Six-dimensional supergravity and lorentzian Lie groups

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HU Berlin, 24 September 2003

Based on work in collaboration with

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- Sonia Stanciu
  - ⋆ `hep-th/0303212` (*JHEP* 06 (2003) 025)

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- Ali Chamseddine and Wafic Sabra (CAMS, Beirut)
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- Ali Chamseddine and Wafic Sabra (CAMS, Beirut)
  - ⋆ `hep-th/0306278`
- George Papadopoulos (King's College, London)
  - ⋆ `hep-th/0211089` (*JHEP* 03 (2003) 048)
  - ⋆ `math.AG/0211170` (*J Geom Phys* to appear)

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Equivalently, they are parallel sections of the bundle

$$\mathcal{E}(M) = TM \oplus \mathfrak{so}(TM)$$

relative to the connection

$$D_X \begin{pmatrix} \xi \\ A \end{pmatrix} = \begin{pmatrix} \nabla_X \xi - A(X) \\ \nabla_X A - R(X, \xi) \end{pmatrix}$$

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$\implies M$ has constant sectional curvature κ.

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The flat and spherical cases are solved (culminating in the work of Wolf in the 1970s), but the hyperbolic and lorentzian cases remain largely open despite many partial results.

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Note: A maximally supersymmetric supergravity background will be abbreviated *vacuum*.

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Extremals of this action—namely, Ricci-flat manifolds—are called *spacetimes*.

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The maximally symmetric solutions are the lorentzian space forms: smooth discrete quotients of Minkowski space and (the universal covers of) de Sitter and anti de Sitter spaces, depending on the sign of λ.

... and Supergravity

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What is so interesting about this action?

It is *invariant*

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Also... this really only works as written in four dimensions. In other dimensions supergravity theories might have *other fields* and both the action and supersymmetry transformations become *more complicated*. **But** supergravity theories are *uniquely* determined by representation theory (of relevant superalgebras).

Supergravities

	32			24		20	16		12	8	4
11	M										
10	IIA	IIB					I				
9	$N=2$						$N=1$				
8	$N=2$						$N=1$				
7	$N=4$						$N=2$				
6	$(2,2)$	$(3,1)$	$(4,0)$	$(2,1)$	$(3,0)$		$(1,1)$	$(2,0)$		$(1,0)$	
5	$N=8$			$N=6$			$N=4$			$N=2$	
4	$N=8$			$N=6$		$N=5$	$N=4$		$N=3$	$N=2$	$N=1$

[Van Proeyen, hep-th/0301005]

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- S a real vector bundle of spinors (i.e., modules over $C\ell(TM)$)

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defined by the supersymmetry variation of the gravitino:

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(putting all fermions to zero)

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Maximal supersymmetry $\Longrightarrow D$ is flat and $A = 0$.

Typically $A = 0$ sets some fields to zero, and the flatness of D constrains the geometry and any remaining fields. The strategy is therefore to study the flatness equations for D.

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- $D = 10$ IIB and I [FO–Papadopoulos]

Classifications of supergravity vacua

In the table we have highlighted the "top" theories whose vacua are known already:

- $D = 4$ $N = 1$ [Tod (1984)]
- $D = 6$ $(1, 0)$, $(2, 0)$ [Chamseddine–FO–Sabra, Gutowski–Martelli–Reall]
- $D = 10$ IIB and I [FO–Papadopoulos]
- $D = 11$ M [FO–Papadopoulos]

