On Grassman's

Affine Invariant Measures

Lecture 5

Before we proceed with trying to answer some of the questions, we make some remarks.

<u>Remark 1</u>: The first is that if $\underline{t} \in \mathbb{R}^d$ are local coordinates near some point $p \in \Sigma$, then

$$\underbrace{\frac{d\nu_{\mathcal{A}}}{dt}}_{\text{Lebesgue measure}}(p) = \mu^p_{\mathcal{A}}\left(\partial_{t_1}, \dots, \partial_{t_d}\right).$$

Letting $\partial_j := \partial_{t_j}$, observe that by our rules

$$M^T \partial_j = \sum_i M_{ji} \partial_i \, ;$$

(directional derivative in the direction of the j-th row of M)

in evaluating $\mu_{\mathcal{A}}(\partial)$ we find ourselves evaluating many expressions of the form

$$(M^T \partial_{j_1}) \cdots (M^T \partial_{j_\ell}) f$$

with indices possibly repeated, so we introduce some notation:

for multi-index $\underline{\alpha} = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ and $M \in SL(\mathbb{R}^d)$ we let

$$(M^T \partial)^{\underline{\alpha}} := (M^T \partial_1)^{\alpha_1} \cdots (M^T \partial_d)^{\alpha_d}.$$

Then we can write $\mu_{\mathcal{A}}(\partial_1, \dots, \partial_d)$ as

$$\inf_{M \in SL(\mathbb{R}^d)} \left[\left(\sum_{\substack{\underline{\alpha}_1, \dots, \underline{\alpha}_n : \\ |\underline{\alpha}_1| = k_1, \dots, |\underline{\alpha}_n| = k_n}} \left(\frac{k_1! \cdots k_n!}{\underline{\alpha}_1! \cdots \underline{\alpha}_n!} \right) \left| \det\left((M^T\partial)^{\underline{\alpha}_1} f \cdots (M^T\partial)^{\underline{\alpha}_n} f \right) \right|^2 \right)^{1/2} \right]^{\frac{d}{Q}}$$

(Recall k_j is the largest k such that $(j,k) \in \Lambda_{d,n}$)

$\hookrightarrow$ (e.g. $k_1 = \dots = k_d = 1$, $k_{d+1} = 2$, $\dots$)

$(\underline{\alpha}! = \alpha^{(1)}! \cdots \alpha^{(n)}!)$

The combinatorial factor is just technical:

$$\frac{k_j!}{\underline{\alpha}_j!} = \frac{k_j!}{\alpha_j^{(1)}!\dots\alpha_j^{(d)}!}$$ is simply the number of ways in which the factors of $(M^T\partial)^{\underline{\alpha}_j}$ can be reordered.

As you can see, we have complicated expressions like

$$\det\left((M^T\partial)^{\underline{\alpha}_1}f \quad (M^T\partial)^{\underline{\alpha}_2}f \quad \dots \quad (M^T\partial)^{\underline{\alpha}_n}f\right)$$

to evaluate, involving M in a highly non-trivial way. However, some simplification is possible, which leads us to our second remark.

<u>Remark 2</u>: To state what we want properly, the following <u>notation</u> is useful: given vectors $v, w \in V$

$$v \wedge w := v \otimes w - w \otimes v;$$

iterating, one can see that

$$v_1 \wedge \dots \wedge v_k = \sum_{\sigma\in S_k} \operatorname{sgn}\sigma \cdot v_{\sigma(1)} \otimes \dots \otimes v_{\sigma(k)}.$$

What this notation means is that if $\lambda_1, \dots, \lambda_k$ are k 1-linear functionals on V (that is, elements of V^*), then

$$(v_1 \wedge \dots \wedge v_k)(\lambda_1, \dots, \lambda_k) = \sum_{\sigma\in S_k} \operatorname{sgn}\sigma \; \lambda_1(v_{\sigma(1)}) \cdots \lambda_k(v_{\sigma(k)}).$$

