Supersymmetric space forms

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Equivalently, they are parallel sections of the bundle

$$\mathcal{E}(M) = TM \oplus \mathfrak{so}(TM)$$

relative to the connection

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$\implies M$ has constant sectional curvature κ.

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The flat and spherical cases are solved (culminating in the work of Wolf in the 1970s), but the hyperbolic and lorentzian cases remain largely open despite many partial results.

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Note: A maximally supersymmetric supergravity background will be abbreviated *vacuum*.

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The maximally symmetric solutions are the lorentzian space forms: smooth discrete quotients of Minkowski space and (the universal covers of) de Sitter and anti de Sitter spaces, depending on the sign of λ.

… and Supergravity

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What is so interesting about this action?

It is *invariant*

It is *invariant* under *supersymmetry transformations*

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Also this really only works as written in four dimensions. In other dimensions supergravity theories might have *other fields* and both the action and supersymmetry transformations become *more complicated*. **But** supergravity theories are *uniquely* determined by representation theory (of relevant superalgebras).

Supergravities

	32	24	20	16	12	8	4
11	M						
10	IIA, IIB			I			
9	$N=2$			$N=1$			
8	$N=2$			$N=1$			
7	$N=4$			$N=2$			
6	$(2,2)$, $(3,1)$, $(4,0)$	$(2,1)$, $(3,0)$		$(1,1)$, $(2,0)$		$(1,0)$	
5	$N=8$	$N=6$		$N=4$		$N=2$	
4	$N=8$	$N=6$	$N=5$	$N=4$	$N=3$	$N=2$	$N=1$

[Van Proeyen, hep-th/0301005]

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defined by the supersymmetry variation of the gravitino:

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(putting all fermions to zero)

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Typically $A = 0$ sets some fields to zero, and the flatness of D constrains the geometry and any remaining fields. The strategy is therefore to study the flatness equations for D.

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- $D = 11$ M [FO–Papadopoulos]

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- bosonic fields:
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  - ⋆ gravitino Ψ, a section of $T^*M \otimes S$, where S is an irreducible real 32-dimensional representation of $C\ell(1,10)$.

- the gravitino variation defines the connection

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Understanding D is essential to understand supersymmetry in $D{=}11$ supergravity.

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In the second case, M is a gravitational wave with a $\mathrm{Spin}(7)$-holonomy transverse space.

[FO hep-th/9904124, Bryant math.DG/0004073]

More generally, in the static case

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d	$H \subset \mathrm{SO}(d)$	ν
10	$\mathrm{SU}(5)$	$\frac{1}{16}$
10	$\mathrm{SU}(2) \times \mathrm{SU}(3)$	$\frac{1}{8}$
8	$\mathrm{Spin}(7)$	$\frac{1}{16}$
8	$\mathrm{SU}(4)$	$\frac{1}{8}$
8	$\mathrm{Sp}(2)$	$\frac{3}{16}$

d	$H \subset \mathrm{SO}(d)$	ν
8	$\mathrm{Sp}(1) \times \mathrm{Sp}(1)$	$\frac{1}{4}$
7	G_2	$\frac{1}{8}$
6	$\mathrm{SU}(3)$	$\frac{1}{4}$
4	$\mathrm{SU}(2) \cong \mathrm{Sp}(1)$	$\frac{1}{2}$
0	$\{1\}$	1

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$H \subset \mathrm{SO}(1,10)$	ν
$\left(\mathrm{Spin}(7) \ltimes \mathbb{R}^8\right) \times \mathbb{R}$	$\frac{1}{32}$
$\left(\mathrm{SU}(4) \ltimes \mathbb{R}^8\right) \times \mathbb{R}$	$\frac{1}{16}$
$\left(\mathrm{Sp}(2) \ltimes \mathbb{R}^8\right) \times \mathbb{R}$	$\frac{3}{32}$
$\left(\mathrm{Sp}(1) \ltimes \mathbb{R}^4\right) \times \left(\mathrm{Sp}(1) \ltimes \mathbb{R}^4\right) \times \mathbb{R}$	$\frac{1}{8}$
$\left(G_2 \ltimes \mathbb{R}^7\right) \times \mathbb{R}^2$	$\frac{1}{16}$
$\left(\mathrm{SU}(3) \ltimes \mathbb{R}^6\right) \times \mathbb{R}^3$	$\frac{1}{8}$
$\left(\mathrm{Sp}(1) \ltimes \mathbb{R}^4\right) \times \mathbb{R}^5$	$\frac{1}{4}$
$\mathbb{R}^9$	$\frac{1}{2}$

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These will be subject of Felipe LEITNER's talk tomorrow.

