Higher-dimensional gauge theory

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Outline of first lecture

- Motivation
- Instantons on $\mathbb{R}^4$
- $\mathbb{H}$ reformulation
- $\mathbb{O}$ extension
- Moment map interpretation of $\mathbb{O}$ instantons

Motivation

- The gauge-theoretic apparatus

  - $*$ principal bundles
  - $*$ connections
  - $*$ Yang–Mills equations

  makes sense in any dimension.

- Why then the insistence in low (≤ 4) dimension? i.e. vortices, monopoles, instantons

- Our aim is to exhibit equally natural equations in $d > 4$

Conventions

- G compact Lie group with Lie algebra $\mathfrak{g}$
- Tr an invariant scalar product on $\mathfrak{g}$
- $\mathbb{R}^n$ denotes euclidean space; that is, with the 'dot' product
- A will always denote a gauge field: it only exists locally but the notation shall not reflect it
- $F_A = dA + \frac{1}{2}[A, A]$ will denote the associated field-strength

Instantons on $\mathbb{R}^4$

- Let x_1, x_2, x_3, x_4 be oriented coordinates for $\mathbb{R}^4$
- Hodge $\star : \Omega^2(\mathbb{R}^4) \longrightarrow \Omega^2(\mathbb{R}^4)$
- $\star^2 = \mathrm{id}$, whence
$$\Omega^2(\mathbb{R}^4) = \Omega^2_+ \oplus \Omega^2_-$$
- A is (anti)self-dual if $\star F_A = \pm F_A$

- Bianchi identity $d_A F_A = 0$ implies the Yang–Mills equation

$$d_A \star F_A = 0$$

for (anti)self-dual A

- (A)SD connections minimise the Yang–Mills functional

$$\int \mathrm{Tr}\, F_A \wedge \star F_A \geq \left| \int \mathrm{Tr}\, F_A \wedge F_A \right|$$

with equality $\iff$ A is (A)SD

- a first-order equation implies a second-order equation and moreover the solutions of the first-order equation are “minimal”: this is one the signatures of supersymmetry
- can consider instantons on any manifold with an $\mathrm{SO}(4)$ structure; that is, a riemannian orientable 4-manifold
- but still seems very four-dimensional!

Quaternions

- associative non-commutative division algebra
- $\mathbb{H} = \mathbb{R}\left\langle \mathbf{1}, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k} \right\rangle$ obeying

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -\mathbf{1}$$

- $\mathbb{H} \ni \boldsymbol{q} = x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k} + x_4\mathbf{1}$, $x_i \in \mathbb{R}$
- $\boldsymbol{q}^* = -x_1\boldsymbol{i} - x_2\boldsymbol{j} - x_3\boldsymbol{k} + x_4\mathbf{1}$

- $(\boldsymbol{q}_1\boldsymbol{q}_2)^* = \boldsymbol{q}_2^*\boldsymbol{q}_1^*$
- $\mathrm{Im}\,\mathbb{H} = \mathbb{R}\,\langle \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\rangle$, $\mathrm{Re}\,\mathbb{H} = \mathbb{R}\mathbf{1}$
- $\langle q_1, q_2\rangle = \mathrm{Re}\,(q_1^* q_2)$, $|q|^2 = \langle q, q\rangle$
- $|q_1 q_2| = |q_1||q_2|$ (normed algebra)
- $\mathrm{Sp}(1) = \{\boldsymbol{q} \in \mathbb{H} | |\boldsymbol{q}| = 1\} \cong \mathrm{SU}(2)$

