When $r < \delta^{-\varepsilon^3}$??

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⚠

$$\delta^{n-1-\varepsilon/4}|\Pi| < r < \delta^{-\varepsilon^3}$$

$$\delta^{n-1-\varepsilon/4}|\Pi| \lesssim \delta^{n-1-\varepsilon/4} W^{2(n-1)}$$

When $r < \delta^{-\varepsilon^3}$??

⚠

$$\delta^{n-1-\varepsilon/4}|\Pi| < r < \delta^{-\varepsilon^3}$$

$$\delta^{n-1-\varepsilon/4}|\Pi| \gtrsim \delta^{n-1-\varepsilon/4} W^{2(n-1)}$$

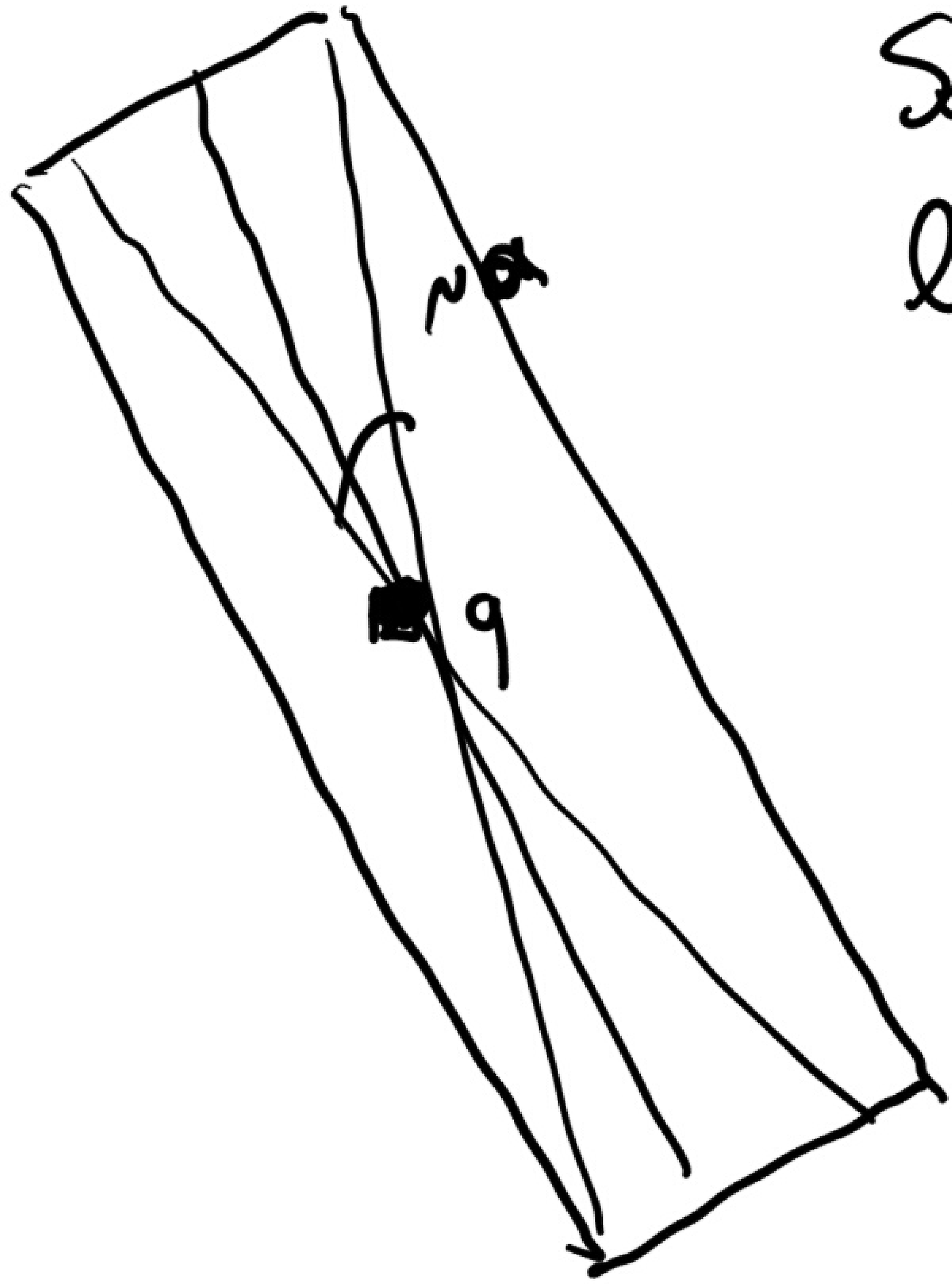
$$\Rightarrow W^{2(n-1)} \lesssim \delta^{n-1+\left(\frac{\varepsilon}{4}-\varepsilon^3\right)}$$

$$\Rightarrow \boxed{W \lesssim \delta^{-1/2+\left(\frac{\varepsilon}{8(n-1)}-\frac{\varepsilon^3}{2(n-1)}\right)}}$$

$$\lesssim \delta^{-1/2+\varepsilon/20}.$$

$\forall q \in P_r(\Pi)$, consider the max angle between any 2 tubes passing through q.