In particular, if the vector space is $V=\mathbb{R}^n$

and we take n vectors $v_1, \ldots, v_n \in \mathbb{R}^n$, $v_1 \wedge \ldots \wedge v_n$ can be identified with the determinant

$$\det \left[v_1 \; \cdots \; v_n \right],$$

because, letting $\lambda_j(v) := \langle v, \underline{e}_j \rangle$ (the j-th component of v) we have

$$(v_1 \wedge \ldots \wedge v_n)(\lambda_1, \ldots, \lambda_n) = \sum_{\sigma \in S_n} \operatorname{sgn} \sigma \, v^{(1)}_{\sigma(1)} \cdots v^{(n)}_{\sigma(n)}$$

$$= \det \left(v_1 \; \cdots \; v_n \right)$$

(and in general $(v_1 \wedge \ldots \wedge v_n)(\lambda_{\tau(1)}, \ldots, \lambda_{\tau(n)}) = \operatorname{sgn} \tau \cdot \det(v_1 \; \cdots \; v_n)$, and if two indices are repeated the result is zero...)

All this painful algebraic notation is useful to let us "factorise" expressions like the determinant into blocks: in particular we can write in general

$$\mathcal{A}^f_t ((X_\lambda)_{\lambda \in \Lambda_{d,n}}) = \det \left(\cdots \; (X_{(j,1)} \cdots X_{(j,k_j)} f(t)) \cdots \right)$$

$$= X_{(1,1)} f \wedge \ldots \wedge X_{(d,1)} f \wedge \ldots \wedge (X_{(j,1)} \cdots X_{(j,k_j)} f) \wedge \ldots$$
$$\cdots \wedge (X_{(n,1)} \cdots X_{(n,k_n)} f).$$

For the quantity we are interested in:

$$\underbrace{(M^T \partial)^{\alpha_1} f \wedge \ldots \wedge (M^T \partial)^{\alpha_d} f}_{\substack{\text{complete block of } |\alpha| = 1 \\ \text{(maximal)}}} \wedge \underbrace{(M^T \partial)^{\alpha_{d+1}} f \wedge \ldots .}_{\substack{\text{complete block of} \\ \text{order } 2 = |\alpha|}}$$

$$\cdots \wedge \underbrace{(M^T \partial)^{\alpha_\ell} f \wedge \ldots}_{\substack{\text{complete block of order} \\ |\alpha| = k-1}} \wedge \underbrace{(M^T \partial)^{\alpha_m} f \wedge \ldots \wedge (M^T \partial)^{\alpha_n} f}_{\underline{\text{incomplete}} \text{ block! (order } k)}$$

The point we want to make is that

> We can replace M with the identity in any complete block as before

In other words, we claim the following:

> **Lemma 1**: Let $\underline{\alpha}_1, \dots, \underline{\alpha}_L$ be an enumeration of the elements of $\mathbb{N}^d$ such that $|\underline{\alpha}| = k$, that is
> $$\{\alpha \in \mathbb{N}^d : |\underline{\alpha}| = k\} = \{\underline{\alpha}_1, \underline{\alpha}_2, \dots, \underline{\alpha}_L\}.$$
> Then for every $M \in SL(\mathbb{R}^d)$ we have
> $$(M^T\partial)^{\underline{\alpha}_1} f \wedge \dots \wedge (M^T\partial)^{\underline{\alpha}_L} f = \partial^{\underline{\alpha}_1} f \wedge \dots \wedge \partial^{\underline{\alpha}_L} f.$$

The wedge "$\wedge$" notation makes it easier to show this fact.

Proof: While the above is only part of the big determinant that defines $\mathcal{A}_t f$, it still behaves like a determinant over a vector space of the right dimension...

