On Grassman's

Affine Invariant Measures

Lecture 4

With these technical details out of the way we can finally use the construction of Theorem 1 on the Affine Curvature Tensor to obtain a density on the submanifold.

Gressman's Affine Invariant Measure

Let $f : \Sigma \to \mathbb{R}^n$ be an embedding of Σ (of dimension d); with $\Delta_{d,n}$ the diagram introduced before, we let $Q := \# \Delta_{d,n}$ (it plays the rôle of a homogeneous dimension).

If $\{\partial_1, \dots, \partial_d\}$ is a fixed basis of the tangent space $T_t \Sigma$ (at $t \in \Sigma$) and $X = \sum_{j=1}^{d} x_j \partial_j$ is a tangent vector, we write

$$M^T X = \sum_i \Big[\sum_j M_{ji} \, x_j \Big] \partial_i$$

(exactly as if we replaced ∂_j with $\underline{e}_j$); then $\rho_M \mathcal{A}^f_t$ is defined as before.

We define the density

$$\mu^t_{\mathcal{A}}(X_1, \dots, X_d) := \inf_{M \in SL(\mathbb{R}^d)} \| \rho_M \mathcal{A}^f_t \|_X^{d/Q}$$

$$= \inf_{M \in SL(\mathbb{R}^d)} \left[\left(\sum_{j_1=1}^{d} \cdots \sum_{j_Q=1}^{d} \left| \mathcal{A}^f_t (M^T X_{j_1}, \dots, M^T X_{j_Q}) \right|^2 \right)^{1/2} \right]^{d/Q}.$$

Using the density, one then defines a measure of Σ itself by push-forward:

If $\varphi : B \subseteq \mathbb{R}^d \to \Sigma$ (B a ball) is a coordinate chart, we let

$$\int_{\varphi(B)} g \, d\nu_{\mathcal{A}} := \int_B g(\varphi(y)) \, \mu_{\mathcal{A}}\left(d\varphi(\partial_{y_1}), \dots, d\varphi(\partial_{y_d}) \right) dy_1 \cdots dy_d .$$

Thus we have defined a measure $\nu_{\mathcal{A}}$ on the submanifold Σ. There are many things to check about this $\nu_{\mathcal{A}}$, but first an obvious one (which we have already seen):

Claim: The functional $\Sigma \longmapsto \nu_{\mathcal{A}}$ constructed above is <u>equi-affine invariant</u>.

<u>Proof</u>: Ignoring translations (clearly irrelevant), let $A \in SL(\mathbb{R}^n)$. We see directly that by linearity

$$\mathcal{A}_t^{Af}\left((X_{(j,k)})_{(j,k) \in \Lambda_{d,n}} \right) = \det \left[\cdots (X_{(j,1)} \cdots X_{(j,k_j)}) Af(t) \cdots \right]$$

$$= \det \left(\cdots A(X_{(j,1)} \cdots X_{(j,k_j)} f(t)) \cdots \right) \quad \leftarrow \text{(this happens in every column)}$$

$$= \det A \cdot \mathcal{A}_t^f \left((X_{(j,k)})_{(j,k) \in \Lambda_{d,n}} \right),$$ which is enough to conclude. ∎

<u>Remark</u>: The equi-affine invariance of $\nu_{\mathcal{A}}$ has nothing to do with the fact that we are taking $\inf_{M \in SL(\mathbb{R}^d)}$ in the definition of $\nu_{\mathcal{A}}$! It's purely a consequence of the structure of $\mathcal{A}_t^f$.

Now $\nu_{\mathcal{A}}$ looks like a reasonable candidate, but there are many questions that need answers:

1. Is $\nu_{\mathcal{A}} \not\equiv 0$ ever? More in general, given a submanifold Σ, can we establish whether $\nu_{\mathcal{A}} \not\equiv 0$ for Σ?

To answer the first part, Gressman provides a large class of examples for which $\mu_\gamma = \text{constant} \neq 0$; they are all of the form seen before, with polynomial coordinates of increasing degrees, but they must satisfy certain conditions (corresponding to a certain derivative vanishing).

The second part is more delicate and there isn't a definitive answer. Gressman provides a series of criteria to verify whether $\mu_\gamma \neq 0$, with various degrees of generality and applicability. We will see one of the most interesting ones.

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(2) Is the measure ν_γ good for our purposes, that is, L^p-smoothing and Fourier Extension/Restriction?

