Symplectic Reduction
-
The Modern, the Classic and the Postmodern

Carlos Zapata-Carratala

Abstract

In this note we review symplectic reduction in the context of modern differential geometry and the theory of Poisson manifolds. Our discussion will cover its historical origins in analytical mechanics as well as an introduction to Dirac geometry, a relatively recent branch of differential geometry that sprang out of it.

Contents

1 The Modern: Symplectic Reduction as an example of Coisotropic Reduction

Let us begin by stating the original result[1] of J. Marsden and A. Weinstein [MW74]. Let (M, ω) be a symplectic manifold and let us denote the musical isomorphism induced by the non-degenerate 2-form by

$$TM \underset{\omega^\sharp}{\overset{\omega^\flat}{\rightleftarrows}} T^*M \ , \qquad \text{where } \omega^\flat(X) := i_X\omega.$$

Consider the smooth action of a (connected) Lie group G on the symplectic manifold

$$\phi : G \times M \to M$$

and its infinitesimal counterpart

$$\psi : \mathfrak{g} \to \Gamma(TM)$$

[1] Our statement of the Marsden-Weinstein theorem will be a somewhat simplified version of what appeared in their original paper, since they considered possibly infinite-dimensional symplectic manifolds and looked at arbitrary regular values of the moment map. We will only consider finite-dimensional smooth manifolds and use the fact that, assuming regularity, the proof of symplectic reduction via arbitrary preimages of the moment map can be constructed from symplectic reduction via de preimage of 0, see Remark 1.22 of [HH04] for a proof.

which is a Lie algebra morphism. The action is called a **symplectic action** when

$$\phi_g^*\omega = \omega \quad \forall g \in G,$$

or equivalently when

$$\mathcal{L}_{\psi(a)}\omega = 0 \quad \forall a \in \mathfrak{g}.$$

If this is the case, Cartan's magic formula implies that the 1-forms corresponding to the infinitesimal action $\omega^\flat(\psi(a)) \in \Omega^1(M)$, are automatically closed

$$d(\omega^\flat(\psi(a))) = di_{\psi(a)}\omega = \mathcal{L}_{\psi(a)}\omega - i_{\psi(a)}d\omega = 0.$$

The definition of moment map is motivated by the requirement that these 1-forms are closed. Thus, a **moment map** is usually defined as a smooth map of the form

$$\mu : M \to \mathfrak{g}^*$$

with its dual usually called the comoment map

$$\overline{\mu} : \mathfrak{g} \to \mathrm{C}^\infty(M),$$

and the added requirement that it is ϕ-Ad^*-equivariant

$$\mu \circ \phi_g = \mathrm{Ad}_g^* \circ \mu \quad \forall g \in G.$$

We then say that the action ϕ is a G-**Hamiltonian action** on the symplectic manifold (M, ω) with moment map μ when

$$\omega^\flat(\psi(a)) = d\overline{\mu}(a).$$

Theorem (Symplectic Reduction)**.** *Let a Hamiltonian G-action on a symplectic manifold (M, ω) with moment map $\mu : M \to \mathfrak{g}^*$. Assume that $0 \in \mathfrak{g}^*$ is a regular value, so that $M_0 := \mu^{-1}(0)$ is a smooth manifold on which, from equivariance of μ, G acts. Let us further assume that this action is free and proper so we have the following diagram of smooth manifolds*

$$\begin{tikzcd} M_0 \arrow[r, hook, "i"] \arrow[d, two heads, "p"'] & M \\ \tilde{M} & \end{tikzcd}$$

where i is the inclusion of M_0 as a submanifold and p is the principal G-bundle projection onto the orbit space $\tilde{M} := M_0/G$. Then, there exists a unique symplectic form $\tilde{\omega}$ on the orbit space $\tilde{M}$ such that

$$p^*\tilde{\omega} = i^*\omega.$$

The symplectic manifold $(\tilde{M}, \tilde{\omega})$ is usually denoted by $M//G$ and is called the ***symplectic reduction*** *of (M, ω) by the Hamiltonian G-action.*

Proof. For a detailed and well-written discussion about the geometry of moment maps see [HH04], in particular, section 1.3 contains a complete proof of this theorem. The construction proposed there, as well as in the original paper [MW74], for the reduced symplectic manifold $(\tilde{M}, \tilde{\omega})$ is fairly elementary and it is based on three key observations:

