A geometric construction of exceptional Lie algebras

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2007 will be known as the year where E8 (and Lie groups) went mainstream...
An international team of mathematicians has detailed a vast complex numerical "structure" which was described more than a century ago.

Mapping the 248-dimensional structure, called E8, took four years of work and produced more data than the Human Genome Project, researchers said.

E8 is a "Lie group", a means of describing symmetrical objects.

The team said their findings may assist fields of physics which use more than four dimensions, such as string theory.
A geometric construction of the exceptional Lie algebras F4 and E8

José Figueroa-O'Farrill

(Submitted on 19 Jun 2007)

We present a geometric construction of the exceptional Lie algebras F4 and E8 starting from the round 8- and 15-spheres, respectively, inspired by the construction of the Killing superalgebra of a supersymmetric supergravity background. (There is no supergravity in the paper.)

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Link back to: arXiv, form interface, contact.
Surfer dude stuns physicists with theory of everything

By Roger Highfield, Science Editor
Last Updated: 6.01pm GMT 14/11/2007

An impoverished surfer has drawn up a new theory of the universe, seen by some as the Holy Grail of physics, which has received rave reviews from scientists.

Garrett Lisi, 39, has a doctorate but no university affiliation and spends most of the year surfing in Hawaii, where he has also been a hiking guide and bridge builder (when he slept in a jungle yurt).

In winter, he heads to the mountains near Lake Tahoe, Nevada, where he snowboards. "Being poor sucks," Lisi says. "It's hard to figure out the secrets of the universe when you're trying to figure out where you and your girlfriend are going to sleep next month."

Despite this unusual career path, his proposal is remarkable because, by the arcane standards of particle physics, it does not require highly complex mathematics.
Introduction

Hamilton

Cayley

Lie

Killing

É. Cartan

Hurwitz

Hopf

J.F. Adams
This talk is about a relation between exceptional objects:

- **Hopf bundles**
- **exceptional Lie algebras**

using a **geometric** construction familiar from **supergravity**: the **Killing (super)algebra**.
Real division algebras

\[
\begin{array}{cccc}
R & C & H & O \\
abla \\
ab ab = ba & ab = ba & ab \neq ba \\
(ab)c = a(bc) & (ab)c = a(bc) & (ab)c = a(bc) & (ab)c \neq a(bc)
\end{array}
\]

These are all the euclidean normed real division algebras. [Hurwitz]
Hopf fibrations

$S^1 \to S^0 \to S^1$

$S^3 \to S^1 \to S^2$

$S^7 \to S^3 \to S^4$

$S^{15} \to S^7 \to S^8$

$S^0 \subset \mathbb{R}$

$S^1 \subset \mathbb{C}$

$S^3 \subset \mathbb{H}$

$S^7 \subset \mathbb{O}$

$S^1 \subset \mathbb{R}^2$

$S^3 \subset \mathbb{C}^2$

$S^7 \subset \mathbb{H}^2$

$S^{15} \subset \mathbb{O}^2$

$S^1 \cong \mathbb{R}P_1$

$S^2 \cong \mathbb{C}P_1$

$S^4 \cong \mathbb{H}P_1$

$S^8 \cong \mathbb{O}P_1$

These are the only examples of fibre bundles where all three spaces are spheres. [Adams]
**Simple Lie algebras**

(over \( \mathbb{C} \))

<table>
<thead>
<tr>
<th>4 classical series:</th>
<th>5 exceptions:</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_{n \geq 1} )</td>
<td>( G_2 )</td>
</tr>
<tr>
<td>( SU(n + 1) )</td>
<td>( 14 )</td>
</tr>
<tr>
<td>( B_{n \geq 2} )</td>
<td>( F_4 )</td>
</tr>
<tr>
<td>( SO(2n + 1) )</td>
<td>( 52 )</td>
</tr>
<tr>
<td>( C_{n \geq 3} )</td>
<td>( E_6 )</td>
</tr>
<tr>
<td>( Sp(n) )</td>
<td>( 78 )</td>
</tr>
<tr>
<td>( D_{n \geq 4} )</td>
<td>( E_7 )</td>
</tr>
<tr>
<td>( SO(2n) )</td>
<td>( 133 )</td>
</tr>
</tbody>
</table>

[Lie]  [Killing, Cartan]
Supergravity

Supergravity is a nontrivial generalisation of Einstein’s theory of General Relativity.

