On Gressman's Affine Invariant Measures

Lecture 3

Gressman's Construction: Step II

We thus have at our disposal a very general construction that we can use to produce a density. But how do we choose the multilinear functional to be fed to the construction?

Some preliminary motivation:

D. Oberlin's Hausdorff-like measure suggest looking at rectangles that approximate portions (small) of the submanifolds to understand their "curvature".

If our submanifold is given by embedding $f: \mathbb{R}^d \to \mathbb{R}^n$ we could consider the smallest (in volume) rectangle R_δ that contains

$$f([-\delta, \delta]^d);$$

we expect a dependence of the form

$$|R_\delta| \sim \delta^\alpha$$

for some $\alpha \geq 0$ as $\delta \to 0$, unless the submanifold is "flat": an extreme example is when $f(\mathbb{R}^d)$ is a linear subspace of $\mathbb{R}^n$, in which case $|R_\delta| = 0$.

The "best-curved" submanifolds (of dimension d in $\mathbb{R}^n$) will be those for which α is ~~[illegible]~~ <u>smallest</u> (so $|R_\delta|$ is "large").

How should one construct examples of such well-curved submanifolds?

Since we can always* Taylor-expand, we should consider polynomial coordinates

 * we could assume analyticity if we have to.

Let us look at <u>codimension 1</u> (hypersurfaces)
Typically the hypersurface looks like
$(n=d+1)$ $\qquad (t_1, \dots, t_d, \phi(t_1, \dots, t_d)) = f(\underline{t})$
for $\phi : \mathbb{R}^d \to \mathbb{R}$. If ϕ is linear the surface is a hyperplane and $|R_\delta| = 0$;
if $\nabla^2 \phi = 0$, say $\phi(t_1, \dots, t_d) = t_1^3 + \dots + t_d^3$
we would have near zero

$$f([-\delta, \delta]^d) \subseteq R_\delta \quad \text{with} \quad |R_\delta| \sim \delta^{d+3};$$

but we can do better if $\nabla^2 \phi \neq 0$, namely if $\phi(\underline{t}) = \sum_{j=1}^d \kappa_j t_j^2$ we see that we will have

$$|R_\delta| \sim \delta^{d+2} = \delta^{n+1}.$$

For <u>codimension $n-1$</u> (curves):
typically the curve is

$$\gamma(t) = (t, \gamma_1(t), \dots, \gamma_{n-1}(t));$$

if any γ_j ~~then~~ is linear in t we have $\gamma(\mathbb{R})$ is contained in a subspace and thus totally flat.
Similarly we see that the situation is only optimal when we take (an affine image of) the moment curve

$$(t, t^2, t^3, \dots, t^n)$$

(having any repeated power would collapse us back into a subspace of $\mathbb{R}^n$, which is bad)

To imitate somewhat these constructions for a generic codimension we should build something that looks like the following:

$f(t_1, \dots, t_d) =$

$$\Big(t_1, \dots, t_d, \underbrace{p_1(t), \dots, p_{M_1}(t)}_{\substack{\text{maximal set} \\ \text{of linearly indep.} \\ \text{homogeneous quadratic polynomials} \\ \text{(deg 2)}}}, \underbrace{p_{M_1+1}(t), \dots, p_{M_2}(t)}_{\substack{\text{deg 3,} \\ \text{maximal} \\ \text{lin. indep.} \\ \text{homog.}}},$$

$$\dots, \underbrace{p_{M_{k-3}+1}(t), \dots, p_{M_{k-2}}(t)}_{\substack{\text{degree } k-1, \\ \text{maximal} \\ \text{lin. indep.} \\ \text{homog.}}}, \underbrace{p_{M_{k-2}+1}(t), \dots, p_n(t)}_{\substack{\text{degree } k, \\ \text{not necessarily} \\ \text{maximal but} \\ \text{lin. indep., homog.}}}\Big)$$

(We stop when we fill all n coordinates)

<u>Caveat</u>: not all of surfaces built this way will be "well-curved" — we will see at least one explicit example.

Clearly, a maximal set of linearly independent polynomials of degree k is given by all the monomials

$$\underline{t}^{\alpha} = t_1^{\alpha_1} \cdots t_d^{\alpha_d} \quad \text{with } |\alpha| = k,$$

where $|\alpha| = \alpha_1 + \dots + \alpha_d$ and $\alpha_j \in \mathbb{N}$.