- Given a vector space with a non-degenerate bilinear form (V, β) and a subspace $W \subset V$, the reduced vector space $\tilde{V} := W^\beta/(W \cap W^\beta)$ inherits canonically a bilinear form $\tilde{\beta}$ induced by the restriction of β to the subspace W^β and the projection to the quotient. Here W^β denotes β-orthogonal complement.
- By construction of the reduced manifold $\tilde{M}$, the tangent space at any orbit $[x]$ is given by $T_{[x]}\tilde{M} \cong T_x M_0/\psi(\mathfrak{g})|_x$, where we have $\psi(\mathfrak{g})|_x \subset T_x M_0$ from the fact that G acts on M_0.
- By the defining conditions of the moment map and the infinitesimal group action we have $TM_0 = \psi(\mathfrak{g})^\omega|_{M_0}$ and thus $\psi(\mathfrak{g})|_{M_0} \subset \psi(\mathfrak{g})^\omega|_{M_0}$.

Since the group G acts by symplectomorphisms, the above observations give a well-defined non-degenerate, skewsymmetric bilinear form $\tilde{\omega}$ on $T\tilde{M}$ which satisfies $p^*\tilde{\omega} = i^*\omega$ by construction. Smoothness of this bilinear form follows from smoothness of the pull-back by p, since the restriction $i^*\omega$ is clearly smooth. That $\tilde{\omega}$ is closed follows from the fact that p is a surjective submersion so its pull-back is injective

$$p^* d\tilde{\omega} = dp^*\tilde{\omega} = di^*\omega = i^* d\omega = 0 \quad \Rightarrow \quad d\tilde{\omega} = 0.$$

Recall that symplectic manifolds are always even dimensional and from the construction above one can clearly see that $\dim M_0 = \dim M - 2\dim G$, hence the notion that symplectic reduction amounts to "quotienting by G twice", which motivates the notation $M//G$ for the reduced symplectic manifold. □

In what follows we shall show that symplectic reduction can be understood as the simplest example of coisotropic reduction, the geometric counterpart of a natural construction of reduction for Poisson algebras. Good references for the material presented here are the seminal paper by A. Weinstein [Wei88] and the more recent and very comprehensive notes on Poisson manifolds by R. Fernandes and I. Marcut [FM14].

A **Poisson algebra** is a triple $(A, \cdot, \{,\})$ with A a $\mathbb{R}$-vector space, $(A, \cdot)$ a commutative (unital, associative) algebra and $(A, \{,\})$ a Lie algebra such that the following **Leibniz rule** holds

$$\{a, b \cdot c\} = \{a, b\} \cdot c + b \cdot \{a, c\}.$$

In what follows we shall omit the commutative product. Equivalently, this condition can be rephrased as

$$\mathrm{ad}_{\{,\}} : A \to \mathrm{Der}(A, \cdot)$$

making the adjoint map of the Lie algebra a homomorphism of Lie algebras from A to the derivations of both bilinear products on A, $\mathrm{Der}(A, \{,\}) \cap \mathrm{Der}(A, \cdot)$. The algebra endomorphisms given by the adjoint map are usually called **Hamiltonian derivations** and are denoted by $X_a := \{a, -\}$. A linear map $\psi : A \to B$ is called a **Poisson homomorphism** if $\psi : (A, \cdot) \to (B, \cdot)$ is a commutative algebra homomorphism and $\psi : (A, \{,\}) \to (B, \{,\})$ is a Lie algebra homomorphism. Poisson algebras live on the intersection of commutative algebra and Lie theory and modulo the Leibniz rule; many of the developments in Poisson geometry over the last 40 years can be understood as the interplay between these two branches of mathematics. What is quite remarkable is that understanding this interaction seems to give us insights about the fundamental structure underlying modern physical theories.

An object that serves as an example of this interplay between commutative algebras and Lie algebras is what we call an **coisotrope** $I \subset A$, which is taken to be simultaneously an ideal of the commutative algebra $(A, \cdot)$, or multiplicative ideal for short, and a subalgebra of the Lie algebra $(A, \{,\})$, or Lie subalgebra for short. Coisotropes play a fundamental role in the theory of Poisson algebras, this shall be illustrated below by the Poisson reduction construction and the definition of the Poisson category.

