_A = 2i\delta_{AB} \frac{d}{dt}
  \]

- This implies that $\delta_1 \theta^a = iX'^a$ and
  \[
  \delta_2 \theta^a = -iX'^b \epsilon_b a + \theta^b \theta^c \partial_c \epsilon_b a
  \]
Closing the supersymmetry algebra (II)

Closure also requires

\[ \partial_{[b} \mathcal{E}_{c]}^a - \partial_d \mathcal{E}_{[b} \mathcal{E}^e_{c]} \mathcal{E}^d_e a = 0 \]
Closing the supersymmetry algebra (II)

• Closure also requires

\[ \partial_{[b} E_{c]} \alpha - \partial_d E_{[b} \epsilon_{c]} d E_e \alpha = 0 \]

• This is equivalent to

\[ \partial_{[b} E_{c]} \alpha \epsilon_a f + \partial_d E_{[b} \epsilon_{c]} d = 0 \]
Closing the supersymmetry algebra (II)

- Closure also requires

\[ \partial_{[b} \mathcal{E}_{c]}^a - \partial_d \mathcal{E}_{[b}^e \mathcal{E}_{c]}^d \mathcal{E}^a_e = 0 \]

- This is equivalent to

\[ \partial_{[b} \mathcal{E}_{c]}^a \mathcal{E}^f_a + \partial_d \mathcal{E}_{[b}^f \mathcal{E}_{c]}^d = 0 \]

- In terms of the Frölicher–Nijenhuis bracket: \([\mathcal{E}, \mathcal{E}] = 0\)
Closing the supersymmetry algebra (II)

- Closure also requires

\[
\partial_{[b} \mathcal{E}_{c]}^a - \partial_d \mathcal{E}_{[b}^e \mathcal{E}_{c]}^d \mathcal{E}_e^a = 0
\]

- This is equivalent to

\[
\partial_{[b} \mathcal{E}_{c]}^a \mathcal{E}_a^f + \partial_d \mathcal{E}_{[b}^f \mathcal{E}_{c]}^d = 0
\]

- In terms of the Frölicher–Nijenhuis bracket: \([\mathcal{E}, \mathcal{E}] = 0\)
- This is equivalent to the integrability of \(\mathcal{E}\)
Closing the supersymmetry algebra (II)

- Closure also requires

\[ \partial_{[b} E_{c]}^{\ a} - \partial_{d} E_{[b}^{\ e} E_{c]}^{\ d} E_{e}^{\ a} = 0 \]

- This is equivalent to

\[ \partial_{[b} E_{c]}^{\ a} E_{a}^{\ f} + \partial_{d} E_{[b}^{\ f} E_{c]}^{\ d} = 0 \]

- In terms of the Frölicher–Nijenhuis bracket: \([E, E] = 0\)
- This is equivalent to the integrability of \(E\)
- The closure on the \(\theta^{a}\) gives no further constraints
The pluricomplex structure

We have shown that for all $\eta_R \in K^+$ and $\zeta_R \in K^-$ such that $\zeta_R^\dagger \eta_R = 1$, there is an integrable complex structure $\mathcal{E}$ on $T_{CM}$ acting complex linearly.
The pluricomplex structure

- We have shown that for all $\eta_R \in K^+$ and $\zeta_R \in K^-$ such that $\zeta_R^\dagger \eta_R = 1$, there is an integrable complex structure $\mathcal{E}$ on $T_{CM}$ acting complex-linearly.

- By varying $\eta_R$ and $\zeta_R$, one can exhibit complex structures $I$, $J$ and $K$ obeying a quaternion algebra.
The pluricomplex structure

- We have shown that for all $\eta_R \in K^+$ and $\zeta_R \in K^-$ such that $\zeta_R^\dagger \eta_R = 1$, there is an integrable complex structure $E$ on $T_{CM}$ acting complex linearly.
- By varying $\eta_R$ and $\zeta_R$, one can exhibit complex structures $I$, $J$ and $K$ obeying a quaternion algebra.
- This gives a 2-sphere worth of integrable complex structures acting complex-linearly on $T_{CM}$. 
The pluricomplex structure

- We have shown that for all $\eta_R \in K^+$ and $\zeta_R \in K^-$ such that $\zeta_R^\dagger \eta_R = 1$, there is an integrable complex structure $E$ on $T_{\mathbb{C}} M$ acting complex linearly.