By dyadic pigeonholing, $\exists$ angle α s.t. the max angle between any 2 tubes through a typical cube in $P_r(\Pi)$ is $\sim \alpha \; (\gtrsim W^{-1})$



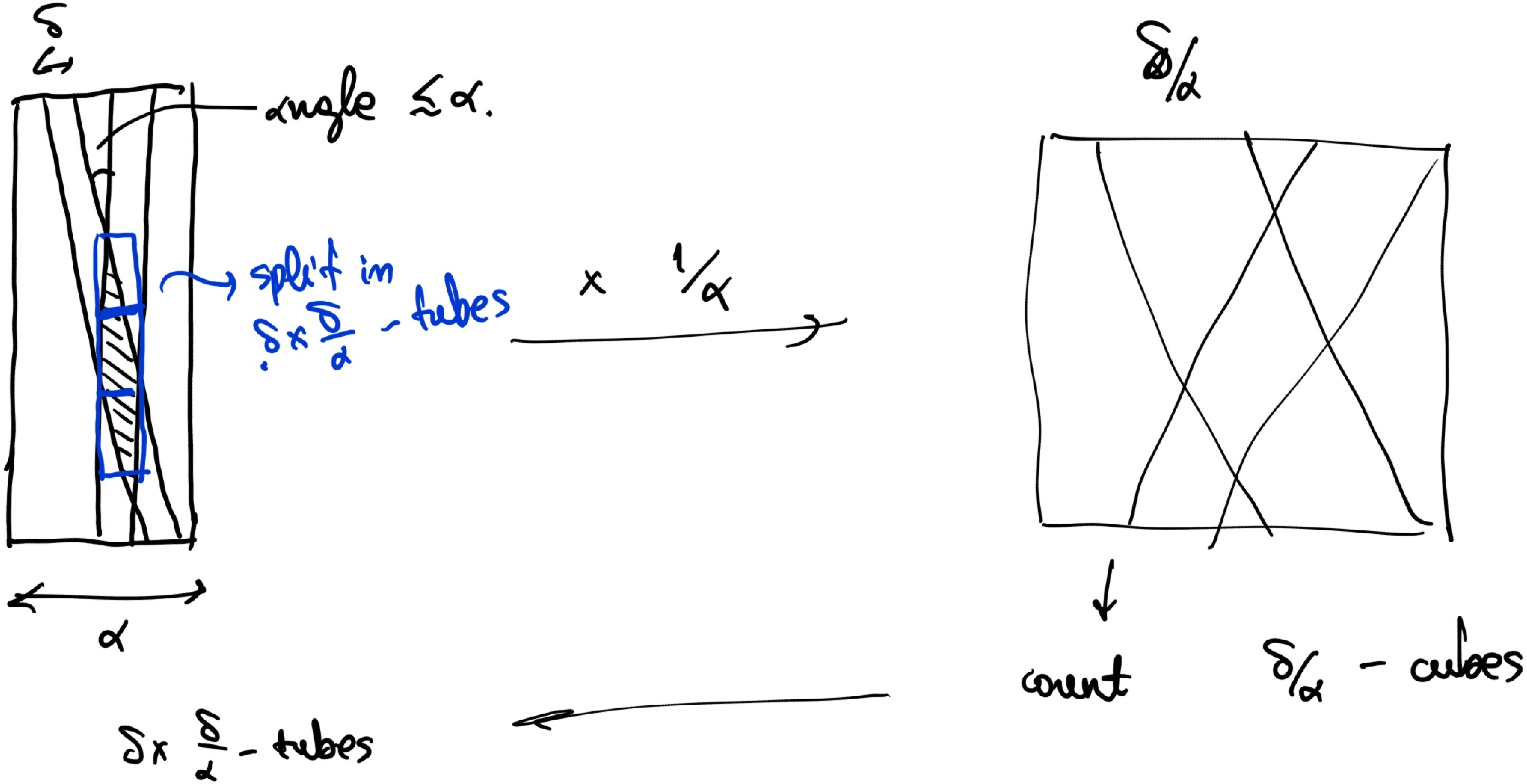
So, if $q \in P_r(\mathbb{T})$, then **all** tubes through q live inside the same α-tube. $\square$

So, consider **all** the α-tubes in $B^n(0,1)$,
($\sim \alpha^{-2(n-1)}$ of them)

Let $\mathbb{T}_\square$ be the set of δ-tubes in $\mathbb{T}$ in $\square$.