The presence of a coisotrope I in a Poisson algebra A enables the construction known as **Poisson reduction**. Recall that the $\mathbb{R}$-vector quotient of an algebra by an ideal canonically inherits an algebra structure, then, if we were to find a subspace of the Poisson algebra $S \subset A$ where I sits as both a multiplicative ideal and a Lie ideal, we could take the quotient S/I to find a new Poisson algebra. The following result makes this construction precise.

Theorem (Algebraic Poisson Reduction)**.** *Let a Poisson algebra A with coisotrope $I \subset A$. Let us denote the Lie idealizer (sometimes called normalizer) of the coisotrope by*

$$N(I) := \{s \in A| \quad \{s, a\} \in I, \forall a \in I\},$$

which is the largest Lie subalgebra of A containing I as a Lie ideal, and define the quotient vector space $A' := N(I)/I$ with the canonical projection $q : N(I) \to A'$. Then, A' inherits a Poisson algebra structure.

Proof. First note that the Jacobi identity of the Lie bracket $\{,\}$ implies that $N(I)$ is a Lie subalgebra, which by construction contains I as a Lie ideal. Let us show that the bracket given by

$$\{q(s_1), q(s_2)\}' := q(\{s_1, s_2\})$$

is well-defined bilinear bracket on A'. Consider $s_3 = s_2 + a$ for any $a \in I$ so that $q(s_3) = q(s_2)$, it will suffice to show that $\{q(s_1), q(s_3)\}' = \{q(s_1), q(s_2)\}'$. Indeed, we have

$$\{q(s_1), q(s_3)\}' = q(\{s_1, s_2 + a\}) = q(\{s_1, s_2\}) + q(\{s_1, a\}) = \{q(s_1), q(s_2)\}' + q(\{s_1, a\})$$

but $\{s_1, a\} \in I$ so $q(\{s_1, a\}) = 0$, thus giving the desired result. The beacket $\{,\}'$ inherits the antisymmetry and Jacobi properties from the restriction of $\{,\}$ to $N(I)$ by construction. Now note the Leibniz rule implies that $N(I)$ is a multiplicative subalgebra and the fact that I is a Lie subalgebra implies that it sits in the idealizer $I \subset N(I)$ as a multiplicative ideal. By an analogous quotient construction, we find a well-defined commutative algebra structure in the quotient vector space $(A', \cdot)$. It only remains to show that these two quotient algebraic structures satisfy the Leibniz rule, indeed this follows directly from the fact that $\{,\}$ satisfies the Leibniz rule:

$$\begin{aligned}\{q(s_1), q(s_2)q(s_3)\} &= \{q(s_1), q(s_2 s_3)\} = q(\{s_1, s_2 s_3\}) \\ &= q(\{s_1, s_2\}s_3 + s_2\{s_1, s_3\}) \\ &= \{q(s_1), q(s_2)\}q(s_3) + q(s_2)\{q(s_1), q(s_3)\}.\end{aligned}$$

□

Let us go back to the geometric realm. A **Poisson manifold** is a smooth manifold P whose commutative algebra of smooth functions has the structure of a Poisson algebra $(\mathrm{C}^\infty(P), \cdot, \{,\})$. A **Poisson map** is a smooth map between Poisson manifolds whose pull-back on functions gives a morphism of Poisson algebras. Recall that derivations of smooth functions are canonically isomorphic to vector fields as $\mathrm{C}^\infty(P)$-modules

$$\mathrm{Der}(\mathrm{C}^\infty(P), \cdot) \cong \Gamma(TP),$$

and thus Hamiltonian derivations will become **Hamiltonian vector fields** and will be denoted by

$$f \in \mathrm{C}^\infty(P) \mapsto X_f = \{f, -\} \in \Gamma(TP).$$

It follows directly from this definition that the assignment of Hamiltonian vector fields is a $\mathbb{R}$-linear Lie algebra homomorphism,

$$X_{\{f,g\}} = [X_f, X_g].$$

Our observation about derivations of smooth functions allows for a more geometric definition of a Poisson manifold as a pair (P, π) with $\pi \in \Gamma(\wedge^2 TM)$ a bivector field, called the **Poisson bivector**, that satisfies an integrability condition[2] which is equivalent to the Jacobi identity of the bracket on smooth functions defined by

