On Gressman's Affine Invariant Measures

Lecture 2

Grassmann's Construction: Step I

To introduce the construction of Grassmann we start by describing

A silly game

(part of this setup is preposterous; don't take this too seriously)

Consider the following situation:

A parallelepiped P in $\mathbb{R}^n$ has been fixed, but you can have only limited knowledge of it. In particular, you have a rather shoddy instrument that can only measure the area of the largest face of P: letting

$$A_j(P) = \text{area of the } j\text{-th face of } P \text{ (according to some ordering)},$$

the instrument returns

$$m(P) := \max_j A_j(P).$$

You are given the task of measuring/estimating the volume of P.

This is clearly impossible using the instrument $m(\cdot)$ ~~only~~ alone, so you are also given a button:

when you press the button, a random* equi-affine transformation of $\mathbb{R}^n$ is applied to the parallelepiped P

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* This is a bit troublesome as there is no uniform probability on $SL(\mathbb{R}^n)$. One could say we take a dense countable set...

In other words, when you press the button P becomes $AP = \{ Ay : y \in P \}$ for some "random" $A \in SL(\mathbb{R}^n)$.

The question then is: <u>can you win this game</u>?

Let's see: if P were a square Q, we would have

$$\mathcal{A}_j(Q) = \ell(Q)^{n-1}$$

and thus $m(Q) = \ell(Q)^{n-1}$, and therefore

$$m(Q)^{\frac{n}{n-1}} = \ell(Q)^n = |Q|.$$

We could keep pressing the button, but how do we know when we have effectively turned P into a square? (or an approximate square)

Observe that

$$|P| \leq \prod_{j=1}^{n} \mathcal{A}_j(P)^{\frac{1}{n-1}},$$

with equality if P is a rectangle (to see the inequality, apply a Gram-Schmidt-like process to P face by face to turn it into a rectangle of equal volume; the RHS can only decrease...)

But then $|P| \leq m(P)^{\frac{n}{n-1}}$ for all parallelepipeds P; and when P is a square we have equality!

This has shown that

$$|P| = \inf_{M \in SL(\mathbb{R}^n)} m(MP)^{\frac{n}{n-1}},$$

So to win the game we keep pressing the button infinitely many times and take the infimum!
In particular, we don't need to know what P looks like - the infimum takes care of that.

Let's change the game a little bit: the setup is the same but the shoddy instrument has been replaced with an equally shoddy one:

if P is identified with the list of its basis vectors,

$$P = \left\{ \sum_{j=1}^{n} t_j v_j \ : \ t_j \in [0,1] \right\},$$

the instrument returns

$$m(P) := \max_{j,k} |\langle v_j, v_k \rangle|.$$

Can we win the game now?

<u>Remark</u>: Clearly we have

$$\max_{j,k} |\langle v_j, v_k \rangle| = \max_j \|v_j\|^2,$$

so equivalently it measures the longest side of P.

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Observe that

$$\max_{j,k} |\langle v_j, v_k \rangle| \sim \Big(\sum_{j,k} |\langle v_j, v_k\rangle|^2 \Big)^{\frac{1}{2}};$$

if we let

$$V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$$

then the RHS above is the Hilbert-Schmidt norm of $V^T V$:

$$(V^T V)_{jk} = \langle v_j, v_k \rangle,$$

$$\Rightarrow \Big(\sum_{j,k} |\langle v_j, v_k \rangle|^2 \Big)^{1/2} = \| V^T V \|_{HS}$$

$$= \mathrm{Tr}\big((V^T V)^T V^T V \big)^{\frac{1}{2}}$$

$$= \mathrm{Tr}\big(V^T V V^T V \big)^{\frac{1}{2}}$$

When we press the button, V turns into MV for some $M \in SL(\mathbb{R}^n)$, and so $\| V^T V \|_{HS}$ turns into

$$\| V^T M^T M V \|_{HS} = \mathrm{Tr}\big((V^T M^T M V)^2 \big)^{\frac{1}{2}}.$$