In particular, consider the vector space of homogeneous differential operators of order k, which has dimension L; because it is spanned by $\partial^{\underline{\alpha}}$ with $|\underline{\alpha}| = k$ and

$$\#\{\underline{\alpha} \in \mathbb{N}^d : |\underline{\alpha}| = k\} = L.$$

We can write any such differential operator as $p(\partial)$, where p is a homogeneous polynomial in d variables of degree k:

$$p(X) = \sum_{\substack{\alpha \in \mathbb{N}^d: \\ |\alpha| = k}} c_\alpha X^\alpha$$

$$\to p(\partial) = \sum_{\substack{\underline{\alpha} \in \mathbb{N}^d: \\ |\underline{\alpha}| = k}} c_{\underline{\alpha}} \partial^{\underline{\alpha}} = \sum_{\substack{\underline{\alpha} \in \mathbb{N}^d: \\ |\underline{\alpha}| = k}} c_{\underline{\alpha}} \partial_1^{\alpha(1)} \cdots \partial_d^{\alpha(d)}$$

We can then see the map $\partial^{\underline{\alpha}} \mapsto (M^T\partial)^{\underline{\alpha}}$ as a linear operator on this vector space:

$$T_M(p(\partial)) := p(M^T\partial) = \sum_{\substack{\underline{\alpha} \in \mathbb{N}^d: \\ |\underline{\alpha}| = k}} c_{\underline{\alpha}} (M^T\partial)^{\underline{\alpha}}$$

(here M is fixed).

The operator T_M maps the vector space to itself, so $\det T_M$ is well-defined; but now notice that ~~xxxxx~~ the dimension of the vector space is L, which is the same as the length of the expression $(M^T\partial)^{\underline{\alpha}_1} f \wedge \ldots \wedge (M^T\partial)^{\underline{\alpha}_L} f$!

This means that

$$(M^T\partial)^{\underline{\alpha}_1} f \wedge \ldots \wedge (M^T\partial)^{\underline{\alpha}_L} f = (\det T_M) \cdot \left(\partial^{\underline{\alpha}_1} f \wedge \ldots \wedge \partial^{\underline{\alpha}_L} f\right)$$

(If you are not convinced, expand the definition of $\wedge$)

So it suffices to establish that $\det T_M = 1$. The simplest way to do this is to appeal to the Singular Value Decomposition of M, so that we find $U, V \in O(\mathbb{R}^n)$ orthogonal matrices and D diagonal such that

$$M = U D V$$

(and $\det D = 1$, since $\det M = 1$).

One can check easily that

$$T_M = T_U \circ T_D \circ T_V,$$

so that it suffices to check separately that $\det T_U = 1$ and $\det T_D = 1$.

For T_U its a technical calulation: use the inner product on polynomials

$$\langle p, q \rangle_k := \sum_{\substack{\underline{\alpha} \in \mathbb{N}^d: \\ |\underline{\alpha}| = k}} \frac{k!}{\underline{\alpha}!} \partial^{\underline{\alpha}} p \big|_0 \, \partial^{\underline{\alpha}} q \big|_0$$

$$= k! \left[(p(\partial)) \, q(X) \right] \Big|_{X=0}$$

(it's just an inner product on the coefficients) to check that T_U is orthogonal as well, that is

$$\langle T_U p, T_U q \rangle_k = \langle p, q \rangle_k \, ;$$

thus $\det T_U = \pm 1$ and that's fine.
(details left as an exercise!)

The interesting bit is T_D; let us write

$$D = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_d \end{bmatrix}$$

then

$$T_D(\partial^{\underline{\alpha}}) = \underline{\lambda}^{\underline{\alpha}}\partial^{\underline{\alpha}} = \left(\lambda_1^{\alpha^{(1)}}\cdot\ldots\cdot\lambda_d^{\alpha^{(d)}}\right)\partial^{\underline{\alpha}},$$

so T_D is ~~the~~ itself diagonal!

It's determinant is therefore

$$\det T_D = \prod_{|\underline{\alpha}|=k}\underline{\lambda}^{\underline{\alpha}} = \prod_{|\underline{\alpha}|=k}\lambda_1^{\alpha^{(1)}}\cdot\ldots\cdot\lambda_d^{\alpha^{(d)}}$$

$$= \left(\prod_{|\underline{\alpha}|=k}\lambda_1^{\alpha^{(1)}}\right)\cdot\ldots\cdot\left(\prod_{|\underline{\alpha}|=k}\lambda_d^{\alpha^{(d)}}\right);$$

by symmetry then all the resulting exponents are the same, say N, and we have

$$\det T_D = \lambda_1^N\cdot\ldots\cdot\lambda_d^N = (\lambda_1\cdot\ldots\cdot\lambda_d)^N = 1.$$

■

Remark: Observe how we have used crucially the fact that the expression $(M^T\partial)^{\alpha_1}\wedge\ldots$ $\ldots\wedge(M^T\partial)^{\alpha_L}$ contains $\boxed{\text{all}}$ multiindices $\underline{\alpha}$ of length k.