$$\{f, g\}_\pi := \pi(df, dg).$$

Note that this bracket satisfies the Leibniz rule automatically by construction. The bivector gives a musical map

$$\pi^\sharp : T^*P \to TP$$

which is a vector bundle morphism covering the identity but not necessarily an isomorphism of vector bundles. The Hamiltonian vector field of a function $f \in \mathrm{C}^\infty(P)$ can now be given explicitly as

$$X_f = \pi^\sharp(df).$$

[2] In the Schouten algebra of multivector fields extending the Lie algebra of vector fields $(\Gamma(\wedge^\bullet TM), [,])$, a Poisson bivector must satisfy $[\pi, \pi] = 0$, which locally translates into some quadratic 1st order PDEs for the components of π in a coordinate chart.

Vector spaces modelling the tangent spaces of Poisson manifolds, that is, pairs (V, π) where V is a $\mathbb{R}$-vector space and $\pi \in \wedge^2 V$ is a bivector, are called **Poisson vector spaces**. Some subspaces of Poisson vector spaces may exhibit some compatibility condition with the bivector. A subspace $W \subset V$ is called **coisotropic** if its annihilator $W^0 \subset V^*$ is isotropic with respect to π, that is $\pi|_{W^0} = 0$. Equivalently, a coisotropic subspace satisfies

$$\pi^\sharp(W^0) \subset W.$$

A submanifold $i : C \hookrightarrow P$ is called a **coisotropic submanifold** of the Poisson manifold (P, π) if $T_x C \subset (T_x P, \pi_x)$ is a coisotropic subspace for all $x \in C$. Denoting by $I_C := \ker(i^*) \subset \mathrm{C}^\infty(P)$ the vanishing ideal of the submanifold C we formulate the following result giving alternative equivalent definitions of coisotropic submanifolds.

Proposition. *Let $i : C \hookrightarrow P$ be a closed submanifold of the Poisson manifold (P, π), then the following are equivalent:*

1. *C is coisotropic,*
2. *I_C is a coisotrope in the Poisson algebra $(C^\infty(P), \cdot, \{,\}_\pi)$,*
3. *for all $c \in I_C$ the Hamiltonian vector field X_c is tangent to C: $X_c|_C \in \Gamma(TC)$.*

Proof. Equivalence between *1.* and *2.* is shown from the observation that for any two $c, c' \in I_C$, the differentials dc, dc' restricted to C give sections of TC^0, in fact all sections arise as linear combinations in this way, and that the definition of the bracket $\{,\}_\pi$ is such that $\{c, c'\}_\pi \in I_C$ iff $\pi(i^* dc, i^* dc') = 0$. Equivalence between *2.* and *3.* follows from the fact that a vector field X is tangent to C iff $X[I_C] \subset I_C$. $\square$

It follows from this proposition that any coisotropic submanifold C carries an involutive tangent distribution given by the images of the Hamiltonian vector fields of the (locally generating) elements of the vanishing ideal I_C. Let us denote this distribution by $X_{I_C} \subset TC$ and note that $[X_{I_C}, X_{I_C}] \subset X_{I_C}$ from the fact that I_C is a coisotrope. This means that there will be a (singular) foliation in C, denoted by $\mathcal{X}_C$, integrating the tangent distribution of Hamiltonian vector fields of vanishing functions. The following theorem gives the standard construction of what is known as **coisotropic reduction**.

Theorem (Coisotropic Reduction)**.** *Let (P, π) be a Poisson manifold and $i : C \hookrightarrow P$ a closed coisotropic submanifold. Assume that the tangent distribution X_{I_C} integrates to a regular foliation in such a way that there is a surjective submersion from C to the space of leaves*

$$q : C \to P' := C/\mathcal{X}_C.$$

Then, the manifold P' inherits a Poisson bracket on functions $(C^\infty(P'), \{,\}')$ uniquely determined by the condition

$$q^*\{f, g\}' = i^*\{F, G\} \quad \forall f, g \in C^\infty(P')$$

and for all leaf-wise constant extensions $F, G \in C^\infty(P)$, i.e functions of the ambient Poisson manifold satisfying

$$q^* f = i^* F, \quad q^* g = i^* G.$$

Proof. A complete proof of the more general Marsden-Ratiu reduction theorem for Poisson manifolds can be found in [MR86]. We can work locally to see how algebraic Poisson reduction models coisotropic reduction. Restricting to a sufficiently small neighbourhood $U \subset P$ with non-empty intersection with C, we can faithfully identify C with its vanishing ideal I_C and regard $\mathrm{C}^\infty(C) \cong \mathrm{C}^\infty(U)/I_C$. Functions on the leaf space are identified with leaf-wise constants thus, since the foliation $\mathcal{X}_C$ is generated by the Hamiltonian vector fields, we can regard

$$\mathrm{C}^\infty(P') \cong \{r \in \mathrm{C}^\infty(C) | \quad X_c[r] = \{c, r\} \in I_C, \forall c \in I_C\}/I_C.$$