Since $(V^T M^T M V)^2$ is symmetric and positive semi-definite, its eigenvalues $\lambda_1, \dots, \lambda_n$ are real and ≥ 0; we have

$$\mathrm{Tr}\big((V^T M^T M V)^2 \big)^{\frac{1}{2}} = \Big(\sum_j \lambda_j \Big)^{\frac{1}{2}}.$$

But by the AM-GM inequality

$$\Big(\sum_j \lambda_j \Big)^{\frac{1}{2}} \geq n^{\frac{1}{2}} \Big(\prod_{j=1} \lambda_j^{1/n} \Big)^{\frac{1}{2}},$$

and $\prod_{j=1}^n \lambda_j = \det\big((V^T M^T M V)^2 \big) = (\det V)^4$,

so we have shown

$$m(MP) \sim \| V^T M^T M V \|_{HS} \gtrsim \det(V)^{\frac{2}{n}} = |P|^{\frac{2}{n}};$$

but it's easy to see that equality can actually be achieved in the AM-GM inequality (choosing M such that MP is a square) and so we have shown

$$\inf_{M \in SL(\mathbb{R}^n)} m(MP)^{n/2} \sim |P|,$$

and we can win the game (to some approximation).

One last (important) example:

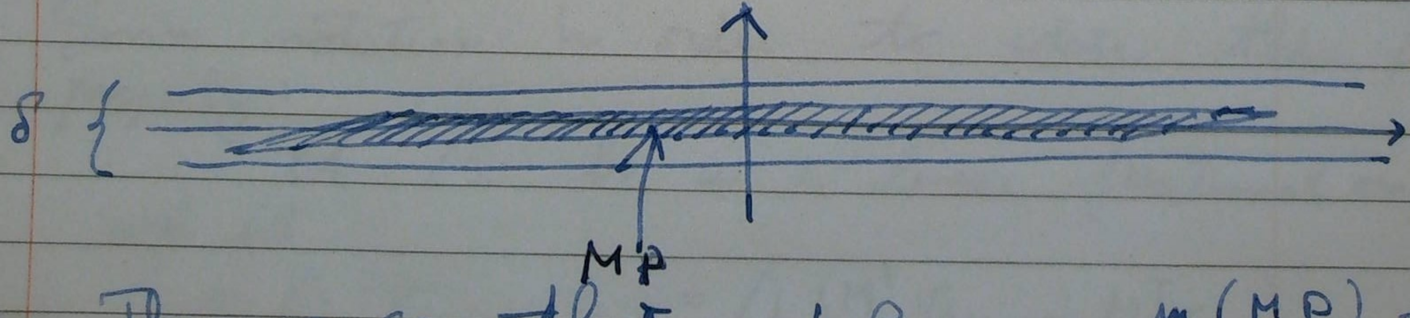
We change the ~~instrument~~ instrument once again; this time our instrument returns

$$m(P) = \max_j |\langle v_j, v \rangle|$$

for some fixed unit vector v. It looks like a small modification of the previous instrument, but...

This time we cannot win!

Indeed, there are plenty of $M \in SL(\mathbb{R}^n)$ such that MP is squeezed in the δ-neighbourhood of the hyperplane $\langle v \rangle^{\perp}$:



Thus we see that $\inf_{M \in SL(\mathbb{R}^n)} m(MP) = 0$; there is no useful information here.

Taking stock:

What have we learned?

- in the 1st case, 2nd case and 3rd case the instrument is measuring a norm of a multilinear functional:

$$\Lambda(v_1, \dots, v_k), \quad \|\Lambda\|_V$$

1st case: $\Lambda(v_1, \dots, v_{n-1})$ = area spanned by $v_1, \dots, v_{n-1}$

2nd case: $\Lambda(v_1, v_2) = \langle v_1, v_2 \rangle$, usual bilinear product

3rd case: $\Lambda(v) = \langle v, \nu \rangle$, linear

- in 1st and 2nd case our procedure of taking

$$\inf_{M \in SL(\mathbb{R}^n)} \|\Lambda(M\cdot, \dots, M\cdot)\|_V$$

produced something $\sim |P|^{\text{constant}}$;

in the 3rd case, the procedure simply returned identically 0.

The point is that this construction works in great generality and the two cases above are the only possibilities!