The supergravity universe consists of a lorentzian spin manifold with additional geometric data, together with a notion of Killing spinor.

These spinors generate the Killing superalgebra.

This is a useful invariant of the universe.
Applying the Killing superalgebra construction to the **exceptional Hopf fibration**, one obtains a triple of **exceptional Lie algebras**:

```
S¹⁵
↓
S⁷
↓
S⁸
```

```
E₈
B₄
F₄
```

“Killing superalgebra”
plane of numbers.

Rules of Multiplication in an Algebra of n units.

In general, if we consider an algebra of n units, \( t_1, t_2, \ldots, t_n \), such that \( t_r^2 = -1 \), \( t_r t_s = -t_s t_r \), a product of \( m \) linear factors will contain terms which are all of even order if \( m \) is even, and all of odd order if \( m \) is odd; for the plane of numbers.

Spinors

Rules of Multiplication in an Algebra of n units.

In general, if we consider an algebra of n units, \( t_1, t_2, \ldots, t_n \), such that \( t_r^2 = -1 \), \( t_r t_s = -t_s t_r \), a product of \( m \) linear factors will contain terms which are all of even order if \( m \) is even, and all of odd order if \( m \) is odd; for the
Clifford algebras

$V^n \langle -, - \rangle$ real euclidean vector space

$C\ell(V) = \frac{\bigotimes V}{\langle v \otimes v + |v|^21 \rangle}$ filtered associative algebra

$C\ell(V) \cong \Lambda V$ (as vector spaces)

$C\ell(V) = C\ell(V)_0 \oplus C\ell(V)_1$

$C\ell(V)_0 \cong \Lambda^{\text{even}}V$  \quad $C\ell(V)_1 \cong \Lambda^{\text{odd}}V$
orthonormal frame $e_1, \ldots, e_n$

$$e_i e_j + e_j e_i = -2 \delta_{ij} 1$$

$C\ell (\mathbb{R}^n) =: C\ell_n$

Examples:

$C\ell_0 = \langle 1 \rangle \cong \mathbb{R}$

$C\ell_1 = \langle 1, e_1 | e_1^2 = -1 \rangle \cong \mathbb{C}$

$C\ell_2 = \langle 1, e_1, e_2 | e_1^2 = e_2^2 = -1, e_1 e_2 = -e_2 e_1 \rangle \cong \mathbb{H}$
From this table one can read the type and dimension of the irreducible representations.

\[
\begin{array}{|c|c|}
\hline
n & C\ell_n \\
\hline
0 & \mathbb{R} \\
1 & \mathbb{C} \\
2 & \mathbb{H} \\
3 & \mathbb{H} \oplus \mathbb{H} \\
4 & \mathbb{H}(2) \\
5 & \mathbb{C}(4) \\
6 & \mathbb{R}(8) \\
7 & \mathbb{R}(8) \oplus \mathbb{R}(8) \\
\hline
\end{array}
\]

**Bott periodicity:**

\[C\ell_{n+8} \cong C\ell_n \otimes \mathbb{R}(16)\]

e.g.,

\[C\ell_9 \cong \mathbb{C}(16)\]

\[C\ell_{16} \cong \mathbb{R}(256)\]
\( \text{Cl}_{n} \) has a unique irreducible representation if \( n \) is even and two if \( n \) is odd.

They are distinguished by the action of

\[ e_1 e_2 \cdots e_n \]

which is central for \( n \) odd.

Notation: \( M_n \) or \( M_n^{\pm} \)

Clifford modules

\[ \dim M_n = 2^\lfloor n/2 \rfloor \]
Spinor representations

\[ so_n \rightarrow C\ell_n \]
\[ e_i \wedge e_j \mapsto -\frac{1}{2} e_i e_j \]
\[ \exp \]
\[ \Spin_n \subset C\ell_n \]

\[ s \in \Spin_n, \quad v \in \mathbb{R}^n \quad \Rightarrow \quad sv s^{-1} \in \mathbb{R}^n \]

which defines a 2-to-1 map
\[ \Spin_n \rightarrow \SO_n \]

with archetypical example
\[ \Spin_3 \cong SU_2 \subset \mathbb{H} \]
\[ \SO_3 \cong SO(\text{Im}\mathbb{H}) \]
By restriction, every representation of $C\ell_n$ defines a representation of $\text{Spin}_n$:

$C\ell_n \subset \text{Spin}_n$

$M = \Delta = \Delta_+ \oplus \Delta_-$ \hspace{1cm} $\Delta_\pm$ \hspace{1cm} chiral spinors