Vacua of $D = 11$ supergravity

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$\mathrm{AdS}_4 \times S^7$

$S^4 \times \mathrm{AdS}_7$

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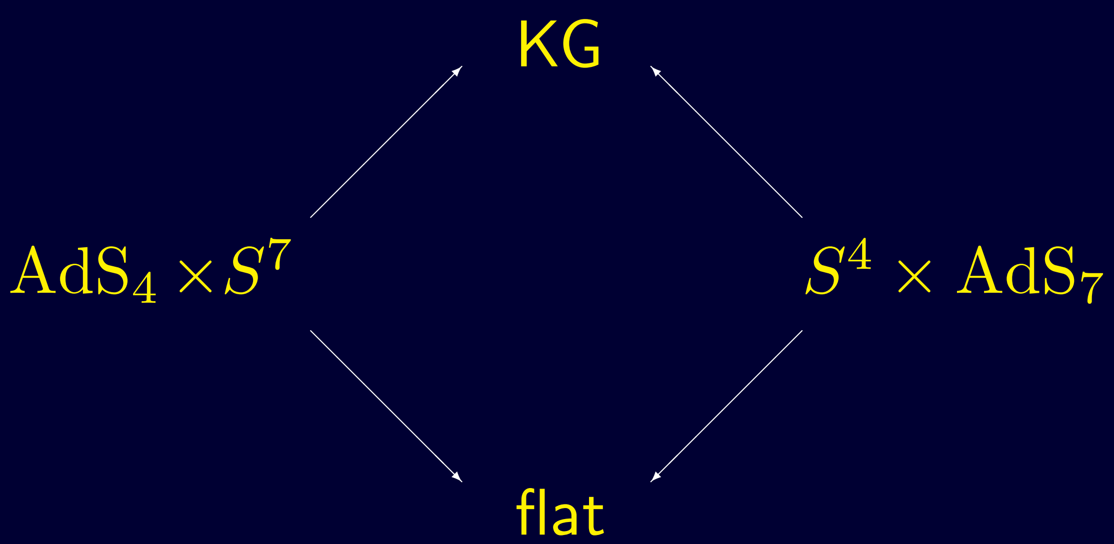
KG

$\mathrm{AdS}_4 \times S^7$ $\qquad$ $S^4 \times \mathrm{AdS}_7$

Vacua of $D = 11$ supergravity



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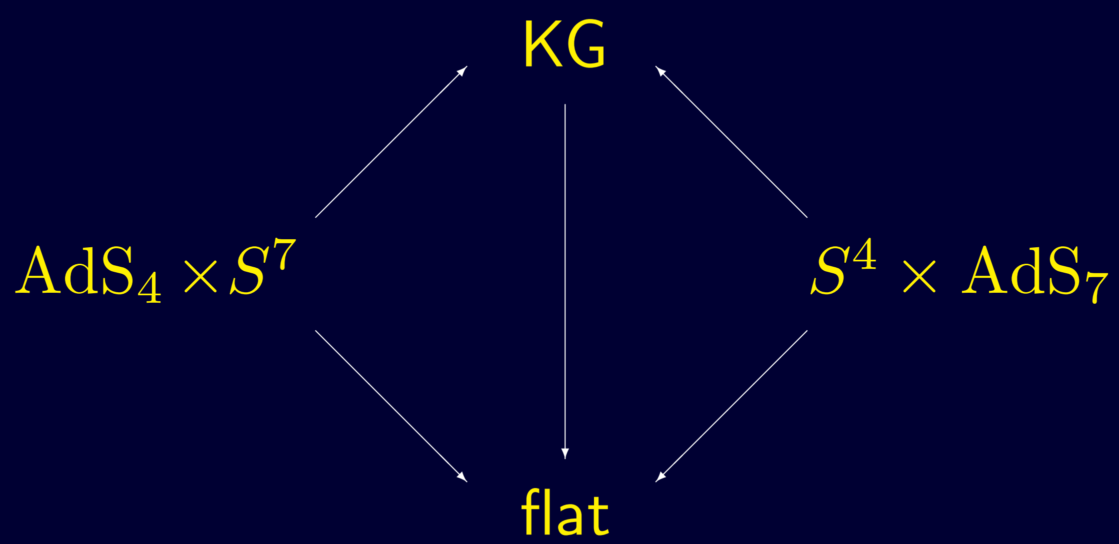