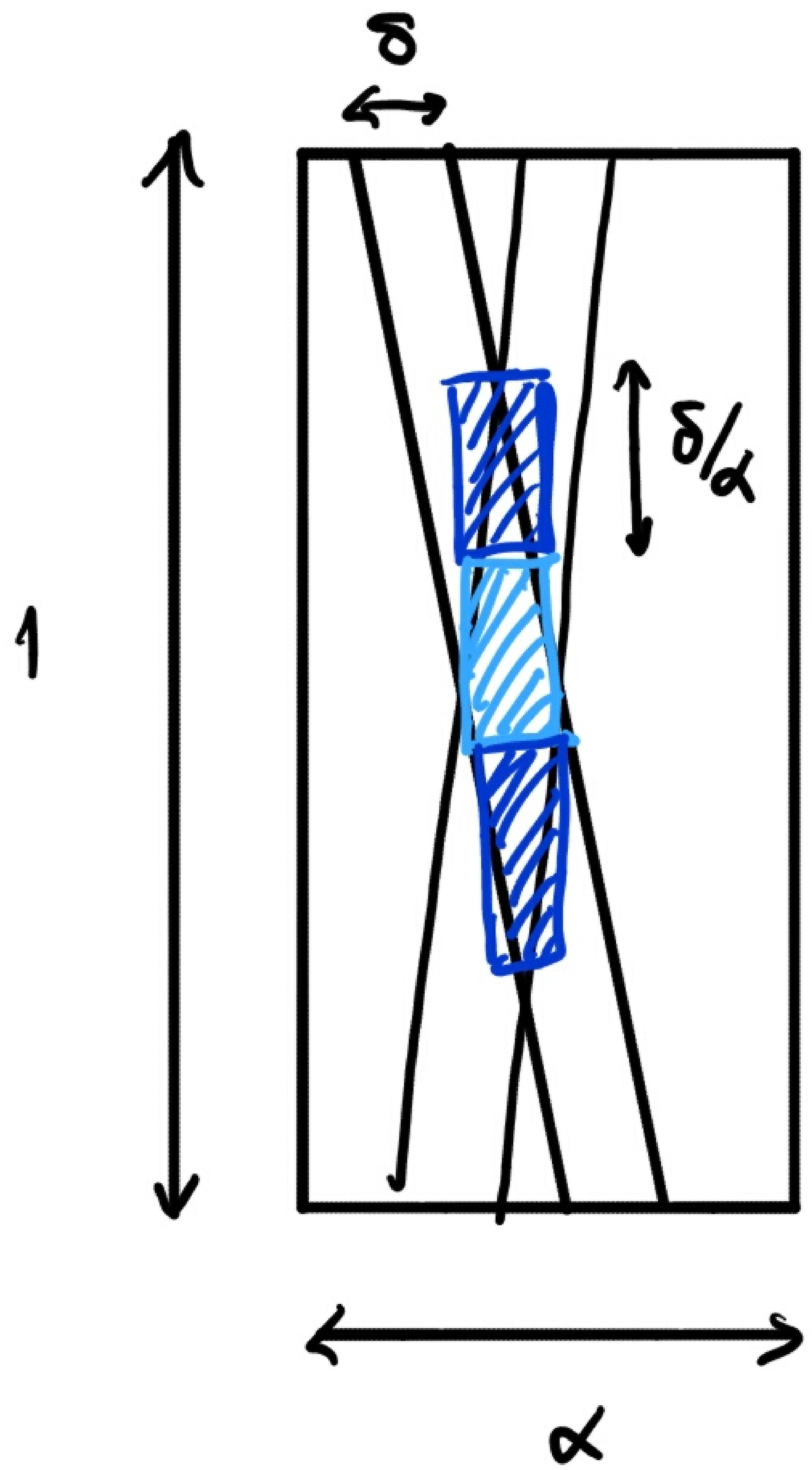
$$|P_r(\mathbb{T})| \lessapprox \sum_\square |P_r(\mathbb{T}_\square)| \lessapprox \sum_\square |P_2(\mathbb{T}_\square)|$$

