On Gressman's Affine Invariant Measures

Lecture 7

The Oberlin Curvature Condition of $\nu_{\mathcal{A}}$

We finally address question (2), that is, we will show that for any (nice) submanifold Σ we have

$$\nu_{\mathcal{A}}(R \cap \Sigma) \lesssim |R|^{\frac{d}{Q}}$$

for any rectangle $R \subseteq \mathbb{R}^n$ (the implicit constants depends only on some multiplicity assumptions on Σ). This is the Oberlin curvature condition of exponent d/Q.
[Recall that conditions of the form above are necessary for an L^p-smoothing estimate to hold.]

We will start with the example of curves to provide some motivation. Recall that for curves $\gamma : [0,1] \to \mathbb{R}^n$ we have

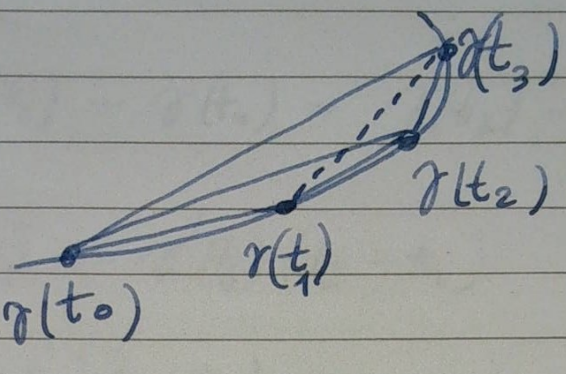
$$\frac{d\nu_{\mathcal{A}}}{dt} = |L_\gamma(t)|^{\frac{2}{n(n+1)}}$$

where

$$L_\gamma(t) := \det\begin{pmatrix} \gamma'(t) & \gamma''(t) & \dots & \gamma^{(n)}(t) \end{pmatrix}.$$

Pick a rectangle $R \subseteq \mathbb{R}^n$ and assume for simplicity that there is an interval I such that $R \cap \gamma = \gamma(I)$
(R is assumed very small and thus so is I)

If we pick points $t_0, t_1, \ldots, t_n \in I$ and consider the parallelepiped generated by vertices $\gamma(t_0), \gamma(t_1), \ldots, \gamma(t_n)$, this will certainly be contained in R.



How should one pick these points though? We should pick them at random but we need to choose a distribution. It is natural to ask ourselves what would happen if we picked $\nu_{\mathcal{A}}|_{\gamma(I)}$ itself as the distribution.

What we end up considering is therefore the quantity

$$\frac{1}{\left(\nu_{\mathcal{A}}(\gamma(I))\right)^n} \int_{I^n} \left| \det\left(\gamma(t_1) - \gamma(t_0) \ \cdots \ \gamma(t_n) - \gamma(t_0)\right) \right| \times d\nu_{\mathcal{A}}(t_1) \cdots d\nu_{\mathcal{A}}(t_n)$$

(with a little abuse of notation here).

This quantity is $\leq |R|$ as said above. We estimate the quantity by assuming that the density $\frac{d\nu_{\mathcal{A}}}{dt}$ is approximately constant over I (equivalently, $|L_\gamma|$ is approximately constant).

Observe that the determinants in the expression and L_γ are related: assuming I small we have

$$\gamma(t_1) - \gamma(t_0) \approx \gamma'(t_0)(t_1 - t_0) ;$$

then

$$\gamma(t_2) - \gamma(t_0) = \gamma(t_2) - \gamma(t_1) + \underbrace{\gamma(t_1) - \gamma(t_0)}$$

linearly dependent, so we remove it

and

$$\gamma(t_2) - \gamma(t_1) \approx \gamma'(t_1)(t_2 - t_1)$$

$$\approx \gamma''(t_0)(t_1 - t_0)(t_2 - t_1) + \underbrace{\gamma'(t_0)(t_2 - t_1)}$$

also linearly dependent, so we remove this as well.