$M^{\pm} = \Delta$ \hspace{1cm} $\Delta$ \hspace{1cm} spinors

One can read off the type of representation from

$\text{Spin}_n \subset (C\ell_n)_0 \cong C\ell_{n-1}$

$\dim \Delta = 2^{(n-1)/2}$ \hspace{1cm} $\dim \Delta_\pm = 2^{(n-2)/2}$
Spinor inner product

\((-,-)\) nondegenerate form on \(\Delta\)

\[(\varepsilon_1, \varepsilon_2) = (\varepsilon_2, \varepsilon_1)\]

\[(\varepsilon_1, e_i \cdot \varepsilon_2) = - (e_i \cdot \varepsilon_1, \varepsilon_2) \quad \forall \varepsilon_i \in \Delta\]

\[\implies (\varepsilon_1, e_i e_j \cdot \varepsilon_2) = - (e_i e_j \cdot \varepsilon_1, \varepsilon_2)\]

which allows us to define

\[
[-,-] : \Lambda^2 \Delta \to \mathbb{R}^n
\]

\[
\langle [\varepsilon_1, \varepsilon_2], e_i \rangle = (\varepsilon_1, e_i \cdot \varepsilon_2)
\]
Spin geometry
Spin manifolds

$M^n$ differentiable manifold, orientable, spin

$g$ riemannian metric

GL($M$) $\leftarrow$ O($M$) $w_1 = 0$ $\leftarrow$ SO($M$) $w_2 = 0$ $\leftarrow$ Spin($M$)

$GL_n$ $\leftarrow$ O$_n$ $\leftarrow$ SO$_n$ $\leftarrow$ Spin$_n$
Possible $\text{Spin}(M)$ are classified by $H^1(M; \mathbb{Z}/2)$.

e.g., $M = S^n \subset \mathbb{R}^{n+1}$

$$O(M) = O_{n+1}$$
$$\text{SO}(M) = \text{SO}_{n+1}$$
$$\text{Spin}(M) = \text{Spin}_{n+1}$$

$$S^n \cong O_{n+1}/O_n \cong \text{SO}_{n+1}/\text{SO}_n \cong \text{Spin}_{n+1}/\text{Spin}_n$$

$$\pi_1(M) = \{1\} \implies \text{unique spin structure}$$
**Spinor bundles**

\[ C\ell(TM) \]
\[ \downarrow \]
\[ M \]

\[ C\ell(TM) \cong \Lambda TM \]

\[ S(M) := \text{Spin}(M) \times_{\text{Spin}_n} \Delta \]

\[ S(M)_\pm := \text{Spin}(M) \times_{\text{Spin}_n} \Delta_\pm \]

We will assume that \( C\ell(TM) \) acts on \( S(M) \).
The Levi-Cività connection allows us to differentiate spinors

\[ \nabla : S(M) \rightarrow T^* M \otimes S(M) \]

which in turn allows us to define

**parallel spinor** \( \nabla \epsilon = 0 \)

**Killing spinor** \( \nabla_X \epsilon = \lambda X \cdot \epsilon \)

**Killing constant**
If $(M,g)$ admits parallel spinors, $(M,g)$ is \textbf{Ricci-flat}.

$(M,g)$ is \textbf{Einstein} if

$$R = 4\lambda^2 n(n - 1)$$

$$\implies \lambda \in \mathbb{R} \cup i\mathbb{R}$$

Today we only consider \textbf{real} $\lambda$. 
Killing spinors have their origin in **supergravity**.

The name stems from the fact that they are “**square roots**” of Killing vectors.

\[
\begin{align*}
\epsilon_1, \epsilon_2 & \quad \text{Killing} \\
\left[\epsilon_1, \epsilon_2\right] & \quad \text{Killing}
\end{align*}
\]
Which manifolds admit real Killing spinors?

\((M, g)\)

\((\overline{M}, \overline{g})\) \hspace{1cm} \text{metric cone}

\[\overline{M} = \mathbb{R}^+ \times M\]

\[\overline{g} = dr^2 + r^2 g\]

Killing spinors in \((M, g)\)

\((\lambda = \pm \frac{1}{2})\)

\[1-1\]

parallel spinors in the cone
More precisely...

If $n$ is odd, Killing spinors are in one-to-one correspondence with chiral parallel spinors in the cone: the chirality is the sign of $\lambda$.