Quaternionic instantons

- imaginary units: $(\mathrm{Im}\,\mathbb{H})_1 = \mathrm{Im}\,\mathbb{H} \cap \mathrm{Sp}(1)$
- $\boldsymbol{u} \in (\mathrm{Im}\,\mathbb{H})_1$, $L_{\boldsymbol{u}} : \mathbb{H} \to \mathbb{H}$ is skew-symmetric, $\omega_{\boldsymbol{u}} \in \Omega^2(\mathbb{R}^4)$
- $\star\omega_{\boldsymbol{u}} = -\omega_{\boldsymbol{u}}$
- a gauge field A is SD $\iff$ $F_A \perp \omega_{\boldsymbol{u}}$ for all $u \in (\mathrm{Im}\,\mathbb{H})_1$
- enough to impose this for $u = \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, obtaining

$$F_{12} = F_{34} \qquad F_{13} = -F_{24} \qquad F_{14} = F_{23}$$

- The ASD equations are obtained using right multiplication $R_{\boldsymbol{u}}$

Octonions

- non-associative non-commutative real division algebra
- $\mathbb{O} = \mathbb{R}\left\langle \boldsymbol{e}_1, \ldots, \boldsymbol{e}_7, \boldsymbol{e}_8 = \boldsymbol{1} \right\rangle$
- for $1 \leq i \neq j \leq 7$,

$$\boldsymbol{e}_i^2 = -\boldsymbol{1} \qquad \text{and} \qquad \boldsymbol{e}_i \boldsymbol{e}_j = \sum_{k=1}^{7} \varphi_{ijk} \boldsymbol{e}_k \ ,$$

where $\varphi \in \Omega^3(\mathbb{R}^7)$ is given by

$$\varphi = dx^{125} + dx^{136} + dx^{147} - dx^{237} + dx^{246} - dx^{345} + dx^{567}$$

- $\boldsymbol{e}_i^* = -\boldsymbol{e}_i$, for $i = 1, \ldots, 7$, $\mathbf{1}^* = \mathbf{1}$
- $(\boldsymbol{o}_1\boldsymbol{o}_2)^* = \boldsymbol{o}_2^*\boldsymbol{o}_1^*$
- $\langle \boldsymbol{o}_1, \boldsymbol{o}_2 \rangle = \mathrm{Re}\,(\boldsymbol{o}_1^*\boldsymbol{o}_2)$
- $|\boldsymbol{o}_1\boldsymbol{o}_2| = |\boldsymbol{o}_1||\boldsymbol{o}_2|$

Octonionic instantons

- $\boldsymbol{u} \in (\operatorname{Im}\mathbb{O})_1$ imaginary units
- $L_{\boldsymbol{u}} : \mathbb{O} \to \mathbb{O}$ skew-symmetric, $\omega_{\boldsymbol{u}} \in \Omega^2(\mathbb{R}^8)$
- Define a gauge field A to be an octonionic instanton if $F_A \perp \omega_{\boldsymbol{u}}$ for all $\boldsymbol{u} \in (\operatorname{Im}\mathbb{O})_1$
- (there is a similar notion of anti-instanton using right multiplication)

- enough to impose this for $\boldsymbol{u} = \boldsymbol{e}_i$, $i = 1, \ldots, 7$:

$$F_{12} - F_{34} - F_{58} + F_{67} = 0$$

$$F_{13} + F_{24} - F_{57} + F_{68} = 0$$

$$F_{14} - F_{23} + F_{56} - F_{78} = 0$$

$$F_{15} + F_{28} + F_{37} - F_{46} = 0$$

$$F_{16} - F_{27} + F_{38} + F_{45} = 0$$

$$F_{17} + F_{26} - F_{35} + F_{48} = 0$$

$$F_{18} - F_{25} - F_{36} - F_{47} = 0$$

- just the right number of equations to have a chance at a finitedimensional moduli space

- these equations imply the Yang–Mills equation and also minimize the Yang–Mills functional

- they also allow for a moment-map interpretation

Recap of first lecture

- instantons on $\mathbb{R}^4 \leftrightarrow \mathbb{H}$
- "instantons" on $\mathbb{R}^8 \leftrightarrow \mathbb{O}$
- moment map interpretation still holds
- Octonionic instanton equations still seem rather "exceptional"
- The aim of the second lecture is to place them in their natural geometric context, which reveals the existence of other instanton-like equations in any dimension