Continuing in this fashion we see that we can replace $\gamma(t_j) - \gamma(t_0)$ with

$$\gamma^{(j)}(t_0)\,(t_j - t_{j-1})(t_{j-1} - t_{j-2}) \cdots (t_1 - t_0),$$

Therefore we have

$$\det\big(\gamma(t_1) - \gamma(t_0) \;\; \ldots \;\; \gamma(t_n) - \gamma(t_0)\big)$$

$$\approx L_\gamma(t_0) \cdot (t_n - t_{n-1})(t_{n-1} - t_{n-2})^2 \cdots (t_1 - t_0)^n.$$

Restricting the integration to $t_1 < t_2 < \ldots < t_n$ we are seeking to estimate

$$\frac{1}{(\nu_\gamma(\gamma(I)))^n} \int_{\{(t_1,\ldots,t_n)\in I^n:\; t_1 < \cdots < t_n\}} |L_\gamma(t_0)| \prod_{j=1}^{n} (t_j - t_{j-1})^{n+1-j} \, d\nu_\gamma(t_1) \ldots d\nu_\gamma(t_n)$$

Observe then that by our (generous) assumptions we have

$$d\nu_{\mathcal{A}}(t_j) \sim |L_\gamma(t_0)|^{\frac{2}{n(n+1)}} dt_j ,$$

and therefore

$$\int_{I^n \cap \{t_1 < t_2 < \dots < t_n\}} |L_\gamma(t_0)| \prod_{j=1}^{n} (t_j - t_{j-1})^{n+1-j} \, d\nu_{\mathcal{A}}(t_1) \cdots d\nu_{\mathcal{A}}(t_n)$$

$$\sim |L_\gamma(t_0)|^{1 + n \frac{2}{n(n+1)}} |I|^{\frac{n(n+1)}{2} + n} .$$

On the other hand

$$\nu_{\mathcal{A}}(\gamma(I)) \approx |L_\gamma(t_0)|^{\frac{2}{n(n+1)}} |I| ,$$

so we estimate the average in question to be ~~as~~ bounded from below by

$$\frac{|L_\gamma(t_0)|^{1 + n \cdot \frac{2}{n(n+1)}} |I|^{\frac{n(n+1)}{2} + n}}{\left(|L_\gamma(t_0)|^{\frac{2}{n(n+1)}} |I| \right)^n}$$

$$= |L_\gamma(t_0)| |I|^{\frac{n(n+1)}{2}} \sim \left(\nu_{\mathcal{A}}(\gamma(I)) \right)^{\frac{n(n+1)}{2}} .$$

Therefore

$$\underset{\substack{\| \\ R \cap \gamma}}{\nu_{\mathcal{A}}(\gamma(I))} \lesssim |R|^{\frac{2}{n(n+1)}}$$

as desired.

What one should take away from this simplistic discussion is that things are more approachable when there is some uniformity.

The uniformity we need for the general case will be provided by the aptly named <u>non-concentration inequalities</u>.

Here is a baby version of the ones we need:

(assume Σ compact)

<u>Basic Non-concentration Lemma</u>

Let Σ be a submanifold and let $\mathcal{F}$ be a finite-dimensional vector space of functions $\Sigma \to \mathbb{R}$ such that at any point $\mathrm{Span}\left((df|_p)_{f \in \mathcal{F}}\right) = T_p\Sigma$.
[will be used later]

For any positive measure μ on Σ (absolutely continuous) and measurable $E \subset \Sigma$ there exists measurable $E' \subseteq E$ s.t.

i) $\mu(E') \gtrsim \mu(E)$

ii) $\sup_{p \in E'} |f(p)| \lesssim_{\dim \mathcal{F}} \dfrac{1}{\mu(E)} \displaystyle\int_E |f| \, d\mu$ for all $f \in \mathcal{F}$.

Property (ii) is a very explicit non-concentration statement – the functions are never too large on E' with respect to their average.