But this is just the Poisson algebra reduction $(\mathrm{C}^\infty(P'), \cdot, \{,\}')$ of the (local) Poisson algebra $(\mathrm{C}^\infty(U), \cdot, \{,\}|_U)$. □

Let us now connect back to symplectic geometry. Symplectic manifolds are precisely Poisson manifolds with non-degenerate Poisson bivectors. Indeed, a symplectic manifold (M, ω) carries a bivector $\omega^{\sharp\sharp} \in \Gamma(\wedge^2 TM)$ defined via the musical isomorphism as

$$\omega^{\sharp\sharp}(\alpha, \beta) := \omega(\omega^\sharp(\alpha), \omega^\sharp(\beta)).$$

This defines a bilinear antisymmetric bracket bracket on functions the Leibniz rule $\{,\}_\omega$, as is the case for any smooth bivector field. It is a simple computation to show that

$$\mathrm{Jac}_{\{,\}_\omega} = 0 \quad \Leftrightarrow \quad d\omega = 0,$$

then all symplectic manifolds are Poisson manifolds with non-degenerate bivector. Conversely, all non-degenerate Poisson bivectors clearly define symplectic forms. Under these definitions, the Hamiltonian vector fields of symplectic manifolds now satisfy the following defining property

$$\omega(X_f, X_g) = \{f, g\}_\omega.$$

A submanifold C of a symplectic manifold (M, ω) is called **coisotropic** when for all $x \in C$ the tangent space T_xC is a coisotropic subspace of the symplectic vector space (T_xM, ω_x). It is straightforward to check that C is a coisotropic submanifold in the symplectic manifold (M, ω) iff C is a coisotropic submanifold in the Poisson manifold $(M, \omega^{\sharp\sharp})$.

Hamiltonian G-actions can be similarly defined for general Poisson manifolds simply by requiring the group to act via Poisson maps. All definitions involving moment maps and infinitesimal actions are left unchanged and we see that G-equivariance together with the Hamiltonian property of the moment map implies that the comoment map is a Lie algebra homomorphism

$$\{\overline{\mu}(a), \overline{\mu}(b)\}_\omega = \overline{\mu}([a, b]).$$

This clearly shows that $\mu^{-1}(0)$ is a coisotropic submanifold since $I_{\mu^{-1}(0)} = \langle \overline{\mu}(\mathfrak{g}) \rangle_{\mathrm{C}^\infty(M)}$ by construction. The symplectic reduction conditions guarantee that the involutive distribution of Hamiltonian vector fields integrates to a regular foliation, which in this case becomes simply the orbits of the group action. The reduced symplectic form is then easily identified with the non-degenerate reduced Poisson bivector.

2 The Classic: origins of Symplectic Reduction in Analytical Mechanics

There are two main streams of development in the history analytical mechanics where symplectic and Poisson reduction procedures were implicitly used or, at least, hinted at:

- Symmetries of mechanical systems and Noether's theorem in Hamiltonian mechanics - a modern formulation with historical notes is found in the Introduction and Chapter 4 of [AM78].
- Dirac's theory for constrained Hamiltonian systems - original paper [Dir50] and a modern treatment in Section 1.0.1 of [LiB12].

The starting point here is that a Poisson or symplectic manifold acts as the phase space of a mechanical systems, e.g. the canonical symplectic structure on a cotangent bundle (T^*Q, ω_Q) is regarded as the "space of positions and momenta" for a particle moving in Q or the linear Poisson structure on $\mathfrak{so}(3)^*$ represents the angular velocities of a spinning top. In many practical situations, a particular mathematical description of a system may have some formal ambiguities that cannot be observed physically and it is desirable to obtain a faithful description that deals with redundancies. Typical examples of this situation are phase spaces with a group action implementing a form of "gauge" symmetry between states that one wishes to mod out or submanifolds of the phase space to which the states of the physical system are constrained. Often, as in the case of symplectic reduction *a la* Marsden-Weinstein, both elements are present.