An immediate consequence of this lemma is that $\rho_M A_t$ simplifies down to (using $\wedge$)

$$\underbrace{(\partial_1 f\wedge\ldots\wedge\partial_d f)}_{\text{complete block of order 1}}\wedge\underbrace{\left(\bigwedge_{|\underline{\alpha}|=2}\partial^{\underline{\alpha}}f\right)}_{\text{complete block of order 2}}\wedge\ldots\wedge\underbrace{\left(\bigwedge_{|\underline{\alpha}|=k-1}\partial^{\underline{\alpha}}f\right)}_{\text{complete block of order } k-1}\wedge$$

$$\wedge\underbrace{\left((M^T\partial)^{\beta_1}f\wedge\ldots\wedge(M^T\partial)^{\beta_m}f\right)}_{\text{incomplete block of order } k \text{ } (\boxed{m} \text{ is its size})}$$

So in general we only need to worry about the derivatives of the highest order (k_n, in the notation of $\Lambda_{d,n}$).

An immediate consequence is that when even <u>the last block is complete</u> ($m=0$, or $k_{n+1} = k_n + 1$) the matrices M^T disappear from the entire expression! In that case <u>there is no ~~supremum~~ infimum to take</u>!

<u>Proposition 1</u>: When d, n are such that $k_{n+1} = k_n + 1$ (i.e., the last block, containing n, is complete) we have for V_α

$$\left.\frac{dV_\alpha}{dt}\right|_p = C \cdot \underbrace{\left| \bigwedge_{1 \le |\underline{\alpha}| \le k} \partial^{\underline{\alpha}} f(p) \right|}_{\text{(this is a determinant)}}.$$

<u>Remark</u>: This condition depends on both d and n, in general.

Another consequence of the lemma is that we can answer question (4) easily:

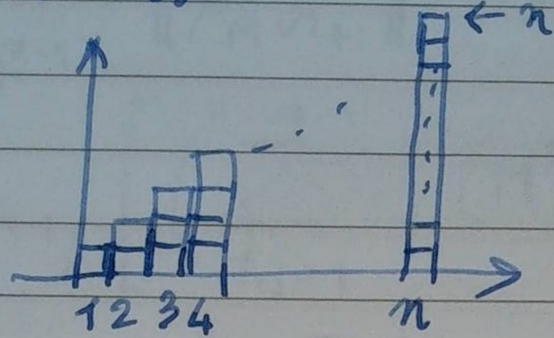
Recall that for curves

$$d\Omega_\gamma = |\tau_\gamma|^{\frac{2}{n(n+1)}} |\gamma'(t)|\, dt = \left|\det(\gamma' \dots \gamma^{(n)})\right|^{\frac{2}{n(n+1)}} dt$$

and for hypersurfaces described by graph $\underline{t} \mapsto (\underline{t}, \phi(\underline{t}))$

$$d\Omega_\Sigma = |K_\Sigma|^{\frac{1}{n+1}} d\sigma_\Sigma = \left|\det \nabla^2 \phi\right|^{\frac{1}{n+1}} dt_1 \dots dt_{n-1}.$$

If we look at $\Lambda_{1,n}$ (curves) we see that it is



, or equivalently $k_j = j$ for all $j \le n$.