<u>Proof</u>:
The key idea is to realise that
$\|f\| := \dfrac{1}{\mu(E)} \displaystyle\int_E |f| \, d\mu$ is a norm on $\mathcal{F}$.

Letting $\dim \mathcal{F} = k$. There must be linearly independent functions $f_1, \dots, f_k$ such that

$$\|f_j\| = 1 \qquad \forall\, j \in \{1, \dots, k\};$$

thus for any $f \in \mathcal{F}$

$$f = \sum_{j=1}^{k} c_j f_j \qquad \text{with} \quad |c_j| \leq \|f\|.$$

Pointwise we have then

$$|f(p)| \leq \sum_{j=1}^{k} |c_j| |f_j(p)| \leq \|f\| \underbrace{\sum_{j=1}^{k} |f_j(p)|}_{\text{fixed function}},$$

so if we take

$$E' := \{ p \in E : \sum_{j=1}^{k} |f_j(p)| < 2k \}$$

we see by Chebyshev's inequality that

$$\mu(E \setminus E') \leq \frac{\int_E \sum_j |f_j| \, d\mu}{2k} = \frac{\mu(E)\, k}{2k} = \frac{\mu(E)}{2}$$

$$\Rightarrow \quad \mu(E') \geq \frac{1}{2} \mu(E), \text{ and}$$

$$\sup_{p \in E'} |f(p)| \leq 2k \|f\| = 2k \frac{1}{\mu(E)} \int_E |f| \, d\mu.$$

To prove what we want we need a differential version of this result.

Let us use the following notation: given vector fields $X_1, \dots, X_d$, we let $\mu(X_1 \wedge \dots \wedge X_d)$ denote the density of μ with respect to the basis $\{X_1, \dots, X_d\}$. One can see that

$$\mu(X_1 \wedge \dots \wedge X_d)(p) = \frac{d\mu}{dt}(p) \cdot |X_1(p) \wedge \dots \wedge X_d(p)|.$$

Consider some open set $U \supset E'$ now.

We re-define the norm a little to be

$$\|f\| := \frac{1}{\mu(E \cap U)} \int_{E \cap U} |f| \, d\mu$$

and pick $f_1, \ldots, f_k$ as before but with respect to this norm (so $\|f_j\| = 1 \;\; \forall j$ and they are linearly independent.)

Recall that we have assumed that the df's span $T_p\Sigma$ for any $p \in \Sigma$. In particular if $\beta_1, \ldots, \beta_d \in \{1, \ldots, k\}$ there is at least one choice of such indices such that

$$df_{\beta_1} \wedge \ldots \wedge df_{\beta_d}(p) \neq 0$$

(for any p fixed). We introduce the set of acceptable indices $\underline{\beta}$:

$$I := \{ \underline{\beta} \in \{1, \ldots, k\}^d : \beta_1 < \ldots < \beta_d \};$$

then by the above any $p \in \Sigma \cap U$ belongs to an open set of the form

$$V_{\underline{\beta}} := \Big\{ p \in U : |df_{\beta_1} \wedge \cdots \wedge df_{\beta_d}(p)| > \frac{1}{2} |df_{\beta_1'} \wedge \ldots \wedge df_{\beta_d'}(p)| \text{ for all } \underline{\beta}' \in I \setminus \{\underline{\beta}\} \Big\}.$$

<u>Remark</u>: V_β is simply the set where $|df_{\beta_1} \wedge \cdots \wedge df_{\beta_d}|$ is "largest", but defined in such a way that the resulting set is open.

One sees that $\# I \lesssim_{d, \dim \mathcal{F}} 1$ and therefore by pigeonholing there is an index

$\underline{\beta} \in \mathcal{I}$ such that

$$\mu(E \cap V_{\underline{\beta}}) \gtrsim_{d, \dim \mathcal{F}} \mu(E \cap U)$$

$(\gtrsim \mu(E))$.