If $n$ is even, Killing spinors with both signs of $\lambda$ are in one-to-one correspondence with the parallel spinors in the cone, and the sign of $\lambda$ enters in the relation between the Clifford bundles.
This reduces the problem to one (already solved) about the holonomy group of the cone.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Holonomy</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$\text{SO}_n$</td>
</tr>
<tr>
<td>$2m$</td>
<td>$\text{U}_m$</td>
</tr>
<tr>
<td>$2m$</td>
<td>$\text{SU}_m$</td>
</tr>
<tr>
<td>$4m$</td>
<td>$\text{Sp}_m \cdot \text{Sp}_1$</td>
</tr>
<tr>
<td>$4m$</td>
<td>$\text{Sp}_m$</td>
</tr>
<tr>
<td>7</td>
<td>$G_2$</td>
</tr>
<tr>
<td>8</td>
<td>$\text{Spin}_7$</td>
</tr>
</tbody>
</table>

Or else the cone is flat and $M$ is a sphere.
Killing superalgebra
Construction of the algebra

\((M, g)\) riemannian spin manifold

\[ \mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1 \]

\[ \mathfrak{k}_1 = \{ \text{Killing spinors} \} \]

(with \( \lambda = \frac{1}{2} \))

\[ \mathfrak{k}_0 = [\mathfrak{k}_1, \mathfrak{k}_1] \subset \{ \text{Killing vectors} \} \]
\([[-, -]] : \Lambda^2 \mathfrak{k} \rightarrow \mathfrak{k} \) ?

\([[-, -]] : \Lambda^2 \mathfrak{k}_0 \rightarrow \mathfrak{k}_0 \quad \checkmark \quad [\cdot, \cdot] \text{ of vector fields}

\([[-, -]] : \Lambda^2 \mathfrak{k}_1 \rightarrow \mathfrak{k}_0 \quad \checkmark \quad g([\varepsilon_1, \varepsilon_2], X) = (\varepsilon_1, X \cdot \varepsilon_2)

\([[-, -]] : \mathfrak{k}_0 \otimes \mathfrak{k}_1 \rightarrow \mathfrak{k}_1 \quad ? \quad \text{spinorial Lie derivative!}

\[\text{Kosmann} \quad \text{Lichnerowicz}\]
\[ X \in \Gamma(TM) \quad \text{Killing} \quad \iff \quad \mathcal{L}_X g = 0 \]

\[ A_X := Y \mapsto -\nabla_Y X \]

\[ \varrho : \mathfrak{so}(TM) \to \text{End}S(M) \quad \text{spinor representation} \]

\[ \mathcal{L}_X := \nabla_X + \varrho(A_X) \quad \text{spinorial Lie derivative} \]

\[ \mathcal{L}_X Y = \nabla_X Y + A_X Y = \nabla_X Y - \nabla_Y X = [X, Y] \]
Properties

\[ \forall X, Y \in \mathfrak{k}_0, \quad Z \in \Gamma(TM), \quad \varepsilon \in \Gamma(S(M)), \quad f \in C^\infty(M) \]

\[ \mathcal{L}_X(Z \cdot \varepsilon) = [X, Z] \cdot \varepsilon + Z \cdot \mathcal{L}_X \varepsilon \]

\[ \mathcal{L}_X(f \varepsilon) = X(f) \varepsilon + f \mathcal{L}_X \varepsilon \]

\[ [\mathcal{L}_X, \nabla Z] \varepsilon = \nabla [X, Z] \varepsilon \]

\[ [\mathcal{L}_X, \mathcal{L}_Y] \varepsilon = \mathcal{L}_{[X,Y]} \varepsilon \]

\[ \forall \varepsilon \in \mathfrak{k}_1, X \in \mathfrak{k}_0 \]

\[ \mathcal{L}_X \varepsilon \in \mathfrak{k}_1 \]

\[ [\cdot, \cdot] : \mathfrak{k}_0 \otimes \mathfrak{k}_1 \to \mathfrak{k}_1 \]

\[ [X, \varepsilon] := \mathcal{L}_X \varepsilon \]
The Jacobi identity

Jacobi: \( \Lambda^3 \mathfrak{k} \rightarrow \mathfrak{k} \)

\( (X, Y, Z) \mapsto [X, [Y, Z]] - [[X, Y], Z] - [Y, [X, Z]] \)

4 components:

