On Gressman's Affine Invariant Measures

Lecture 1

Two important problems in Harmonic Analysis

- Fourier Restriction / Extension:

Given $\Sigma \subseteq \mathbb{R}^n$ submanifold (compact) define

$$E_\Sigma g(x) := \int_\Sigma e^{2\pi i \xi \cdot x} g(\xi)\, d\mu(\xi)$$
$$= \widehat{g\, d\mu}(x)$$

(some measure on Σ)

Want extension estimates

$$\|E_\Sigma g\|_{L^q(\mathbb{R}^n)} \lesssim \|g\|_{L^p(\Sigma, d\mu)}.$$

Naturally Affine invariant!

- L^p-smoothing: Given Σ as before (and $d\mu$) define average

$$\mathcal{A}_\Sigma f := f * d\mu ;$$

want L^p-smoothing estimates

$$\|\mathcal{A}_\Sigma f\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

for $\boxed{q > p}$ (also naturally affine invariant)

($p < q$ is impossible by an argument of Hörmander: essentially because $\mathcal{A}_\Sigma$ is local, i.e. preserves extreme disjointness ...)

Some loose connection between the two:

$$\|E_\Sigma g\|_{L^4(\mathbb{R}^n)} = \|\widehat{g\,d\mu}\|_{L^4} = \|\widehat{g\,d\mu} \cdot \widehat{g\,d\mu}\|_{L^2}^{1/2}$$

1 $$= \|g\,d\mu * g\,d\mu\|_{L^2}^{1/2} = \sup_{\|h\|_{L^2}=1} |\langle h, g\,d\mu * g\,d\mu\rangle|$$

$$= \sup_{\|h\|_2=1} |\langle \underbrace{h * g\,d\mu}_{\text{an average of } h \text{ with respect to } g\,d\mu \text{, a density on } \Sigma \ldots}, g\,d\mu \rangle|$$

For both problems, curvature of Σ (of some sort) is essential: if Σ is a linear subspace, all estimates are false!

However, the "amount of curvature" also matters:

<u>Example</u>: • If $\mathbb{P} := \{(t,t^2) ;\ t \in [-1,1]\}$ we have

$$\|E_{\mathbb{P}}\, g\|_{L^q(\mathbb{R}^2)} \lesssim \|g\|_{L^p(\mathbb{P}, dt)}$$

for

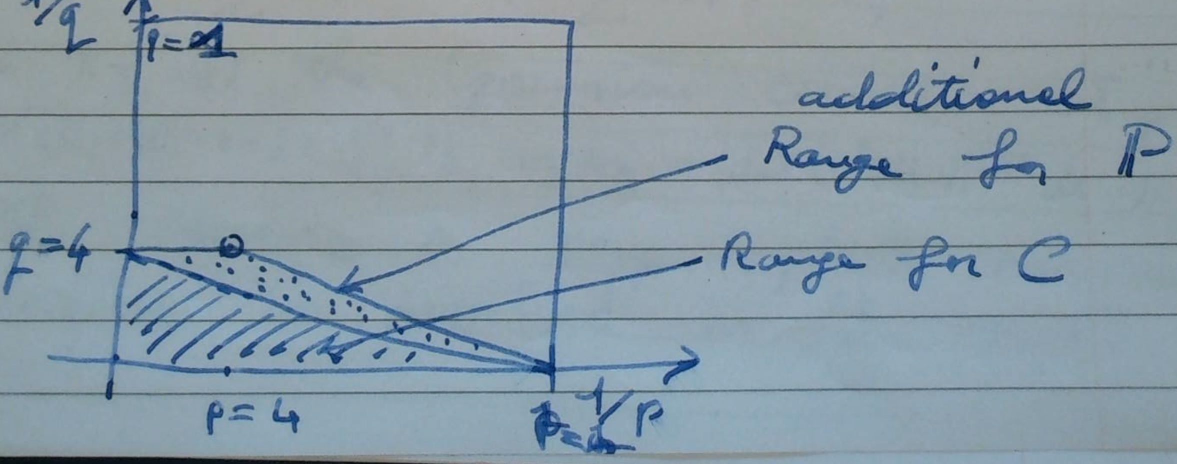
$$q > 4 \quad \text{and} \quad \boxed{q \geq 3p'}.$$

• If $C := \{(t,t^3) : t \in [-1,1]\}$ instead, we only have

$$\|E_C\, g\|_{L^q(\mathbb{R}^2)} \lesssim \|g\|_{L^p(C, dt)}$$

for

$$q > 4 \quad \text{and} \quad \boxed{q \geq 4p'}.$$



People have looked for ways to obtain "universal/uniform" statements and results across large classes of submanifolds