Outline of second lecture

- $\mathbb{O}$ instantons revisited
- Generalised self-duality
- Riemannian holonomy and parallel forms
- Examples in dimensions 8, 7 and 6

The Cayley form in $\mathbb{R}^8$

- $\Omega = -\frac{1}{6}\sum_{i=1}^{7} \omega_{\boldsymbol{e}_i} \wedge \omega_{\boldsymbol{e}_i} \in \Omega^4(\mathbb{R}^8)$
- equivalently, $\Omega = \star_7\varphi + \varphi \wedge dx^8$, $\star\Omega = \Omega$
- explicitly,

$$\Omega = dx^{1234} - dx^{1267} + dx^{1357} - dx^{1456} + dx^{2356} + dx^{2457} + dx^{3467} \\ + dx^{1258} + dx^{1368} + dx^{1478} - dx^{2378} + dx^{2468} - dx^{3458} + dx^{5678}$$

- Ω is left invariant by a subgroup $\mathrm{Spin}(7) \subset \mathrm{SO}(8)$, which still acts irreducibly on $\mathbb{R}^8$

- $\phi_\Omega : \Omega^2(\mathbb{R}^8) \to \Omega^2(\mathbb{R}^8)$, defined by

$$\phi_\Omega(F) = \star(\Omega \wedge F) \ ,$$

is symmetric and traceless $\implies$ can be diagonalised

- $\phi_\Omega^2 + 2\phi_\Omega = 3\,\mathrm{id}$, with eigenspace decomposition

$$\Omega(\mathbb{R}^8) = \Omega^2_7 \oplus \Omega^2_{21}$$

Under $\mathrm{Spin}(7)$, Ω^2_7 corresponds to the defining 7-dimensional representation and Ω^2_{21} is the adjoint representation

$\mathbb{O}$ instantons revisited

- F_A is SD $\iff$ $F_A \perp \omega_{\boldsymbol{u}}$ for all $\boldsymbol{u} \in (\mathrm{Im}\,\mathbb{O})_1$
- F_A is SD $\iff$ $\phi_\Omega(F_A) = F_A$
- Note: ASD is not the other equation $\phi_\Omega(F_A) = -3F_A$, it is the same equation relative to a different Ω obtained from the ω using right multiplication
- Since $d\Omega = 0$, if F_A is SD, then

$$d_A \star F_A = d_A(\Omega \wedge F_A) = d\Omega \wedge F_A + \Omega \wedge d_A F_A = 0$$

$\Longrightarrow$ A is Yang–Mills

- Furthermore

$$\int \mathrm{Tr}\, F_A \wedge \star F_A \geq \int \Omega \wedge \mathrm{Tr}\, F_A \wedge F_A \ ,$$

with equality $\iff$ A is SD

Generalised self-duality

- (M^n, g) oriented, riemannian manifold
- $\Omega \in \Omega^4(M)$ defines symmetric, traceless $\phi_\Omega : \Omega^2(M) \to \Omega^2(M)$ by

$$\phi_\Omega(F) = \star(\star\Omega \wedge F)$$

- Diagonalising ϕ_Ω,

$$\Omega^2(M) = \bigoplus_\lambda \Omega^2_\lambda$$

- We say that A is λ-selfdual if

$$\phi_\Omega(F_A) = \lambda F_A \qquad \exists \lambda \neq 0$$

- A is Yang-Mills if $d \star \Omega = 0$
- canonical example: $\Omega = \mathrm{dvol}$ in four-dimensional manifold
- second canonical example: Ω Cayley form in $\mathrm{Spin}(7)$-holonomy manifold