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- F null: a one parameter $\mu \in \mathbb{R}$ family of *symmetric plane waves* or *indecomposable lorentzian symmetric spaces with solvable transvection group*

[Cahen–Wallach (1970)]

$$g = 2dx^+dx^- - \tfrac{1}{36}\mu^2 \left(4\sum_{i=1}^{3}(x^i)^2 + \sum_{i=4}^{9}(x^i)^2\right)(dx^-)^2 + \sum_{i=1}^{9}(dx^i)^2$$

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[Blau–FO–Hull–Papadopoulos hep-th/0201081]

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KG

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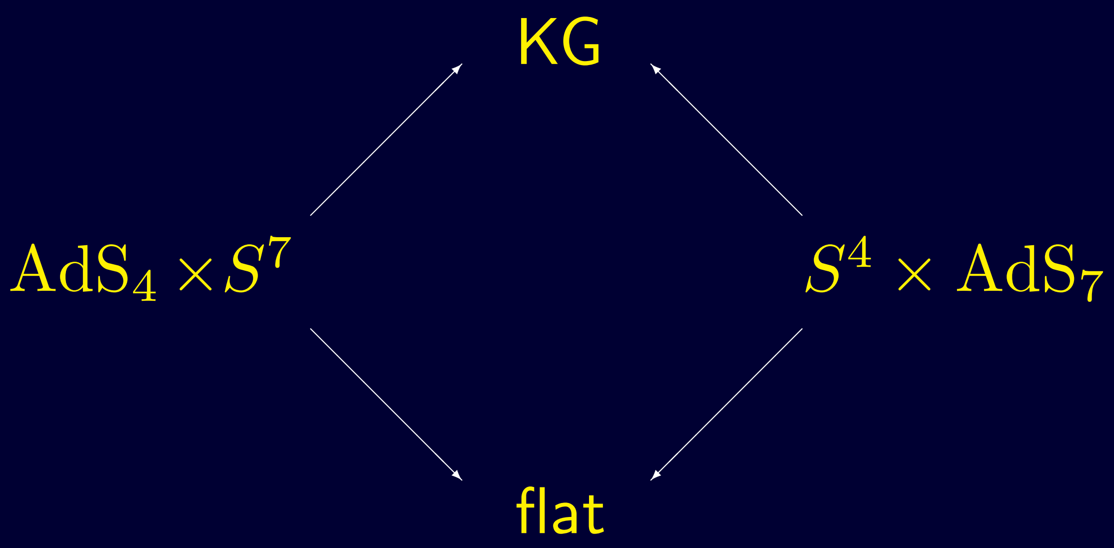

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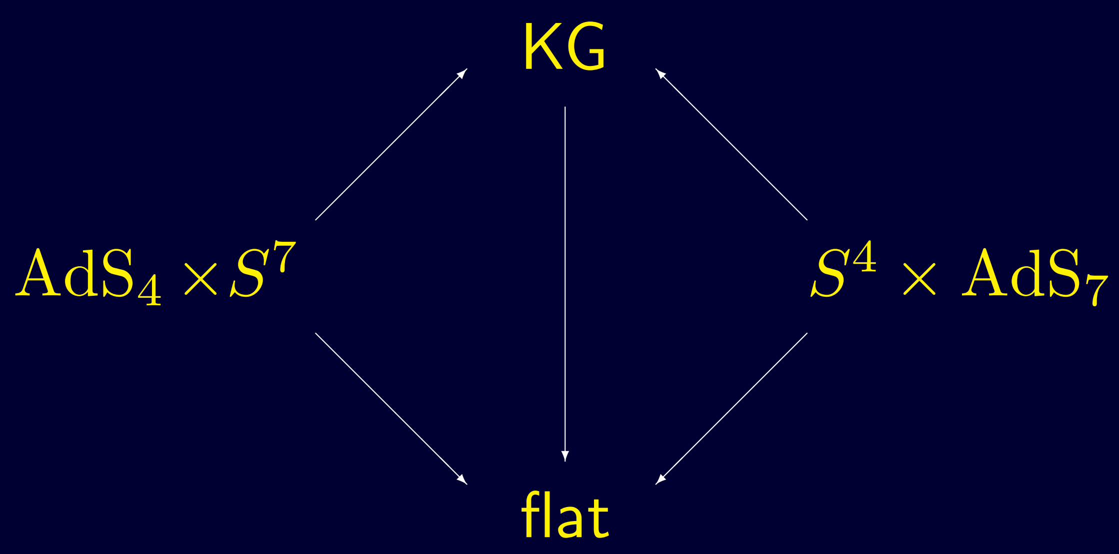