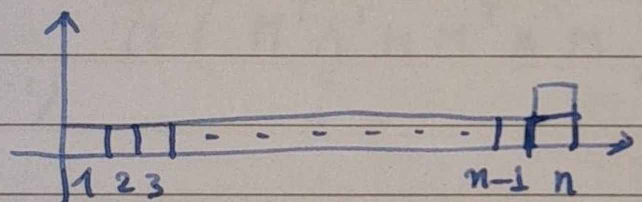
$\to Q = \#\Lambda_{1,n} = \frac{n(n+1)}{2}$

All blocks are ~~a~~ automatically complete and by the Proposition 1 we have directly

$$\frac{d\nu_A}{dt} = \left|\det\left(\gamma' \dots \gamma^{(n)}\right)\right|^{\frac{2}{n(n+1)}},$$

so that $d\nu_A$ and $d\Omega_\gamma$ coincide.

When we look at $\Lambda_{n-1,n}$ (hypersurfaces) it looks like this:



$k_j = 1$ for $j = 1, \dots, n-1$

$k_n = 2$.

Thus every expression for ${}_M A_t$ boils down to

$$\partial_1\left(\frac{t}{\phi(t)}\right) \wedge \dots \wedge \partial_{n-1}\left(\frac{t}{\phi(t)}\right) \wedge (M^T\partial)_i (M^T\partial)_j \left(\frac{t}{\phi(t)}\right)$$

for some i, j; as a matrix, this is

$$\det \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \\ \partial_1\phi & \partial_2\phi & & \partial_{n-1}\phi & (M^T\partial_i)(M^T\partial_j)\phi \end{bmatrix} = (M^T\partial_i)(M^T\partial_j)\phi$$

$$= \left(\sum_k M_{ki}\,\partial_k\right)\left(\sum_\ell M_{\ell j}\,\partial_\ell\right)\phi.$$

It is easy to see that this quantity is exactly the same as the (i,j) entry in the matrix $M^T(\nabla^2\phi)M$!

Thus we have $(d = n-1,\ Q = \#\Lambda_{n-1,n} = n+1)$

$$\inf_{M \in SL(\mathbb{R}^{n-1})} \| \rho_M \mathcal{A}_t \|_{\partial}^{\frac{n-1}{n+1}}$$ (omitting some constants here)

$$= \inf_{M \in SL(\mathbb{R}^{n-1})} \left[\left(\sum_{i,j=1}^{n-1} \left| \left(M^T (\nabla^2 \phi) M\right)_{(i,j)\text{ entry}} \right|^2 \right)^{1/2} \right]^{\frac{n-1}{n+1}}$$

$$= \inf_{M \in SL(\mathbb{R}^{n-1})} \| M^T (\nabla^2 \phi) M \|_{HS}^{\frac{n-1}{n+1}} ;$$

we have essentially already evaluated this infimum when motivating Theorem 1 in Lecture 2, and we had found that (with $A = \nabla^2 \phi$)

$$\inf_{M \in SL(\mathbb{R}^{n-1})} \| M^T A M \|_{HS} = \inf_{M \in SL(\mathbb{R}^{n-1})} \operatorname{Tr}\left((M^T A M)^T M^T A M \right)^{1/2}$$

$$= \inf_{M \in SL(\mathbb{R}^{n-1})} \operatorname{Tr}\left(M^T \underset{A}{\underset{\shortparallel}{A^T}} M M^T A M \right)^{1/2} = c_n \det (A^T A)^{\frac{1}{2} \cdot \frac{1}{n-1}}$$

$$= c_n (\det A)^{\frac{1}{n-1}} .$$

Therefore we have

$$\frac{d\nu_{\mathcal{A}}}{dt} = \inf_{M \in SL(\mathbb{R}^{n-1})} \| \rho_M \mathcal{A}_t \|_{\partial}^{\frac{n-1}{n+1}} = c_n' \left| \det \nabla^2 \phi \right|^{\frac{1}{n+1}} ,$$

which is precisely $d\Omega_\Sigma$ in these local coordinates.

So we have concluded:

<u>Answer to 4</u>: When $d = 1$, we have

$$d\nu_{\mathcal{A}} = d\Omega_\gamma$$

and when $d = n-1$

$$d\nu_{\mathcal{A}} = d\Omega_\Sigma$$

(up to constants)