A nice way to do it is to introduce a **weight/density** on Σ that automatically dampens the degeneracies:

$$d\mu \longrightarrow w_\Sigma \underbrace{d\sigma_\Sigma}_{\text{usual surface measure } = dH^d|_\Sigma}$$

- For γ a curve, the "correct" choice is
$$w_\gamma(t) := |\tau_\gamma(t)|^{\frac{2}{n(n+1)}},$$
where τ_γ is the torsion of γ; in terms of coordinates one can calculate
$$w_\gamma(t)\, d\sigma_\gamma(t) = |\tau_\gamma(t)|^{\frac{2}{n(n+1)}} |\gamma'(t)|\, dt$$
$$= \left|\det\left(\gamma'(t)\ \gamma''(t) \cdots \gamma^{(n)}(t)\right)\right|^{\frac{2}{n(n+1)}} dt$$
(for $\gamma(t) = (t, t^2)$, $w_\gamma\, d\sigma_\gamma = \text{constant} \times dt$;
for $\gamma(t) = (t, t^3)$, $w_\gamma\, d\sigma_\gamma = |t|^{1/3} dt$, so the origin contributes very little ...)
- For Σ a hypersurface, the "correct" choice is
$$w_\Sigma(y) := |K_\Sigma(y)|^{\frac{1}{n+1}},$$
with $K_\Sigma(y)$ the Gaussian curvature at $y \in \Sigma$.
(if $\Sigma = \{(t, \phi(t)) : t \in [-1,1]^{n-1}\}$, $w_\Sigma\, d\sigma_\Sigma = |\det(\nabla^2\phi(t))|^{\frac{1}{n+1}} dt$.)

In both cases, a very explicit measure of curvature is involved in the weight.

3)

With these measures one can ~~make~~ prove universal statements, e.g.

Theorem: Let Σ be a hypersurface, then

$$\| f * w_\Sigma \, d\sigma_\Sigma \|_{L^{n+1}(\mathbb{R}^n)} \lesssim_{m(\Sigma)} \|f\|_{L^{\frac{n+1}{n}}(\mathbb{R}^n)}$$

(where $m(\Sigma)$ is a certain multiplicity)

Remark: ($L^{\frac{n+1}{n}} \to L^{n+1}$ is the endpoint)

As you can see, we did not have to specify anything about Σ being "well-curved": the weight takes care of that for us.

An analogous one for Fourier extension:

Theorem: Let γ be a convex curve in $\mathbb{R}^2$. Then

$$\|E_\gamma g\|_{L^q(\mathbb{R}^2)} \lesssim \|g\|_{L^p(\underbrace{w_\gamma d\sigma_\gamma}_{= |\det(\gamma' \cdots \gamma^{(n)})|^{\frac{2}{n(n+1)}} dt \; = |\det(\gamma' \gamma'')|^{1/3} dt})}$$

$$\|E_\gamma g\|_{L^q(\mathbb{R}^2)} = \| \widehat{g(w_\gamma d\sigma_\gamma)} \|_{L^q(\mathbb{R}^2)}$$

for all $q > 4$ and $q \geq 3p'$.

Remark: This is the same range that one has for $\gamma(t) = (t, t^2)$, which is the best possible case

These weights have a fundamental property: they are affine-invariant, in a sense made precise by the following:

<u>Definition</u>: Let Ω be a functional that maps a submanifold (in a certain given class, closed under affine transformations of $\mathbb{R}^n$) to a measure on the submanifold itself. We say that Ω is <u>equi-affine invariant</u> if $\forall\ \varphi: \mathbb{R}^n \to \mathbb{R}^n$ equi-affine (invertible) transformation of $\mathbb{R}^n$ and $\forall\ E \subseteq \Sigma$ (Borel measurable) we have

$$\Omega_\Sigma(E) = \Omega_{\varphi(\Sigma)}(\varphi(E)).$$

<u>Remark</u>: The "equi" in "equi-affine" means volume-preserving; in other words, $|\det d\varphi| = 1$

One can verify:

- $d\Omega_\gamma = |\tau_\gamma|^{\frac{2}{n(n+1)}} d\sigma_\gamma(t) = |\tau_\gamma|^{\frac{2}{n(n+1)}} |\gamma'|\, dt$ is equi-aff. inv.
- $d\Omega_\Sigma = |K_\Sigma|^{\frac{1}{n+1}} d\sigma_\Sigma$ is equi-aff. inv.
- the powers $\frac{2}{n(n+1)}$ and $\frac{1}{n+1}$ above are the unique ones for which this happens.
- $\Omega_\gamma, \Omega_\Sigma$ are also invariant under reparametrisations of γ, Σ.