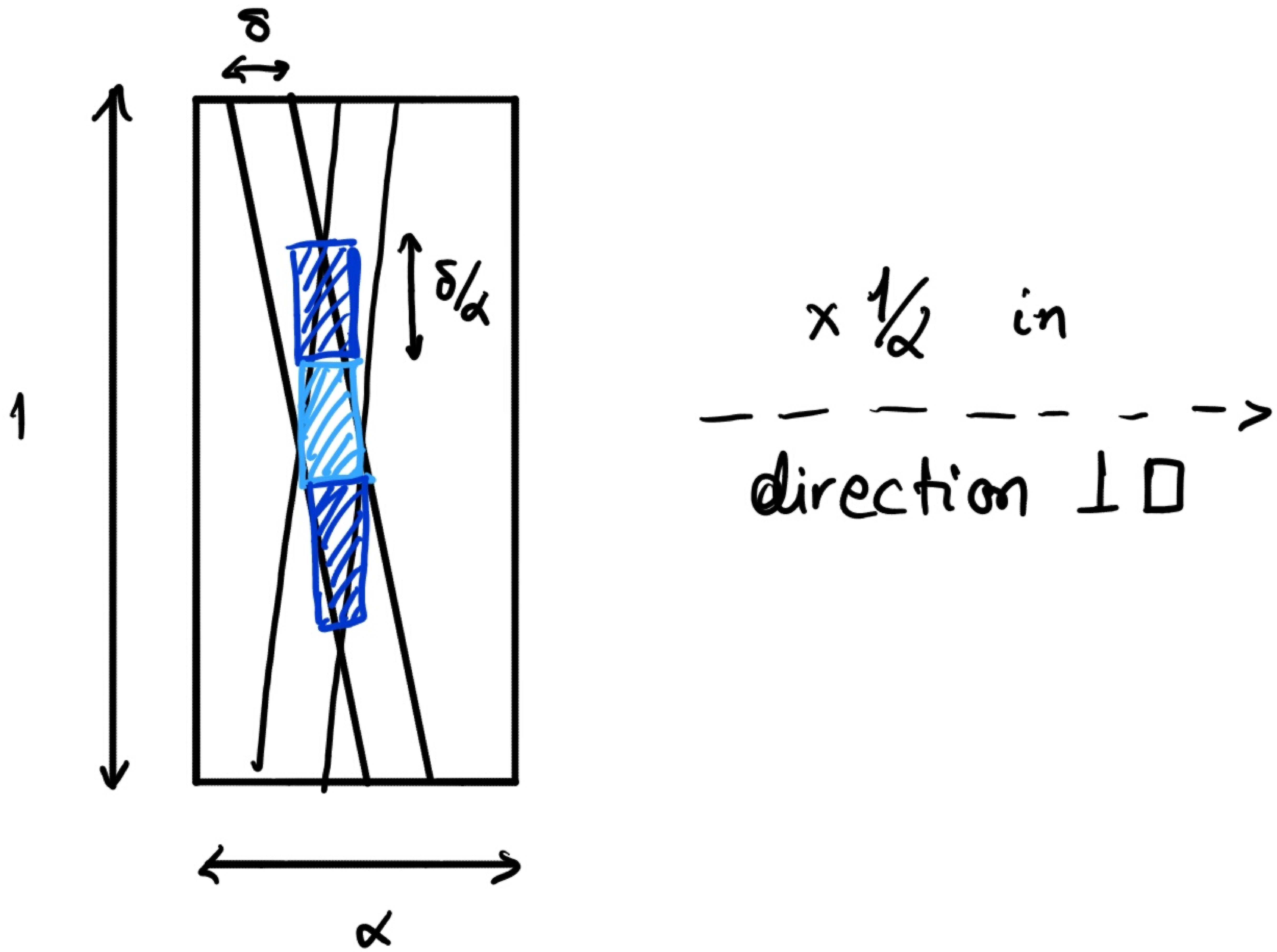
means $P_{\geq 2}(\mathbb{T}_\square)$.