On $V_{\underline{\beta}}$ it is easy to define a special class of vector fields: we want $X_1, \ldots, X_d$ with the property that

$$X_j f_{\beta_i}(p) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

for all $p \in V_{\underline{\beta}}$. (they are "dual" to our functions $f_{\beta_1}, \ldots, f_{\beta_k}$).

The way to construct these is simply to define, for any $g : V_{\underline{\beta}} \to \mathbb{R}$,

j-th position

$$X_j g(p) := \frac{df_{\beta_1} \wedge \ldots \wedge dg \wedge \ldots \wedge df_{\beta_d}(p)}{df_{\beta_1} \wedge \ldots \wedge df_{\beta_j} \wedge \ldots \wedge df_{\beta_d}(p)}$$

for $j \in \{1, \ldots, d\}$. It's immediate to verify that indeed $X_i f_{\beta_i} = \delta_{ij}$ as desired.

Observe the following: for any $i \in \{1, \ldots, k\}$ we have for all $p \in V_{\underline{\beta}}$

$$|X_j f_i(p)| \leq 2$$

by definition of $V_{\underline{\beta}}$ itself!

With this we have then the following: taken $f \in \mathcal{F}$ we have

$$f = \sum_{i=1}^{k} c_i f_i \quad \text{with } |c_i| \leq \|f\|$$

as before, and when we apply X_j to f

we have that

$$|X_j f(p)| \le \|f\| \sum_{i=1}^{k} |X_j f_i(p)|$$
$$\le 2k\|f\| \lesssim_k \frac{1}{\mu(E\cap U)} \int_{E\cap U} |f|\, d\mu$$
$$\lesssim_k \frac{1}{\mu(E)} \int_E |f|\, d\mu$$

for all $p \in V_\beta$ and $f \in \mathcal{F}$, which is a differential form of the baby non-concentration result (ii).

More is true about these vector fields X_j: they are <u>non-degenerate</u> in a quantifiable way, in particular we claim that on "most of" $V_\beta \cap E$ they satisfy

$$\underbrace{\mu(X_1 \wedge \ldots \wedge X_d)(p)}_{\substack{\text{density of } \mu \\ \text{at } p \text{ w.r.t. the} \\ X_1, \ldots, X_d \text{ basis}}} \gtrsim \mu(E) .$$

(some parameters here...)

To see this, consider the expression

$$\int_{E\cap V_\beta} \left[\mu(X_1 \wedge \ldots \wedge X_d)(p)\right]^{-1} d\mu(p)$$

We multiply and divide by $|df_{\beta_1} \wedge \ldots \wedge df_{\beta_d}(p)|$ and look at the denominator.

The denominator is

$$\mu(X_1 \wedge \dots \wedge X_d)\, |df_{\beta_1} \wedge \dots \wedge df_{\beta_d}|$$

$$= \frac{d\mu}{dt} \cdot |X_1 \wedge \dots \wedge X_d|\, |df_{\beta_1} \wedge \dots \wedge df_{\beta_d}| \,;$$

the last two factors are determinants and we have then

$$\det\begin{pmatrix} df_{\beta_1} & \dots & df_{\beta_d} \end{pmatrix} \cdot \det\begin{pmatrix} X_1 & \dots & X_d \end{pmatrix}$$

$$= \det\begin{pmatrix} \nabla f_{\beta_1} \\ \vdots \\ \nabla f_{\beta_d} \end{pmatrix} \cdot \det\begin{pmatrix} X_1 & \dots & X_d \end{pmatrix}$$

$$= \det\begin{pmatrix} X_1 f_{\beta_1} & \dots & X_d f_{\beta_1} \\ \vdots & \ddots & \vdots \\ X_1 f_{\beta_d} & \dots & X_d f_{\beta_d} \end{pmatrix} \,;$$

by construction of the X_i vector fields the latter is the identity matrix and so the above product is equal to 1.
Thus we have