One of the earliest instances of symplectic reduction can be found already in the seminal works on analytical mechanics by W.R. Hamilton and C.G.J. Jacobi in the first half of the 19^{th} century. In great generality, those first examples are today encapsulated in what we call **canonical symplectic reduction**. This is the instance of a free and proper action of a symmetry group on a configuration space $G \circlearrowright Q$, indicating that the system's position is faithfully described by the smooth manifold Q/G. The group acts on the cotangent bundle via cotangent lifts $G \circlearrowright T^*Q$ and there is a canonical comoment map given by

$$\overline{\mu} = l \circ \psi : \mathfrak{g} \to \mathrm{C}^\infty(T^*Q)$$

where ψ denotes the infinitesimal action and $l : \Gamma(TM) \hookrightarrow \mathrm{C}^\infty(T^*Q)$ is the natural inclusion of vector fields as fibre-wise linear functions of the cotangent bundle. Then it is easy to show that

$$T^*(Q/G) \cong \mu^{-1}(0)/G = T^*Q//G$$

A similar situation is when the positions of a physical system are constrained to a submanifold $i : S \hookrightarrow Q$. On the one hand, taking S as an intrinsic manifold itself, the phase space of the constrained system will be T^*S, on the other hand S induces a submanifold on the cotangent bundle $T^*Q|_S$ which is easily shown to be coisotropic. What we could call **canonical coisotropic**

reduction then gives the following consistency result for the description of the phase space of the constrained system:

$$T^*S \cong T^*Q|_S/TS^0|_S.$$

About a century later, P.A.M. Dirac was interested in the problem of quantization of mechanical systems with constraints. The early approaches to quantization relied heavily on the explicit expression of the Poisson bracket of classical observables so that it could be assigned concrete commutators of self-adjoint operators of the quantum Hilbert space. Dirac found conditions for the existence of Poisson brackets on the functions restricted to the constraint submanifold and gave a formula for the bracket, the celebrated Dirac bracket, when the constraint manifold was defined as the level set of a finite family of functions on phase space. In terms of modern differential geometry, Dirac's results correspond to the theory of admissible functions in Poisson and presymplectic manifolds (see the section below for details).

The first appearance of the modern notion of moment map is agreed to happen in S. Lie's work, albeit in a fairly implicit manner, in the 1890s. The first use of a moment map as we know it today is due to B. Kostant in the 1960s, who employed the concept, without giving it any name, to prove the symplectic covering theorem that bears his name. Around the same time, in his treaty on dynamics [Sou70], J-M. Souriau coined the French term *application moment* for what is known today as the moment map. The name was trying to emphasize the fact this notion is a generalization of conventional physical quantities such as angular momentum, *moment cinétique* in French. The first English usage of the term "moment map" occurred in the original symplectic reduction paper by J.E. Marsden and A. Weinstein [MW74], in which they translated the French term in a textual manner instead of using the equivalent English term "momentum map", which is more physically accurate as "moment" has another meaning in mechanics already, e.g. moment of inertia. Given the great importance of these original publications the name has stuck and to this day there are proponents of both options. We have chosen "moment map" in our text simply for brevity.

3 The Postmodern: from Symplectic Reduction to Dirac Geometry

T. Courant first introduced the notion of Dirac manifold in the late 1980s, see [Cou90], in an attempt to develop a geometric framework that systematically accounted for the structures found in "intermediate obejects" in reduction schemes, such as preimages of moment maps and coisotropic submanifolds, and the Dirac bracket, the latter motivating the original name of Dirac manifold or Dirac structure that shall be motivated and defined in this section. A modern and concise introduction to this topic can be found in [Bur13], a set of notes by H. Bursztyn, one of the main contributors to the latest developments in Dirac geometry.

The technical difficulty common to identifying the geometric structures of intermediate objects in reduction schemes and the Dirac theory of constraints is that one starts with a symplectic structure which after restriction to a submanifold or quotient by some foliation becomes degenerate. Since a symplectic manifold (M, ω) can be equivalently seen as a Poisson manifold with non-degenerate bivector (M, π), there are two ways to "go degenerate" from a symplectic structure.