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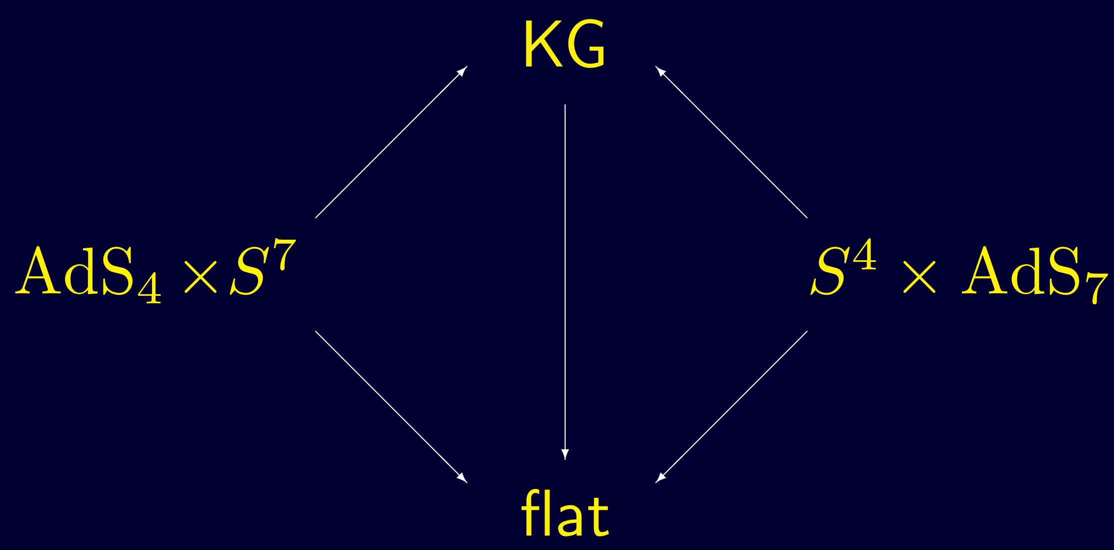

where KG is an indecomposable lorentzian symmetric space with solvable tranvection group (cf. Cahen–Wallach)

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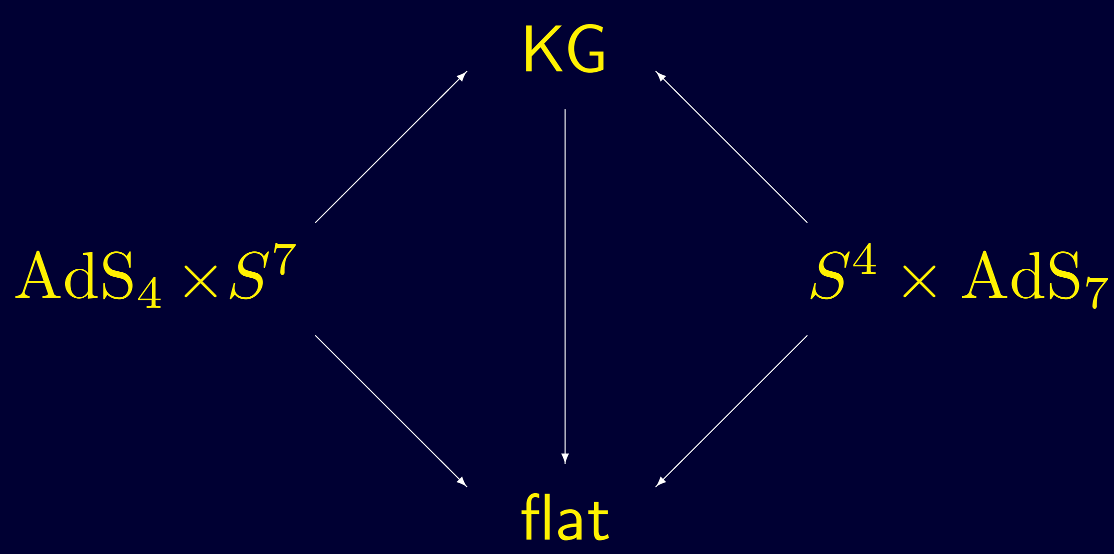

where KG is an indecomposable lorentzian symmetric space with solvable tranvection group (cf. Cahen–Wallach), discovered in this context by Kowalski-Glikman.