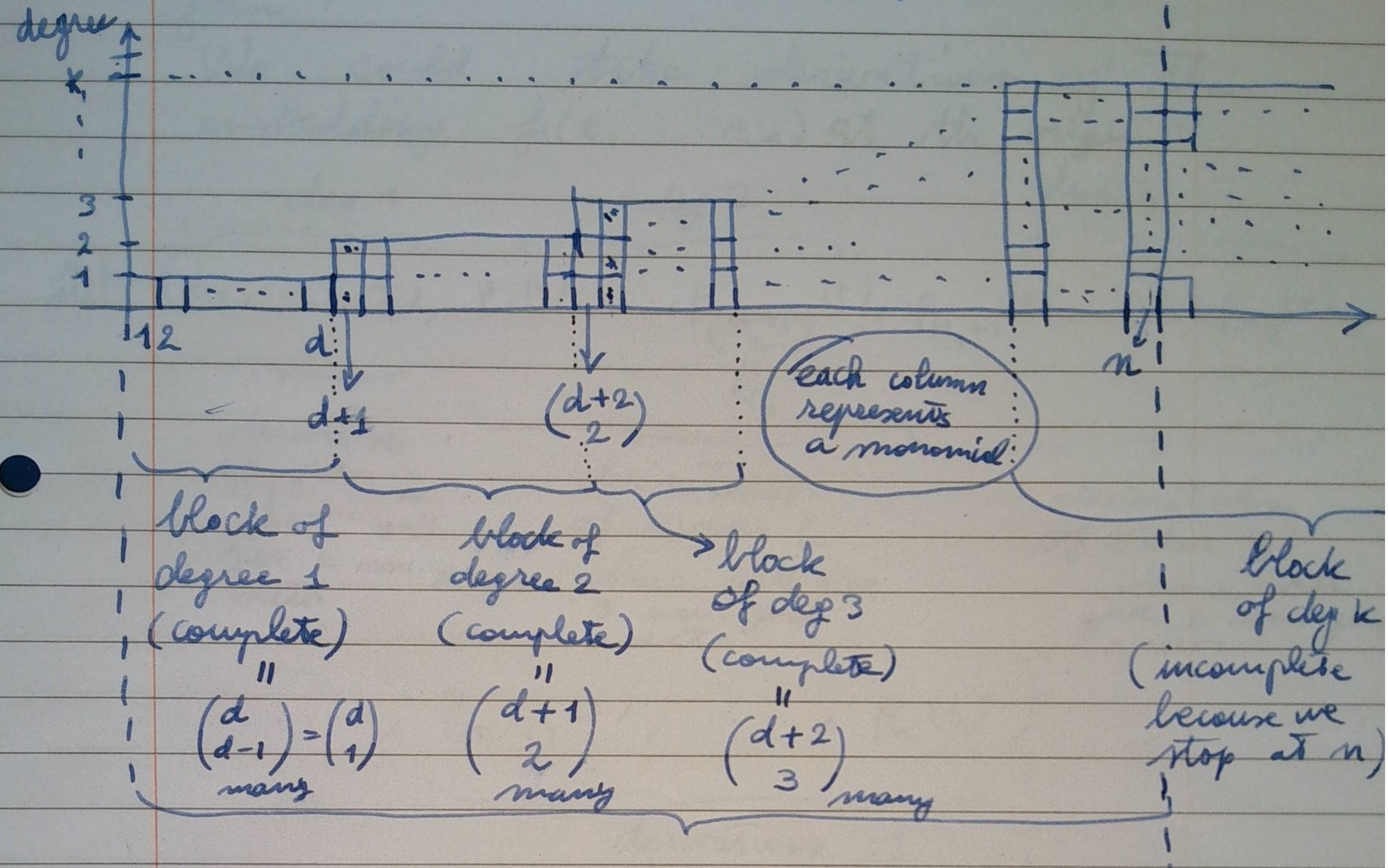
We can count precisely how many there are:

$$\binom{d+k-1}{d-1} = \# \text{ of monomials in } d \text{ variables and degree } k$$

while

$$\binom{d+k}{d} = \dim \text{ of vector space of polynomials } P \in \mathbb{R}[X_1, \dots, X_d] \text{ with } \deg P \leq k.$$

Let us introduce a graphical representation that will be notationally useful later:



The collection of squares inside here is denoted $\Delta_{d,n}$

- Each square can be identified with coordinates (j, k)
- For each j (a column) the highest square is (j, k_j)

As stated above, each column represents a monomial of degree given by its height (which monomial is not important; just choose an ordering of your liking)

Going back to our model surfaces, how do we detect whether they are of that form?