The first is to assume that the bivector is (potentially) degenerate, thus giving a general **Poisson manifold** (M, π).

The second is to assume that the 2-form is (potentially) degenerate, thus giving a **presymplectic manifold** (M, ω). The **characteristic distribution** of a presymplectic manifold is defined as $K_\omega := \ker(\omega^\flat) \subset TM$. We can define the space of admissible functions as

$$\mathrm{C}^\infty_\omega(M) := \{f \in \mathrm{C}^\infty(M) | \quad \exists X_f \in \Gamma(TM) : i_{X_f}\omega = df\}.$$

We call X_f a Hamiltonian vector field for $f \in \mathrm{C}^\infty_\omega(M)$, which, when it exists, is uniquely determined up to sections of the characteristic distribution K_ω. It follows that the bracket

$$\{f, g\}_\omega := \omega(X_f, X_g) = \mathcal{L}_{X_f} g = -\mathcal{L}_{X_g} f$$

is well-defined on admissible function and since products of admissible functions are admissible, it is easy to show that $(\mathrm{C}^\infty_\omega(M), \{,\}_\omega)$ is a Poisson algebra. Because ω is closed, K_ω is an involutive distribution. Assuming that K_ω is integrated by a regular foliation so that there is a surjective submersion $q : M \to \tilde{M}$ and noting that projecting with q corresponds quotienting by the kernel of ω, $\tilde{M}$ inherits a symplectic structure $\tilde{\omega}$. Under this conditions it follows by construction that $\mathrm{C}^\infty_\omega(M) = q^*\mathrm{C}^\infty(\tilde{M})$ and the Poisson structure of admissible functions corresponds to the Poisson bracket of the symplectic structure on $(\tilde{M}, \tilde{\omega})$.

Let us consider the **standard Courant algebroid**, also known as the Pontryagin bundle, the generalized tangent bundle or double tangent bundle

$$\mathbb{T}M := TM \oplus T^*M$$

endowed with an anchor map $\rho := \mathrm{pr}_1$, a non-degenerate, split-signature bilinear form $\langle,\rangle$ defined from the dual pairing and the Dorfman bracket

$$[\![X \oplus \alpha, Y \oplus \beta]\!] := [X, Y] \oplus \mathcal{L}_X\beta - i_Y d\alpha$$

A **Dirac structure** on M is defined as a subbundle $L \subset \mathbb{T}M$ that is maximally isotropic, i.e. $\langle,\rangle|_L = 0$ and $\mathrm{rk}(L) = \dim M$, and involutive with respect to the Dorfman bracket, $[\![\Gamma(L), \Gamma(L)]\!] \subset \Gamma(L)$. We can define the admissible functions of a Dirac structure as

$$\mathrm{C}^\infty_L(M) := \{f \in \mathrm{C}^\infty(M) | \quad \exists X_f \in \Gamma(TM) : X_f \oplus df \in \Gamma(L)\}$$

that can be shown to form a Poisson algebra with the bracket

$$\{f, g\}_L := \mathcal{L}_{X_f} g = -\mathcal{L}_{X_g} f.$$

The graphs of a 2-form $\omega \in \Gamma(\wedge^2 T^*M)$ and a bivector $\pi \in \Gamma(\wedge^2 TM)$ are clearly maximally isotropic subbundles of the standard Courant algebroid

$$L_\omega, L_\pi \subset \mathbb{T}M.$$

A direct computation shows that

$$\begin{aligned} [\![\Gamma(L_\omega), \Gamma(L_\omega)]\!] \subset \Gamma(L_\omega) \quad &\Leftrightarrow \quad d\omega = 0 \\ [\![\Gamma(L_\pi), \Gamma(L_\pi)]\!] \subset \Gamma(L_\pi) \quad &\Leftrightarrow \quad [\pi, \pi] = 0 \end{aligned}$$

and thus Dirac structures encompass both presymplectic and Poisson structures.