Vacua of $D = 10$ IIB supergravity

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BFHP

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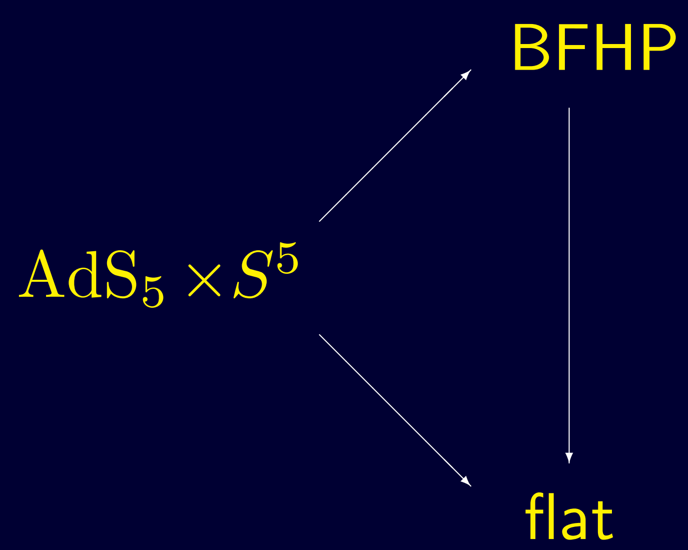
Vacua of $D = 10$ IIB supergravity



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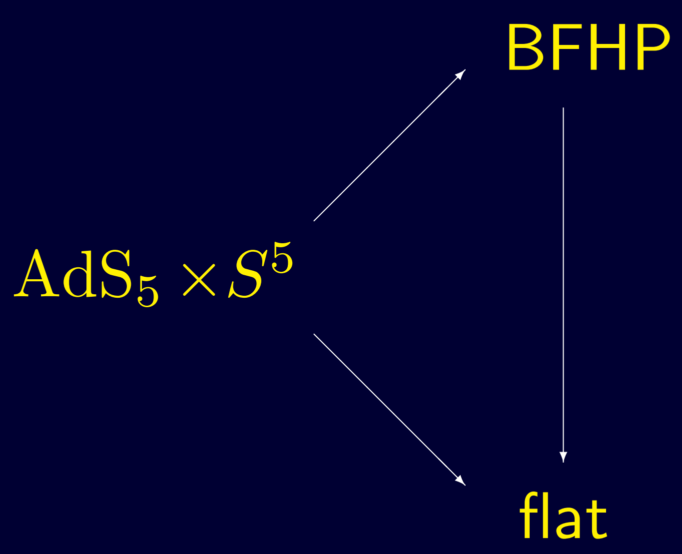

where BFHP is a Cahen–Wallach space, locally isometric to a lorentzian Lie group.

Vacua of $(1,0)$ $D=6$ supergravity

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is a real 8-dimensional representation of $\mathrm{Spin}(1,5) \times \mathrm{Sp}(1)$.

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This construction is due to Medina and Revoy.

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Every metric Lie algebra is obtained as an orthogonal direct sum of indecomposables.

[See also FO–Stanciu `hep-th/9506152`]

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$$\mathbb{R} \to \Lambda^2\mathbb{R}^4 \cong \mathfrak{so}(4)$$

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The third case is a six-dimensional version of the Nappi-Witten spacetime, NW_6, discovered by Meessen. [Meessen `hep-th/0111031`]

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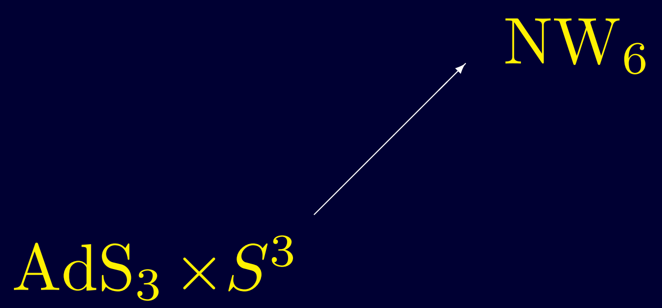