$$\int_{E\cap V_\beta} [\mu(X_1 \wedge \dots \wedge X_d)]^{-1}\, d\mu = \int_{E\cap V_\beta} |df_{\beta_1} \wedge \dots \wedge df_{\beta_d}| \left(\frac{d\mu}{dt}\right)^{-1} d\mu$$

$$= \int_{E\cap V_\beta} |df_{\beta_1} \wedge \dots \wedge df_{\beta_d}|\, dt.$$

Since $df_{\beta_1}, \dots, df_{\beta_d}$ span $T_p\Sigma$ for $p \in V_\beta$, we can think of them as locally being coordinates around every point (though not on the whole of V_β in general).
Consider then the map $\Phi : V_\beta \longrightarrow \mathbb{R}^d$

given by
$$\Phi(p) := (f_{\beta_1}(p), \dots, f_{\beta_d}(p));$$
its Jacobian determinant is precisely $|df_{\beta_1} \wedge \dots \wedge df_{\beta_d}|$, so if Φ were bijective we'd have
$$\int_{E\cap V_\beta} |df_{\beta_1} \wedge \dots \wedge df_{\beta_d}|\, dt = \int_{\Phi(E\cap V_\beta)} 1 \, dx = |\Phi(E\cap V_\beta)|.$$
This is not generally the case, so we introduce the <u>multiplicity assumption</u>
$$\# \Phi^{-1}(\{x\}) \leq M \quad \text{for all } x \in \Phi(E);$$
with this one can at least say
$$\int_{E\cap V_\beta} |df_{\beta_1} \wedge \dots \wedge df_{\beta_d}|\, dt \leq M \, |\Phi(E\cap V_\beta)|.$$

<u>Remark</u>: The multiplicity assumption is satisfied for a finite M if we assume e.g. that the $f \in F$ are analytic (in a suitable sense).

Now observe that, crudely,
$$|\Phi(E\cap V_\beta)| \leq \prod_{j=1}^{d} |f_{\beta_j}(E\cap V_\beta)|;$$
but recall that by the baby non-concentration result we can find $E'' \subset E\cap U$ such that (ii) holds, that is (for f_{β_j} in particular)
$$\sup_{p\in E''} |f_{\beta_j}(p)| \lesssim \frac{1}{\mu(E\cap U)} \int_{E\cap U} |f_{\beta_j}|\, d\mu = 1.$$
(by construction)

If we replace $E\cap V_\beta$ with E'' in the argument above we have $|f_{\beta_j}(E'')| \lesssim 1$ and thus $|\Phi(E'')| \lesssim 1$ as well.
Overall we have shown

that

$$\int_{E''} [\mu(X_1 \wedge \cdots \wedge X_d)]^{-1} \, d\mu \lesssim_{d, \dim F, M} 1$$

and so by Chebyshev's inequality we can further refine E'' to E''' of comparable μ-measure with the property that

$$\mu(X_1 \wedge \cdots \wedge X_d)(p) \gtrsim \mu(E) \quad \text{for all } p \in E'''$$

(as claimed).

Iterating these arguments leads to a fully differential version of the non-concentration result:

Differential Non-Concentration Lemma:

Let Σ, F as before and take $N \in \mathbb{N}$. For every (abs. cont.s) measure μ on Σ and $E \subseteq \Sigma$, measurable, there exist:

- $E' \subseteq E$ with $\mu(E') \gtrsim_{d, \dim F} \mu(E)$,
- U open set $\supseteq E'$ $\quad (E' \subseteq E \cap U)$
- N lists of d vector fields each:

$$\begin{matrix} X_{1,1}, & \cdots & , X_{1,d}, \\ \vdots & \vdots & \vdots \\ X_{N,1}, & \cdots & , X_{N,d}, \end{matrix}$$

such that:

i) $\mu(X_{j,1} \wedge \cdots \wedge X_{j,d})(p) \gtrsim_{d, \dim F, M} \mu(E)$

for all $p \in E'$ and $j \in \{1, \ldots, N\}$;

ii) each vector field can be written as a linear combination of vector fields in the previous list with uniformly small (variable) coefficients; more precisely, one can always write

$$X_{j,i}(p) = \sum_{i'=1}^{d} c_{j,i,i'}(p)\, X_{j-1,i'}(p)$$

with $|c_{j,i,i'}(p)| \lesssim 1$.