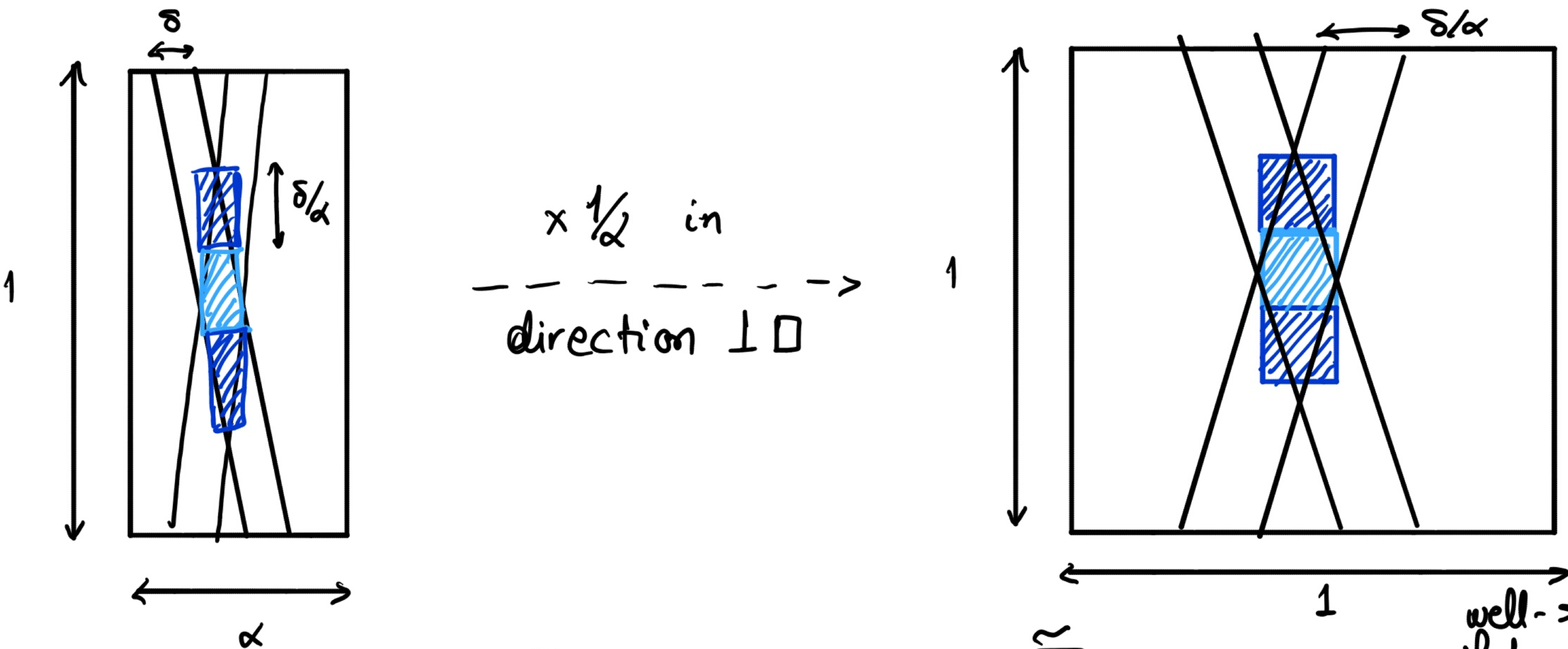


$\bar{P}_r(\mathbb{T}_\square) :=$ the $\geq r$ - rich $\delta \times \frac{\delta}{\alpha}$ tubes ↪ for $\mathbb{T}_\square$.

$|P_r(\mathbb{T}_\square)| \lesssim |\bar{P}_r(\mathbb{T}_\square)| \cdot \alpha^{-1}$.


δ
δ/α
1
α


δ
δ/α
1
α
× 1/2 in
direction ⊥ □



- δ-tubes in $T_\square$ $\longrightarrow$ $\tilde{\delta} := \delta/\alpha$ - tubes in $\tilde{T}_\square$.
- Each w^{-1}- tube contains ≤ 1 δ-tube $\longrightarrow$ Each w^{-1}/α -tube contains ≤ 1 δ/α - tube in $\tilde{T}_\square$.

  well-spaced, if I add as many as needed.
- $\delta \times \delta/\alpha$ tubes in $\underline{P_2(T_\square)}$ $\longrightarrow$ δ/α - cubes in $\underline{P_2(\tilde{T}_\square)}$.

So: $\underline{|P_q(T_\square)|} = |P_q(\tilde{T}_\square)| \lesssim_\varepsilon \tilde{\delta}^{-\varepsilon} \cdot |\tilde{T}_\square|^{n/(n-1)}$

$$\lesssim_\varepsilon \frac{\delta^{-\varepsilon}}{\alpha^{-\varepsilon}} \cdot \left[\frac{\left((W_\alpha)^{q(n-1)}\right)^{n/(n-1)}}{r^{\frac{n+1}{n-1}}}\right] \cdot \left(\delta^{-\varepsilon^3}\right)^{\frac{n+1}{n-1}} \sim$$

$$-qn + q + \varepsilon - 1 + qn$$

$$\sim_\varepsilon \delta^{-\varepsilon} \frac{|T|^{\frac{n}{n-1}}}{r^{\frac{n+1}{n-1}}} \cdot \left[\alpha^{\varepsilon} \left(\delta^{-\varepsilon^3}\right)^{\frac{n+1}{n-1}}\right] \alpha^{qn}$$

So: $|P_r(T)| \lessapprox \underline{\alpha^{-q(n-1)}}\ \underline{\alpha^{\varepsilon}} \left(\delta^{-\varepsilon^3}\right)^{\frac{n+1}{n-1}} \cdot \left(\delta^{-\varepsilon} \frac{|T|^{n/(n-1)}}{r^{\frac{n+1}{n-1}}}\right) \cdot \underline{\alpha^{-1}}\ \alpha^{qn}$

$$\sim_\varepsilon \underbrace{\alpha^{1+\varepsilon} \left(\delta^{-\varepsilon^3}\right)^{\boxed{\frac{n+1}{n-1}}}}_{<<1} \cdot \left(\delta^{-\varepsilon} \frac{|T|^{n/(n-1)}}{r^{\frac{n+1}{n-1}}}\right).$$