\( \Lambda^3 \mathfrak{k}_0 \rightarrow \mathfrak{k}_0 \) \( \checkmark \) Jacobi for vector fields

\( \Lambda^2 \mathfrak{k}_0 \otimes \mathfrak{k}_1 \rightarrow \mathfrak{k}_1 \) \( \checkmark \) \( [\mathcal{L}_X, \mathcal{L}_Y] \varepsilon = \mathcal{L}_{[X,Y]} \varepsilon \)

\( \mathfrak{k}_0 \otimes \Lambda^2 \mathfrak{k}_1 \rightarrow \mathfrak{k}_0 \) \( \checkmark \) \( \mathcal{L}_X (Z \cdot \varepsilon) = [X, Z] \cdot \varepsilon + Z \cdot \mathcal{L}_X \varepsilon \)

\( \Lambda^3 \mathfrak{k}_1 \rightarrow \mathfrak{k}_1 \) ? but \( \mathfrak{k}_0 \) — equivariant
Some examples

\[ S^7 \subset \mathbb{R}^8 \quad \mathfrak{k}_0 = \mathfrak{so}_8 \quad \mathfrak{k}_1 = \Delta_+ \quad 28 + 8 = 36 \quad \mathfrak{so}_9 \]

\[ S^8 \subset \mathbb{R}^9 \quad \mathfrak{k}_0 = \mathfrak{so}_9 \quad \mathfrak{k}_1 = \Delta \quad 36 + 16 = 52 \quad \mathfrak{f}_4 \]

\[ S^{15} \subset \mathbb{R}^{16} \quad \mathfrak{k}_0 = \mathfrak{so}_{16} \quad \mathfrak{k}_1 = \Delta_+ \quad 120 + 128 = 248 \quad \mathfrak{e}_8 \]

\[ (\mathfrak{k}_1 \otimes \Lambda^3 \mathfrak{k}_1^*)^{\mathfrak{k}_0} = 0 \quad \implies \quad \text{Jacobi} \]

Resulting Lie algebras are simple.
A sketch of the proof

Two observations:

1) The bijection between Killing spinors and parallel spinors in the cone is **equivariant** under the action of isometries.

Use the cone to calculate $\mathcal{L}_X \varepsilon$.

2) In the cone, $\mathcal{L}_X \varepsilon = \varphi(A_X) \varepsilon$ and since $X$ is **linear**, the endomorphism $A_X$ is constant.

It is the natural action on spinors.
A (slight) generalisation

\[ S^9 \subset \mathbb{R}^{10} \quad C\ell_9 \cong \mathbb{C}(16) \quad M = \Delta \otimes \mathbb{C} \]

irreducible real spinor module

\[ \text{Spin}(9) \to \text{SO}(16) \]

complex spinor bundle

\[ S \to S^9 \quad \text{dvol}(S^9) = i \]

complex symmetric inner product

\[ \langle X \cdot \psi_1, \psi_2 \rangle = + \langle \psi_1, X \cdot \psi_2 \rangle \]
Killing spinors

\[ K_\pm = \{ \psi \in \Gamma(S) | \nabla_X \psi = \pm \frac{1}{2} X \cdot \psi \} \]
\[ \mathfrak{k}_0 = \mathfrak{so}_\mathbb{C}(10) \]
\[ \mathfrak{k}_1 = K_+ \oplus K_- \]

Natural brackets well-defined, but Jacobi fails!

(Killing-Yano)

\[ g_0 = \mathfrak{so}_\mathbb{C}(10) \oplus \mathbb{C} \quad g_1 = \mathfrak{k}_1 \]

\[ [\psi_+, \psi_-] = \cdots + \langle \psi_+, \psi_- \rangle \]

\[ z \in \mathbb{C} \implies [z, \psi] = \frac{1}{3} D\psi \]

empirical!
Jacobi:

\[
[[\psi_+, \psi_-], \chi_+] - [[\chi_+, \psi_-], \psi_+] = 0 \quad \forall \psi_\pm, \chi_\pm \in K_\pm
\]

is satisfied, even when

\[
(\Lambda^3 g_1 \rightarrow g_1)^{g_0} \neq 0
\]

The resulting Lie algebra is \textbf{E6} (complexified)
Open questions

• Other **exceptional** Lie algebras? $E_7$ should follow from the $11$-sphere, but this is still work in progress. $G_2$?

• Are the Killing superalgebras of the Hopf spheres related?

• What structure in the $15$-sphere has $E_8$ as automorphisms?