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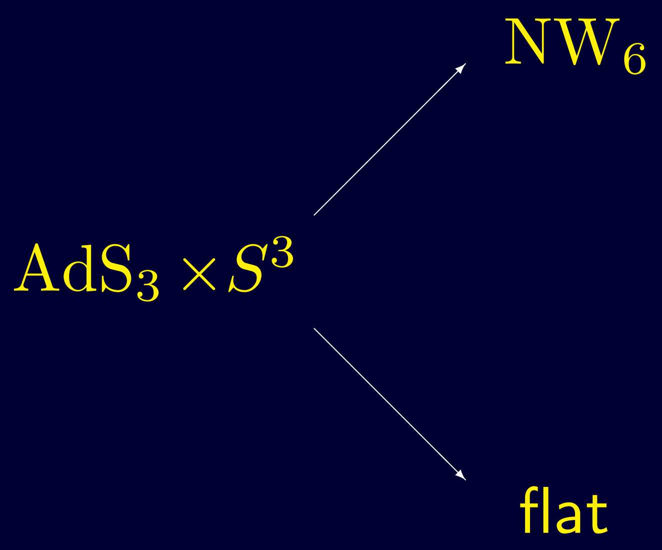

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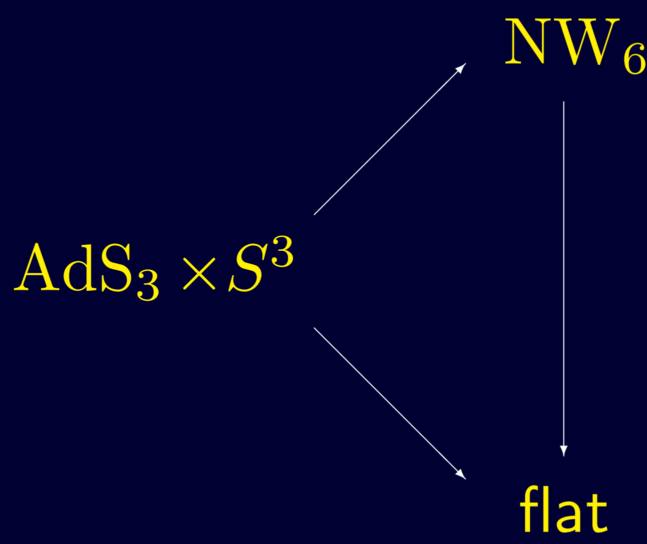

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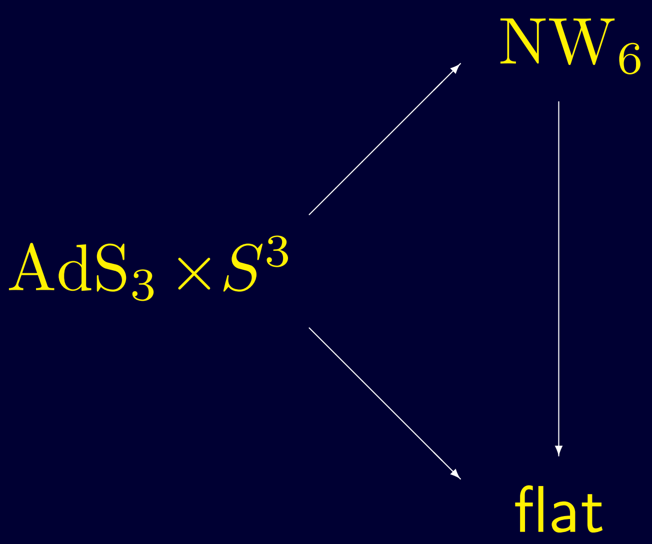

which can be interpreted here as group contractions à la Inönü–Wigner.

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$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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is the group contraction

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In summary, up to the action of the $\mathrm{Sp}(2)$ R-symmetry group, $(2,0)$ vacua are in one-to-one correspondence with $(1,0)$ vacua.

Thank you.

Lunch beckons.