Some notation in order to state this fact properly:

let Λ be a k-linear functional on $\mathbb{R}^n$ and let

$$\rho_M \Lambda(v_1, \dots, v_k) = \Lambda(M^T v_1, \dots, M^T v_k)$$

(we use M^T instead of M so that ρ is a representation: $\rho_{M_1} \rho_{M_2} \Lambda = \rho_{M_1 M_2} \Lambda$.)

Given a list of vectors $V = (v_1, \dots, v_n)$ (n vectors in $\mathbb{R}^n$) we define norm (ℓ^2)

$$\|\Lambda\|_V := \left(\sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} |\Lambda(v_{j_1}, \dots, v_{j_k})|^2 \right)^{1/2}$$

and we let

$$\det V = \det \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}.$$

Then we have

(could even be larger than n!)

Theorem 1: Let Λ be a k-linear functional on $\mathbb{R}^n$. Then there exists a constant $c_\Lambda \geq 0$ (possibly 0) such that for any list of n vectors V we have

$$\inf_{M \in SL(\mathbb{R}^n)} \|\rho_M \Lambda\|_V^{\frac{n}{k}} = c_\Lambda |\det V|.$$

In other words, the quantity

$$\left[\inf_{M \in SL(\mathbb{R}^n)} \left(\sum_{j_1=1}^{n} \cdots \sum_{j_k=1}^{n} |\Lambda(M^T v_{j_1}, \dots, M^T v_{j_k})|^2 \right)^{\frac{1}{2}} \right]^{\frac{n}{k}}$$

is always a <u>density</u>, in the sense that it is a scalar multiple of the determinant of $(v_1, \dots, v_n)$. (maybe the scalar is 0)

<u>Remark</u>: We did not assume anything of Λ; in particular the resulting density is $|\det V|$ but Λ need not be alternating.

<u>Proof of Theorem 1</u>:

Consider first the case in which the vectors $v_1, \dots, v_n$ are linearly dependent.

Then we can find $V \subsetneq \mathbb{R}^n$ with $\dim V = n-1$ and such that

$$\operatorname{Span}(\{v_1, \dots, v_n\}) \leq V.$$

Let then M_δ be invertible matrices s.t.

- M_δ^T contracts every vector in V by δ
- M_δ^T dilates every vector in $V^\perp$ by $\delta^{-(n-1)}$

Then $(M_\delta)_{\delta > 0} \subset SL(\mathbb{R}^n)$ and

$$\Lambda(M_\delta^T v_{j_1}, \dots, M_\delta^T v_{j_k}) \longrightarrow 0$$

for all $j_1, \dots, j_k \in \{1, \dots, n\}$. Thus the inf will be zero.

Assume therefore that $v_1, \dots, v_n$ are linearly independent and let V be the matrix

$$V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}$$

and V' the normalised matrix given by

$$V = |\det V|^{\frac{1}{n}} V'$$

(so that $\det V' = \pm 1$).

Then

$$v_j = V \underline{e}_j, \text{ where } \underline{e}_j \text{ is } \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j\text{-th}$$

and

$$\Lambda(M^T v_{j_1}, \dots, M^T v_{j_k})$$

$$= \Lambda(M^T V \underline{e}_{j_1}, \dots, M^T V \underline{e}_{j_k})$$

$$= \Lambda(M^T |\det V|^{\frac{1}{n}} V' \underline{e}_{j_1}, \dots, M^T V' \underline{e}_{j_k} |\det V|^{\frac{1}{n}})$$

$$= |\det V|^{k/n} \Lambda(M^T V' \underline{e}_{j_1}, \dots, M^T V' \underline{e}_{j_k});$$

if $\det V' = +1$ we have $M^T V' \in SL(\mathbb{R}^n)$ as well and thus

$$\inf_{M \in SL(\mathbb{R}^n)} \| \rho_M \Lambda \|_V^{\frac{n}{k}} = |\det V| \underbrace{\inf_{M \in SL(\mathbb{R}^n)} \| \rho_M \Lambda \|_{\{e_1 \dots e_n\}}^{n/k}}_{=: c_\Lambda} ;$$

if $\det V' = -1$, swap two vectors. ■

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