- By varying $\eta_R$ and $\zeta_R$, one can exhibit complex structures $I$, $J$ and $K$ obeying a quaternion algebra.

- This gives a 2-sphere worth of integrable complex structures acting complex-linearly on $T_{\mathbb{C}} M$.

- This defines a **pluricomplex structure** on $M$. 
The pluricomplex structure

- We have shown that for all $\eta_R \in K^+$ and $\zeta_R \in K^-$ such that $\zeta_R^\dagger \eta_R = 1$, there is an integrable complex structure $E$ on $T_{C\mathcal{M}}$ acting complex linearly.

- By varying $\eta_R$ and $\zeta_R$, one can exhibit complex structures $I$, $J$ and $K$ obeying a quaternion algebra.

- This gives a 2-sphere worth of integrable complex structures acting complex-linearly on $T_{C\mathcal{M}}$.

- This defines a **pluricomplex structure** on $\mathcal{M}$.

- This means that the moduli $X^a$ and $\theta^a$ belong to a multiplet of the $d = 1 \ N = 4$ supersymmetry algebra, as expected for $\frac{1}{2}$-BPS configurations.
1. Hyperbolic monopoles

2. Supersymmetric Yang–Mills–Higgs in hyperbolic space

3. The geometry of the monopole moduli space

4. Conclusions and future directions
Conclusions

- We have presented a construction of a supersymmetric Yang–Mills–Higgs theory in $H^3$. 

José Miguel Figueroa O’Farrill
Conclusions

We have presented a construction of a supersymmetric Yang–Mills–Higgs theory in $H^3$
whose bosonic BPS configurations are in one-to-one correspondence with (complexified) hyperbolic monopoles.
Conclusions

- We have presented a construction of a supersymmetric Yang–Mills–Higgs theory in $H^3$

- whose bosonic BPS configurations are in one-to-one correspondence with (complexified) hyperbolic monopoles

- We have shown that there is a supersymmetry relating the bosonic and fermionic moduli
Conclusions

- We have presented a construction of a supersymmetric Yang–Mills–Higgs theory in $H^3$
- whose bosonic BPS configurations are in one-to-one correspondence with (complexified) hyperbolic monopoles
- We have shown that there is a supersymmetry relating the bosonic and fermionic moduli
- Closing the algebra requires a pluricomplex structure on the moduli space
Future directions

- It would be good to have a more direct construction of the theory: perhaps coupling supersymmetric Yang–Mills to a conformal supergravity theory in $\mathbb{R}^4$.
Future directions

- It would be good to have a more direct construction of the theory: perhaps coupling supersymmetric Yang–Mills to a conformal supergravity theory in $\mathbb{R}^4$.
- What rôle do the Hitchin metrics play? Are they perhaps regularised metrics?
Future directions

- It would be good to have a more direct construction of the theory: perhaps coupling supersymmetric Yang–Mills to a conformal supergravity theory in $\mathbb{R}^4$.
- What rôle do the Hitchin metrics play? Are they perhaps regularised metrics?
- Can the pluricomplex structure be used to analyse the dynamics of hyperbolic monopoles?
Future directions

- It would be good to have a more direct construction of the theory: perhaps coupling supersymmetric Yang–Mills to a conformal supergravity theory in $\mathbb{R}^4$.

- What rôle do the Hitchin metrics play? Are they perhaps regularised metrics?

- Can the pluricompact structure be used to analyse the dynamics of hyperbolic monopoles?

- Pluricompact manifolds have a unique torsion-free connection leaving the complex structures invariant. Are geodesics with respect to that connection perhaps the trajectories of low-energy hyperbolic monopoles?