iii) for any choice of indices

$$1 \le j_1 < \dots < j_\ell \le N$$

(identifying the lists) we have for any choice of indices $i_1, \dots, i_\ell \in \{1, \dots, d\}$

$$\sup_{p \in E'} \left| (X_{j_\ell, i_\ell} \cdots X_{j_1, i_1} f)(p) \right| \lesssim \frac{1}{\mu(E)} \int_E |f| \, d\mu$$

for all $f \in \mathcal{F}$.

[When $\ell = 0$ we have again $\sup_{p \in E'} |f(p)| \lesssim \frac{1}{\mu(E)} \int_E |f| \, d\mu$.]

Remark: The various implicit constants depend on d, $\dim \mathcal{F}$, M in various ways. (and N).

Now we are ready to finish!

Let R be a rectangle in $\mathbb{R}^n$ and let $f : \Sigma \to \mathbb{R}^n$ be the embedding; we will take $E := f^{-1}(R) \subset \Sigma$.

The components of $f = (f_1, \dots, f_n)$ and their polynomial combinations up to degree n form a finite dimensional vector space $\mathcal{F}$ on Σ of the type considered before. The quantity we are interested in is

$$I(E) := \frac{1}{(\nu_A(E))^n} \int_{E^n} \left| \det(f(p_1) - f(p_0) \cdots f(p_n) - f(p_0)) \right| \times d\nu_A(p_1) \cdots d\nu_A(p_n),$$

and we know that $I(E) \leq |R|$ by our initial discussion (it's an average of parallelepipeds/simplices contained in R).

If one applies the Differential Non-Concentration Lemma iteratively (first in p_1, then in p_2, etc, picking k_j derivatives at the j-th step, where k_j is given by $\Lambda_{d,n}$ as usual) we will have by (iii)

$$I(E) \gtrsim \sup_{(p_1,\dots,p_n)\in(E')^n} \left|\det\left(X_{1,i_{(1,1)}} f(p_1) \wedge \dots \wedge \left(X_{k_n,i_{(n,1)}} \cdots X_{1,i_{(n,k_n)}} f(p_n)\right)\right)\right|$$

(we have taken $N = k_n$).
We remark that the choice of indices $i_{(*,*)}$ is arbitrary - the inequality holds for all of them.

Every column is evaluated at a different point p_j, but we are looking for a lowerbound and so we can simply restrict to the diagonal:

$$I(E) \gtrsim \sup_{p\in E'} \left|\det\left(X_{1,i_{(1,1)}} f(p) \dots \left(X_{k_n,i_{(n,1)}} \cdots X_{1,i_{(n,k_n)}} f(p)\right)\right)\right|.$$

This is starting to look more like A_p^f, but there are vector fields from all the different N $(=k_n)$ lists involved, instead of just one...

We can use (ii) of the Lemma to correct this.

Indeed, consider the expression that we desire to have at the RHS:

$$\det\left(X_{k_n, i'_{(1,1)}} f(p) \;\cdots\; (X_{k_n, i'_{(n,1)}} \cdots X_{k_n, i'_{(n,k_n)}} f(p))\right)$$

(all vector fields belonging to the last list).
We can express any $X_{k_n, i'_{(*,*)}}$ by its linear combination
(with variable coefficients) in terms of $X_{k_n - 1, i_{(*,*)}}$. In doing so, by Leibniz's

rule, we will produce terms containing derivatives of the coefficients $c_{j,i,i'}(p)$.
However, ~~exactly~~ exactly like when we showed that $\mathcal{A}^f_t$ is indeed a tensor, we can see that these terms vanish: indeed fewer derivatives fall on f and the resulting column is linearly dependent with respect to the previous columns by design!