[Back]

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(Unfortunate notation: a 2-Lie algebra is a Lie algebra.)

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$$ad_X[Y, Z] = [\mathrm{ad}_X Y, Z] + [Y, \mathrm{ad}_X Z]$$

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If F is totally antisymmetric then $\langle -, - \rangle$ is an *invariant metric*.

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$$F = G + \star G \qquad \text{where} \qquad G = \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \theta^4 \wedge \theta^5$$

[FO–Papadopoulos `math.AG/0211170`]

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- F non-degenerate case: a one-parameter ($R > 0$) family of vacua

$$\mathrm{AdS}_5(-R) \times S^5(R) \qquad F = \sqrt{\frac{4R}{5}}\left(\mathrm{dvol}(\mathrm{AdS}_5) - \mathrm{dvol}(S^5)\right)$$

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[Blau–FO–Hull–Papadopoulos hep-th/0110242]

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The wave is isometric to a solvable lorentzian Lie group

[Stanciu–FO hep-th/0303212]

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BFHP

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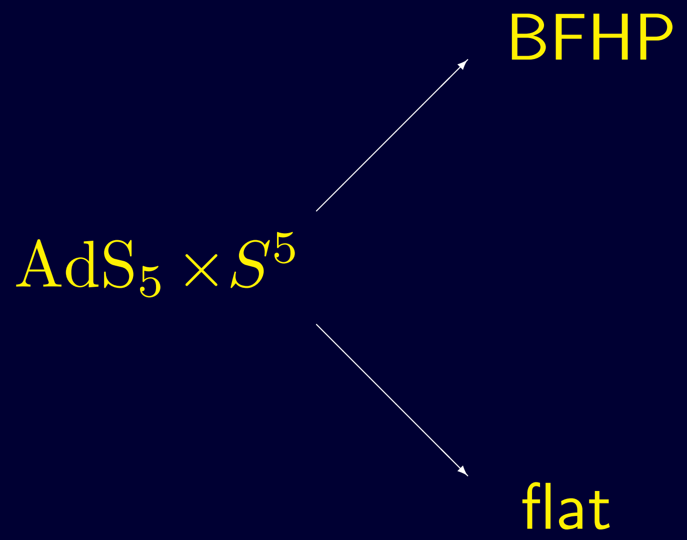

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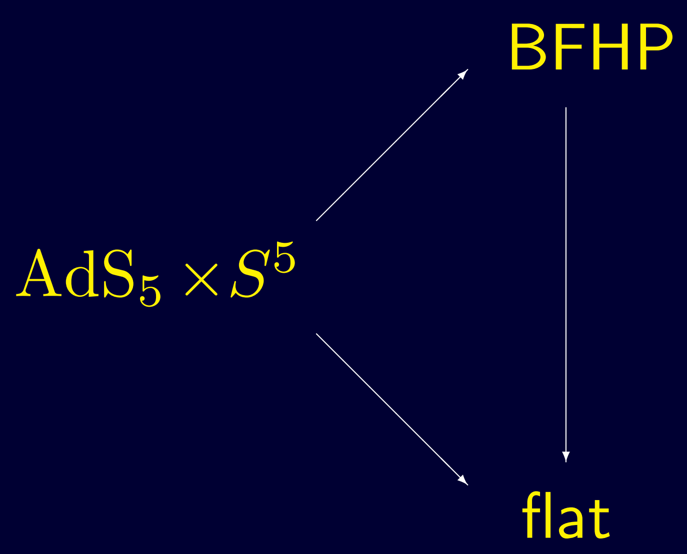

[Back]

Thank you.