Let us go back to a result stated before that involved the affine invariant measure on hypersurfaces:

Theorem (endpoint L^p-smoothing for hypersurfaces)
Let $d\Omega_\Sigma = |\kappa_\Sigma|^{\frac{1}{n+1}} d\sigma_\Sigma$, where $\Sigma \subseteq \mathbb{R}^n$ is a hypersurface; then

$$\|f * d\Omega_\Sigma\|_{L^{n+1}(\mathbb{R}^n)} \lesssim \|f\|_{L^{\frac{n+1}{n}}(\mathbb{R}^n)}.$$

We make the following observation.
The restricted weak-type reformulation of the inequality above is

$$\langle \mathbb{1}_E * d\Omega_\Sigma , \mathbb{1}_F \rangle \lesssim |E|^{\frac{n}{n+1}} |F|^{\frac{n}{n+1}}$$

for measurable sets E, F (notice $(n+1)' = \frac{n+1}{n}$).

If we take $E = F = R$ a given rectangle this reads

$$\langle \mathbb{1}_R * d\Omega_\Sigma , \mathbb{1}_R \rangle = \langle d\Omega_\Sigma , \mathbb{1}_R * \mathbb{1}_R \rangle \lesssim |R|^{\frac{2n}{n+1}};$$

but one can easily see that

$$\mathbb{1}_R * \mathbb{1}_R \gtrsim |R| \, \mathbb{1}_R ,$$

and therefore

$$|R| \underbrace{\langle d\Omega_\Sigma , \mathbb{1}_R \rangle}_{\Omega_\Sigma(R)} \lesssim |R|^{\frac{2n}{n+1}}$$

$$\Rightarrow \boxed{\Omega_\Sigma(R) \lesssim |R|^{\frac{n-1}{n+1}}}.$$

(Recall that the dimension $\frac{\alpha}{n}$ for Oberlin's rectangular Hausdorff measure in the case of hypersurfaces was exactly $\frac{\alpha}{n} = \frac{n-1}{n+1}$.)

The above is known as a "curvature condition" for measures, and the argument given generalises to show that if one has L^p-smoothing for a certain measure μ, that is

$$\| f * d\mu \|_{L^q(\mathbb{R}^n)} \lesssim \| f \|_{L^p(\mathbb{R}^n)} , \qquad (q > p)$$

then it must satisfy

$$\mu(R) \lesssim |R|^{\frac{1}{p} - \frac{1}{q}},$$

which is again a curvature condition.

Remark: Fourier restriction estimates for submanifolds with measures μ similarly imply curvature conditions for μ (using Gaussians instead of rectangles).

So, the above has shown that a necessary condition for ν_ϑ to be "useful" is that it has to be curved for some exponent. We will show that this is indeed the case: in particular, we will show that

$$\boxed{\nu_\vartheta^\Sigma(R) \lesssim |R|^{d/Q}} \qquad (C)$$

for all rectangles $R \subseteq \mathbb{R}^n$ ($d = \dim \Sigma$, $Q = \#\Lambda_{d,n}$)

This discussion has however raised another question.

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(3) Are there "better" measures than ν_ϑ to do what we want? For example, are there measures μ that are $\gtrsim \nu_\vartheta$ (and much larger) but are curved with the same d/Q exponent?

The answer to this question will be essentially NO, in the sense that ν_ϑ is the largest measure that satisfies (C), up to a multiplicative constant.

More formally, one can show that if we let

$$C_\mu := \sup_{R \text{ rectangle}} \frac{\mu(R)}{|R|^{d/Q}},$$

then for every μ such that $C_\mu < \infty$ one has

$$\mu \lesssim_{d,n} C_\mu \, \nu_{\mathcal{A}}.$$

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(4) What is the relationship between $\nu_{\mathcal{A}}$ and the affine invariant measures for curves

$$d\Omega_\gamma = |T_\gamma|^{\frac{2}{n(n+1)}} |\gamma'(t)| \, dt$$

and hypersurfaces

$$d\Omega_\Sigma = |K_\Sigma|^{\frac{1}{n+1}} d\sigma_\Sigma \quad ?$$

These two cases are luckily simple enough that we can show directly that $\nu_{\mathcal{A}}$ actually coincides ~~...~~ with those measures when $d=1$ (curves) or $d=n-1$ (hypersurfaces).

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(5) What is the relationship between $\nu_{\mathcal{A}}$ and Oberlin's measure $\mathcal{R}^\alpha|_\Sigma$?

It turns out (as one would intuitively expect) that the two measures that result from these two functionals are always comparable. However, to see this one has to ~~...~~ do some extra work – this fact indeed follows from Gressman's more recent paper "Geometric Averaging Operators and Nonconcentration Inequalities."