~~Thus~~ Thus, only the terms where all vector fields fall on f survive.
Iterating appropriately ($X_{k_n-1, i_{(*,*)}}$ can be written in terms of vector fields $X_{k_n-2, i_{(*,*)}}$ and so on) we finally see that the expression at the top of the page has become

$$\sum_{\underline{i}} c_{\underline{i}', \underline{i}}(p) \det\left(X_{1, i_{(1,1)}} f(p) \;\cdots\; (X_{k_n, i_{(n,1)}} \cdots X_{1, i_{(n,k_n)}} f(p))\right)$$

for certain coefficients such that $|c_{\underline{i}', \underline{i}}(p)| \lesssim 1$.

Notice that the determinants in the last sum are exactly of the form that we have in our current lowerbound for $I(E)$, moreover, recall the choice of $i_{(*,*)}$ was arbitrary. Therefore we have

$$\sup_{p\in E'} \left|\det\left(X_{k_n, i'_{(1,1)}} f(p) \cdots \left(X_{k_n, i'_{(n,1)}} \cdots X_{k_n, i'_{(n,k_n)}} f(p)\right)\right)\right|$$

$$\lesssim \sum_{i} \sup_{p\in E'} \left|\det\left(X_{1, i_{(1,1)}} f(p) \cdots \right.\right.$$

$$\left.\left.\cdots \left(X_{k_n, i_{(n,1)}} \cdots X_{1, i_{(n,k_n)}} f(p)\right)\right)\right|$$

$$\lesssim I(E)$$ (since the choices of $i_{(*,*)}$ are boundedly many).

Now we have finally an expression that we can identify with $\mathcal{A}^f_p$!
Indeed, the above is simply

$$\mathcal{A}^f_t\left(X_{k_n, i'_{(1,1)}}, X_{k_n, i'_{(2,1)}}, \dots, X_{k_n, i'_{(n,1)}}, \dots, X_{k_n, i'_{(n,k_n)}}\right)$$

Once again we have the freedom to choose the indices $i'_{(*,*)}$; taking then the ℓ^2 sum over all choices of $i'_{(*,*)}$ we get

$$I(E) \gtrsim \sup_{p\in E'} \|\mathcal{A}^f_p\|_{(X_{k_n,1}, \dots, X_{k_n,d})}$$

(recall the definition of norm for a multilinear form associated to a list of vectors).

Clearly then, by definition of $\nu_{\mathcal{A}}$,

$$I(E) \gtrsim \sup_{p \in E'} \inf_{M \in SL(\mathbb{R}^d)} \left\| \rho_M \mathcal{A}_p^f \right\|_{(X_{k_n,1}, \ldots, X_{k_n,d})}$$

$$= \sup_{p \in E'} \left(\nu_{\mathcal{A}} \left(X_{k_{n,1}}^{(p)} \wedge \cdots \wedge X_{k_{n,d}}^{(p)} \right) \right)^{Q/d}$$

$$= \sup_{p \in E'} \left[\left(\frac{d\nu_{\mathcal{A}}}{d\sigma}(p) \right) \left| X_{k_{n,1}} \wedge \cdots \wedge X_{k_{n,d}}(p) \right| \right]^{Q/d} .$$

By (i) of the Lemma we have

$$\left| \nu_{\mathcal{A}} (X_{k_{n,1}} \wedge \cdots \wedge X_{k_{n,d}})(p) \right| \gtrsim \nu_{\mathcal{A}}(E) \quad \text{for all } p \in E',$$

and therefore we have shown

$$|R| \gtrsim I(E) \gtrsim \left(\nu_{\mathcal{A}}(E) \right)^{Q/d}$$

$$\Rightarrow \nu_{\mathcal{A}}(E) \lesssim |R|^{d/Q},$$

as claimed.

The proof is thus concluded.