$<<1$ OK, if $\boxed{\alpha < \delta^{10\varepsilon^3}}$

(!) <u>as long as</u> $\quad 2 > \tilde{\delta}^{\,n-1-\varepsilon/4} \, |\tilde{\pi}_{\square}| \sim$

$$\sim \frac{\delta^{n-1-\varepsilon/4}}{\alpha^{n-1-\varepsilon/4}} \, w^{2(n-1)} \, \alpha^{2(n-1)} \sim$$

$$\sim \delta^{n-1-\varepsilon/4} \, w^{2(n-1)} \, \alpha^{n-1+\frac{\varepsilon}{4}}$$

$$\iff w^{2(n-1)} \lesssim \delta^{-(n-1)+\varepsilon/4} \, \alpha^{-(n-1)-\varepsilon/4}$$

$$\iff w \lesssim \delta^{-1/2 + \frac{\varepsilon}{8(n-1)}} \, \alpha^{-1/2 - \frac{\varepsilon}{8(n-1)}}.$$

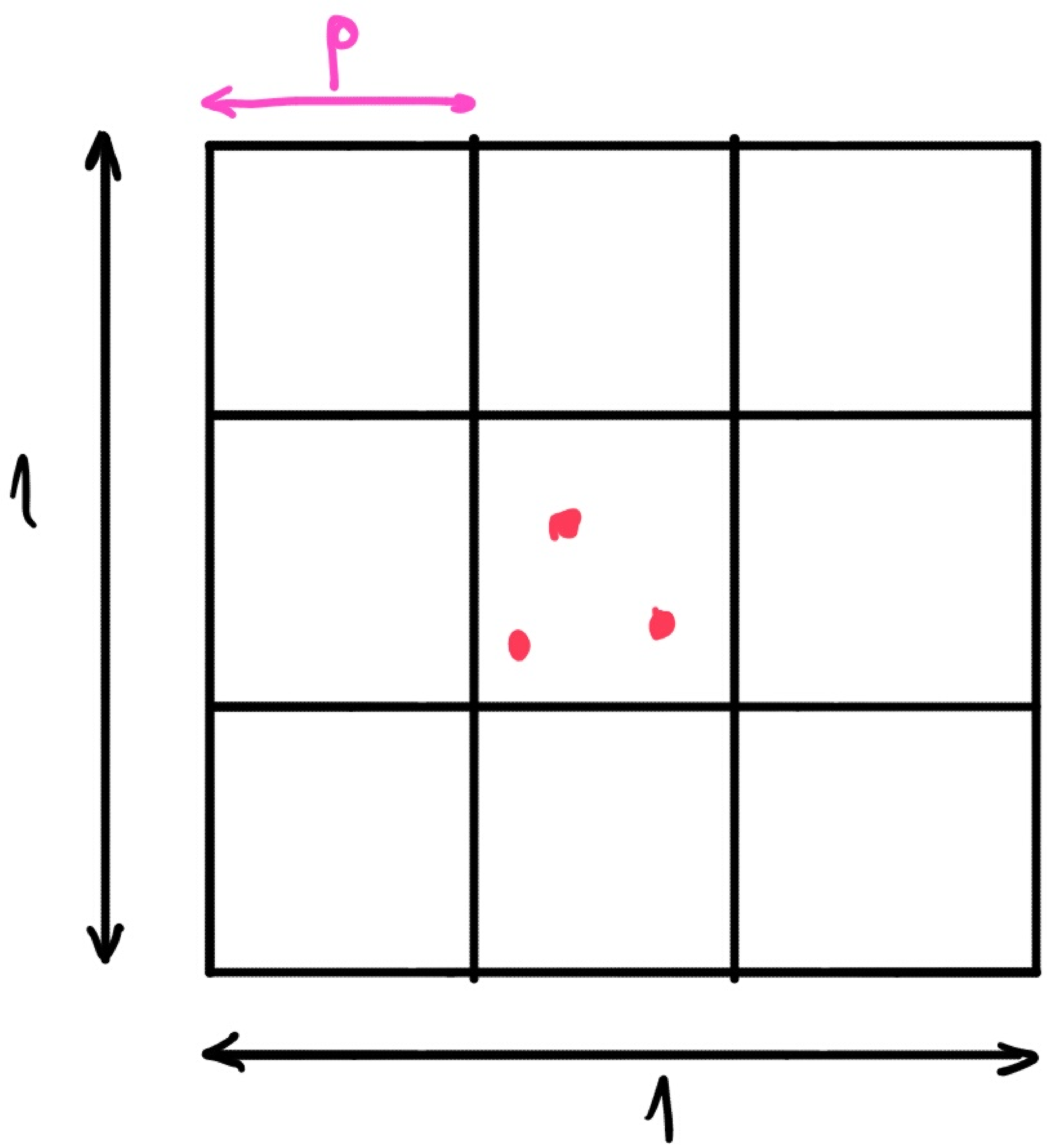
$\alpha < \delta^{10\varepsilon^3}$

Since $w \lesssim \delta^{-1/2 + \left(\frac{\varepsilon}{8(n-1)} - \frac{\varepsilon^3}{2(n-1)}\right)}$, OK if $\delta^{-\frac{\varepsilon^3}{2(n-1)}} \lesssim \alpha^{-1/2 - \frac{\varepsilon}{8(n-1)}}$