We could take derivatives of the embedding $f(t_1, \dots, t_d)$ at the origin:

$$f(\underline{t}) = (\overbrace{t_1, \dots, t_d}^{\text{deg } 1}, \overbrace{p_1(t), \dots, p_{\binom{d+1}{2}}(t)}^{\text{deg } 2}, \overbrace{p_{M_1+1}(t), \dots, p_{M_2}(t)}^{\text{deg } 3}$$

$\binom{d+1}{2} = M_1$

derivatives of order 1 here will give a non-zero vector

derivatives of order 2 here will give a non-zero vector

derivatives of order 3 here will give a non-zero vector

$$\dots, \underbrace{p_{M_{k-1}+1}(t), \dots, p_n(t)}_{\text{derivatives of order } k = k_n \text{ here will give a non-zero vector}})$$

(see previous page)

One can in particular consider the vectors that result: for example

$$\partial_{t_1}\begin{pmatrix} t_1 \\ \vdots \\ t_d \\ \vdots \\ p_n(t) \end{pmatrix}, \dots, \partial_{t_d}\begin{pmatrix} t_1 \\ \vdots \\ t_d \\ \vdots \\ p_n(t) \end{pmatrix}, \dots \quad \partial^{\alpha}_{\underline{t}}\begin{pmatrix} t_1 \\ \vdots \\ t_d \\ \vdots \\ p_n(t) \end{pmatrix}$$

derivatives of order 1

derivative of order $|\alpha| = k$

Somewhat intuitively, there should be some curvature in the submanifold if these vectors turn out to be linearly independent — or at least, this seems like a minimal requirement for this to happen. It is with this intuition in mind that Gressman defines the multilinear functional to be used with the construction described before.

Affine Curvature Tensor

We will use the elements of $\Lambda_{d,n}$ described before as indices.
In particular, we consider collections of vector fields / directional derivatives

$(X_\lambda)_{\lambda \in \Lambda_{d,n}}$, where each X_λ is a vector field / a directional derivative

$(\lambda = (j,k))$

Then the Affine Curvature Tensor $\mathcal{A}_t^f$ is defined to be (for $\boxed{f : \Sigma \to \mathbb{R}^n}$)
(embedding)

$$\mathcal{A}_t^f\left((X_\lambda)_{\lambda \in \Lambda_{d,n}}\right) := \det \begin{pmatrix} X_{(1,1)} f(t) & \cdots & (X_{(j,1)} X_{(j,2)} \cdots X_{(j,k_j)}) f(t) \\ & & \\ & & \cdots (X_{(n,1)} \cdots X_{(n,k_n)}) f(t) \end{pmatrix}$$

Perfect correspondence with the squares/boxes: to each square corresponds a derivative and they are grouped together if they belong to the same column.

The idea is to use the construction of a density on $\mathcal{A}_t^f$, but first we need to check a few things about $\mathcal{A}_t^f$:

1) The tensor $\mathcal{A}_t$ (omit the f) is equi-affine invariant, in the sense that if φ is an affine transformation of $\mathbb{R}^n$ with $|\det \varphi'| = 1$, we have

$$\mathcal{A}_t^{\varphi(f)} = \pm \mathcal{A}_t^f.$$

This is simply because

$$X_\lambda(\varphi \circ f) = d\varphi \cdot X_\lambda f,$$

where $d\varphi$ is a constant element of $SL(\mathbb{R}^n)$.

2) There is no difference between taking the X_λ to be vector fields or directional derivatives: $\mathcal{A}_t$ depends only on the value of X_λ at t. (So it can be fed to the construction...)

To see this, take $(X_\lambda)_\lambda$ and $(X'_\lambda)_\lambda$ so that they agree at t. One can see that we can replace the X_λ by X'_λ one by one by some simple identities: for example, consider j with $k_j = 3$ and let

$$X_{(j,1)} = X \qquad X_{(j,2)} = Y$$

$$X_{(j,3)} = Z \qquad X'_{(j,3)} = Z'$$

We replace Z with Z' in $\mathcal{A}_t$ and look at the difference in the column j we have

$$XYZf - XYZ'f \;;$$

but observe that

$$\underbrace{XYZ - XYZ'}_{\text{order } 3} = \underbrace{X\overbrace{[Y, Z-Z']}^{\text{order } 1}}_{\text{order } 2} + \underbrace{X(Z-Z')Y}_{\text{order } 3}$$

and

$$\underbrace{X(Z-Z')Y}_{\text{order } 3} = \underbrace{[X, Z-Z']Y}_{\text{order } 2} + \underbrace{(Z-Z')XY}_{\text{order } 3} .$$

Since $Z = Z'$ at t, the last term vanishes (this is because $Z - Z'$ is now the last derivative to be applied, being the leftmost one; in effect, the point of the above manipulations was exactly this)

The remaining terms are of order 2 (because the commutator is of order 1); but in A_t there ~~are~~ is already a maximal set of linearly independent differential operators of order 2 applied to f, and therefore adding any of these remaining terms makes the determinant zero.

Thus we have replaced $Z = X_{(j,3)}$ with $Z' = X'_{(j,3)}$; we can repeat for every other component of $(X_\lambda)_\lambda$ (the identities are more cumbersome to write down but are simply elementary iterations of the ones above). ∎