Parallel 4-forms and riemannian holonomy groups

- Parallel forms are in particular co-closed

- The holonomy principle establishes a bijective correspondence between

  * parallel differential forms on (M, g), and
  * invariants in the exterior powers of the holonomy representation

- (M, g) irreducible, simply-connected, complete, non-symmetric riemannian manifold

- Berger list of holonomy groups:

d	$H \subset \mathrm{SO}(d)$	Geometry	Parallel forms
n	$\mathrm{SO}(n)$	generic	dvol
$2n$	$\mathrm{U}(n)$	Kähler	ω
$2n$	$\mathrm{SU}(n)$	Calabi–Yau	$\omega,\ \Lambda_n^{\mathbb{C}}$
$4n$	$\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)$	Quaternionic Kähler	Ξ_4
$4n$	$\mathrm{Sp}(n)$	Hyperkähler	$\omega_i,\ \omega_j,\ \omega_j$
7	G_2		$\varphi_3,\ \star\varphi$
8	$\mathrm{Spin}(7)$		Ω_4

- Almost all have parallel 4-forms!

Examples

- Some famous examples:
  - $*$ $d = 8$: $\mathrm{Spin}(7)$ instanton equation
  - $*$ $d = 7$: G_2 instanton equations
  - $*$ $d = 6$: Kähler Yang–Mills equations

G_2 instanton equations in $d = 7$

- The 4-form now is $\star\varphi$ whose associated map on $\Omega^2(M)$ satisfies

$$(\phi_{\star\varphi} + \mathbf{21})(\phi_{\star\varphi} - \mathbf{1}) = 0$$

  with eigenspace decomposition

$$\Omega^2(M) = \Omega^2{}_{-2}(M) \oplus \Omega^2{}_1(M)$$

  with ranks 7 and 14, respectively

- the last summand corresponds to the embedding $\mathfrak{g}_2 \subset \mathfrak{so}(7)$

- The 1-instanton equations are the G_2 instanton equations

$$\star F_A = \varphi \wedge F_A$$

- The Levi-Cività connection on any manifold with G_2 holonomy furnishes an example of such a G_2 instanton

Kähler–Yang–Mills equations in $d = 6$

- The 4-form now is $\frac{1}{2}\omega^2$ whose associated map on $\Omega^2(M)$ satisfies

$$(\phi_{-\frac{1}{2}\omega^2} - 2\mathbf{1})(\phi_{-\frac{1}{2}\omega^2} - \mathbf{1})(\phi_{-\frac{1}{2}\omega^2} + \mathbf{1}) = 0$$

with eigenspace decomposition

$$\Omega^2(M) = {\Omega^2}_2(M) \oplus {\Omega^2}_{-1}(M) \oplus {\Omega^2}_1(M)$$

with ranks 1, 8 and 6, respectively, where

$*$ ${\Omega^2}_2(M)$ are the multiples of the Kähler form ω,
$*$ ${\Omega^2}_{-1}(M)$ are the forms $F + F^*$, for F a $(0, 2)$-form,

$*$ $\Omega^2{}_1(M)$ are the real primitive $(1,1)$ forms

- The first and last summand correspond to the embedding $\mathfrak{u}(3) = \mathfrak{u}(1) \oplus \mathfrak{su}(3) \subset \mathfrak{so}(6)$
- -1-instantons obey the Kähler–Yang–Mills equations

$$F_A^{0,2} = 0 \qquad \text{and} \qquad F_A \cdot \omega = 0$$

- The Donaldson–Uhlenbeck–Yau theorem relates them to stable holomorphic bundles, whence it is possible in principle to construct many examples

Outline for third and fourth lectures

- supersymmetry
- supersymmetric sigma models
- supersymmetric Yang–Mills on $\mathbb{R}^{9,1}$
- dimensional reductions and cohomological field theories
- supersymmetric Yang–Mills on $\mathrm{Spin}(7)$-holonomy manifolds and octonionic instantons
- octonionic instantons and “aholomorphic” curves

References

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