$$\iff \alpha \lesssim \delta^{\frac{4\varepsilon^3}{4(n-1)+\varepsilon}} \quad ✓$$

for $r < \delta^{-\varepsilon^3}$, $\alpha \geq \delta^{10\varepsilon^3}$:

for $r < \delta^{-\varepsilon^3}$, $\alpha \geq \delta^{10\varepsilon^3}$: Split $B^n(0,1)$ in ρ-cubes, for some ρ slightly larger than δ.

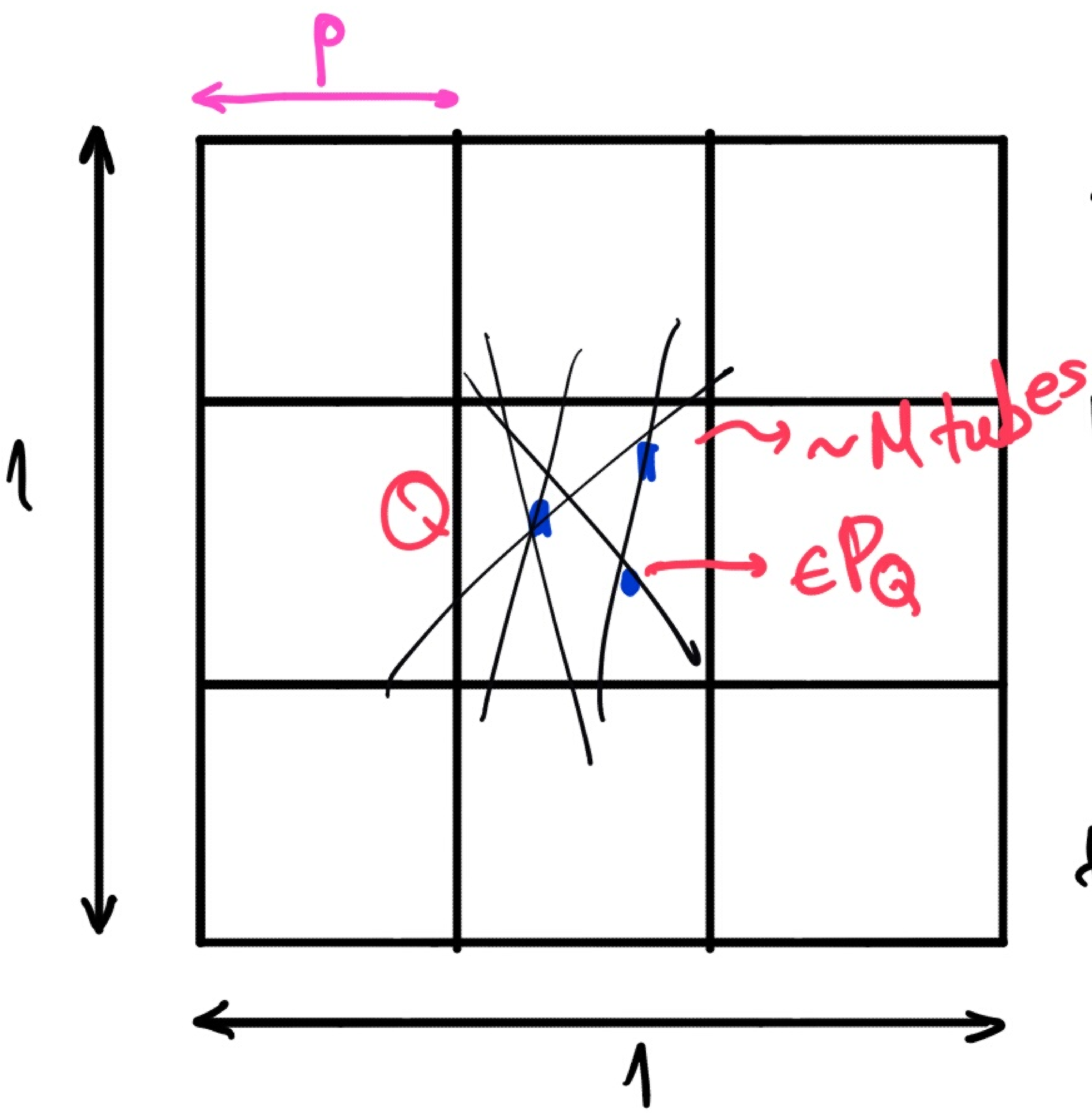


Good pick for ρ (in hindsight):