Affine invariant functionals have been known ~~a~~ outside of harmonic analysis for a long time ...

for example:

<u>Affine Isoperimetric Inequality</u>

~~For every convex body K with C^2 boundary we have~~

$$\Omega_{\partial K}(\partial K)^{n+1} \lesssim |K|^{n-1},$$

~~that is~~

$$\left(\int_{\partial K} |\kappa_{\partial K}(y)|^{\frac{1}{n+1}} \, d\sigma_{\partial K}(y) \right)^{n+1} \lesssim |K|^{n-1}.$$

<u>Remark</u>: The aff. Isop. inequality follows from any affine invariant Fourier extension estimate along the critical line (i.e. for $q = \frac{n+1}{n-1} p'$).

<u>Remark</u>: The usual isoperimetric inequality goes the opposite way:

$$\sigma(E)^n = \left(\int_{\partial E} d\sigma_{\partial E}(y) \right)^n \gtrsim |E|^{n-1}.$$

<u>Question</u>: What is the correct/best choice of weights for arbitrary submanifolds $\Sigma \subseteq \mathbb{R}^n$ of dimension d?

Until 2018, the answer was known only for the two cases above* (curves and hypersurfaces)

Then in 2018 Gressman built a functional that is <u>equi-affine invariant</u> and is defined <u>for arbitrary submanifolds</u> of $\mathbb{R}^n$.

Moreover, he proved that in a sense this functional is the best possible.

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* and also for 2-surfaces ~~in~~ $\mathbb{R}^4$.

Before that, we discuss another candidate for the functional:

Consider the modification of the Hausdorff measure that uses rectangles instead of balls / squares:

$$R^\alpha_\delta(E) := \inf \left\{ \sum_i |R_i|^{\frac{\alpha}{n}} : (R_i)_i \text{ collection of rectangles s.t. } E \subseteq \bigcup_i R_i \text{ and } \operatorname{diam}(R_i) < \delta \right\}.$$

$$R^\alpha(E) := \lim_{\delta \to 0^+} R^\alpha_\delta(E).$$

D. Oberlin had shown that for curves (and hypersurfaces ...) it holds that

$$R^{2/n+1}\Big|_\gamma (E) \sim \Omega_\gamma(E) = \left(\int \mathbb{1}_E(\gamma(t)) \times \left|\det(\gamma' \dots \gamma^{(n)})\right|^{\frac{2}{n(n+1)}} dt \right)$$

and

$$R^{\frac{n(n-1)}{n+1}}\Big|_\Sigma (E) \sim \Omega_\Sigma(E)$$

(constants depend only on multiplicity of curves/hypersurfaces)

It was natural to conjecture that the right candidate would be

$$R^{\alpha_{d,n}}\Big|_\Sigma$$

for arbitrary Σ submanifold. This is correct!

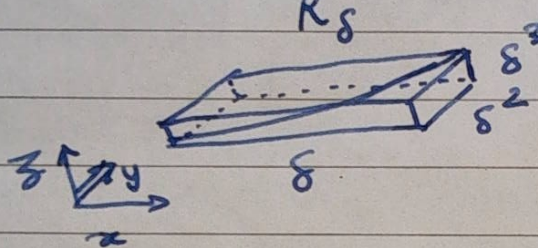
We'll see ...

Where do those exponents α come from?

Consider curves: the prototypical well-curved curve is the moment curve

$$\gamma(t) = (t, t^2, t^3, \dots, t^n)$$

To cover a small portion $\gamma([0,\delta])$ we want to choose the smallest rectangle R_δ that contains it:



$$|R_\delta| = \delta \cdot \delta^2 \cdot \delta^3 \cdots \delta^n = \delta^{\frac{n(n+1)}{2}}$$

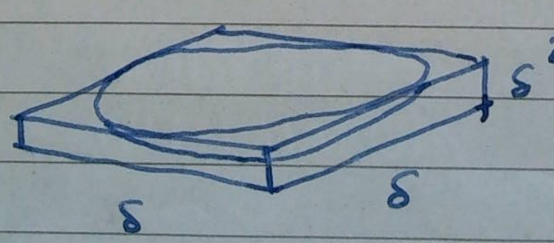
to make it a length (dimensionally) we take

$$\delta = |R_\delta|^{\frac{\alpha}{n}} \iff \alpha = \frac{2}{n+1}.$$

For hypersurfaces: prototypical example is a quadratic surface

$$f(t_1, \dots, t_{n-1}) = \left(t_1, \dots, t_{n-1}, \sum_{j=1}^{n-1} a_j t_j^2 \right)$$

To cover $f([-\delta, \delta]^{n-1})$ with a rectangle R_δ we need at least δ^{n+1} volume:



$$|R_\delta| = \underbrace{\delta \cdot \ldots \cdot \delta}_{n-1} \cdot \delta^2 = \delta^{n+1}$$

To make an area, we need

$$\delta^{n-1} = |R_\delta|^{\alpha/n} \iff \alpha = \frac{n(n-1)}{n+1}.$$