One of the main conceptual strengths of the Courant algebroid formalism is that it allows for a unifying categorical treatment of many known generalizations of symplectic manifolds (Poisson, presymplectic, quasi-presymplectic, quasi-Poisson, integrable distributions with presymplectic foliations...). Inspired by A. Weinstein's definition of the symplectic category, a modern account by the author himself can be found in [Wei09], we define a **Courant morphism** $R : \mathbb{T}N \dashrightarrow \mathbb{T}M$ as a Dirac structure of the product

$$R \subset \mathrm{pr}_1 \mathbb{T}M \oplus \mathrm{pr}_2 \overline{\mathbb{T}}N$$

where $\overline{\mathbb{T}}N$ denotes the standard Courant structure with bilinear form of opposite sign, $\langle , \rangle_{\overline{\mathbb{T}}N} = -\langle , \rangle_{\mathbb{T}N}$. Disregarding potential technical problems due to cleanness of intersection, these morphisms can be shown to be composable as maximally isotropic (Lagrangian) relations. Making the necessary adjustments in our definition of Dirac structure to account for involutive maximally isotropic subbundles over a submanifold of the base, it can be shown that a smooth map $\varphi : N \to M$ induces a Courant morphism $R_\varphi : \mathbb{T}N \dashrightarrow \mathbb{T}M$, which is a Dirac structure naturally defined from the tangent map $T\varphi$ over the graph $\mathrm{Grph}(\varphi) \subset M \times N$. Note that Dirac structures can be regarded as Courant morphisms between a manifold and the point manifold with zero Courant algebroid. Given two Dirac structures (N, L_N) and (M, L_M), a smooth map $\varphi : N \to M$ is called a **forward Dirac map** if

$$R_\varphi \circ L_N = L_M$$

and a **backwards Dirac map** if

$$L_N = (\varphi^* L_M) \circ R_\varphi.$$

These maps can now be used to define the category of Dirac manifolds whose objects are manifolds with a Dirac structure on their standard Courant algebroid $(M, L \subset \mathbb{T}M)$ and whose morphisms are Dirac maps.

To bring back the original motivating example, recall the symplectic reduction scheme for a symplectic manifold (M, ω) via some coisotropic submanifold $i : C \hookrightarrow M$. We can identify the

Dirac structures associated to 2-forms forms to write the reduction diagram

$$\begin{tikzcd} (C, L_{i^*\omega)} \arrow[r, hook, "i"] \arrow[d, two heads, "p"'] & (M, L_\omega) \\ (\tilde{M}, L_{\tilde{\omega}}) & \end{tikzcd}$$

It is easy to see that the conditions for coistropic reduction are tantamount to demanding that i and p are backwards Dirac maps and that

$$(i^* L_\omega) \circ R_i = L_{i^*\omega} = (p^* L_{\tilde{\omega}}) \circ R_p.$$

References

[Dir50] P.A.M. Dirac. "Generalized hamiltonian dynamics". In: *Canadian journal of mathematics* 2 (1950), pp. 129–148.

[Sou70] J-M. Souriau. "Structure des systèmes dynamiques, Dunod, Paris, 1970". In: *MR* 41 (1970), p. 4866.

[MW74] J. Marsden and A. Weinstein. "Reduction of symplectic manifolds with symmetry". In: *Reports on mathematical physics* 5.1 (1974), pp. 121–130.

[AM78] R. Abraham and J.E. Marsden. *Foundations of mechanics*. Vol. 36. Benjamin Publishing Company Reading, Massachusetts, 1978.

[MR86] J.E. Marsden and T. Ratiu. "Reduction of Poisson manifolds". In: *Letters in Mathematical Physics* 11.2 (1986), pp. 161–169.

[Wei88] A. Weinstein. "Coisotropic calculus and Poisson groupoids". In: *Journal of the Mathematical Society of Japan* 40.4 (1988), pp. 705–727.

[Cou90] T.J. Courant. "Dirac manifolds". In: *Transactions of the American Mathematical Society* 319.2 (1990), pp. 631–661.

[HH04] G. Heckman and P. Hochs. "Geometry of the momentum map". In: *Proc. Utrecht Spring School* (2004).

[Wei09] A. Weinstein. "Symplectic categories". In: *arXiv preprint arXiv:0911.4133* (2009).

[LiB12] D. Li-Bland. "LA-Courant algebroids and their applications". In: *arXiv preprint arXiv:1204.2796* (2012).

[Bur13] H. Bursztyn. "A brief introduction to Dirac manifolds". In: *Geometric and topological methods for quantum field theory* (2013), pp. 4–38.

[FM14] R.L. Fernandes and I. Marcut. *Lectures on Poisson Geometry*. Springer, 2014.