$w \lesssim \delta^{-1/2 + \varepsilon/20} \longrightarrow \delta \lesssim \delta^{\varepsilon/10} w^2$

$\longrightarrow \boxed{w^{-2} \gtrsim \delta \cdot \delta^{-\varepsilon/10}}$

$w^{-2} = \rho$



Look at the p-cubes that contain δ-cubes in $P_r(\mathbb{T})$

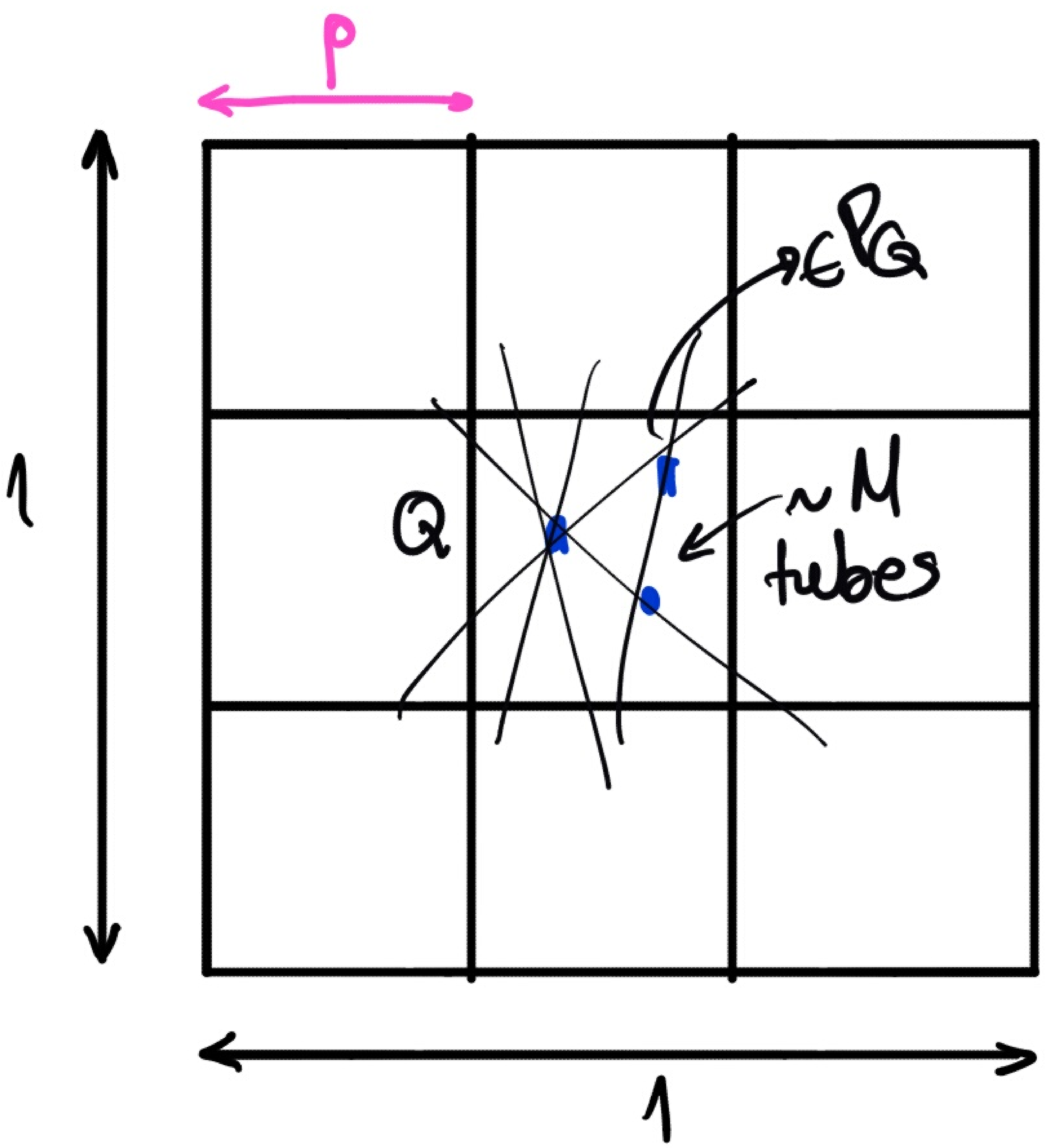
We may assume that all such p-cubes are intersected by the same # of δ-tubes in $\mathbb{T}$:

$\forall$ dyadic M : $Q_M := \{$ p-cubes each intersected by $\sim M$ δ-tubes in $\mathbb{T}\}$.

$\to \exists M$: $P := \{ q \in P_r(\mathbb{T}) : q \subseteq \bigcup_{Q \in Q_M} Q \}$ satisfies

$|P| \gtrapprox |P_r(\mathbb{T})|$. $\forall Q \in Q_M$, $P_Q := \{ q \in P : q \subseteq Q \}$.

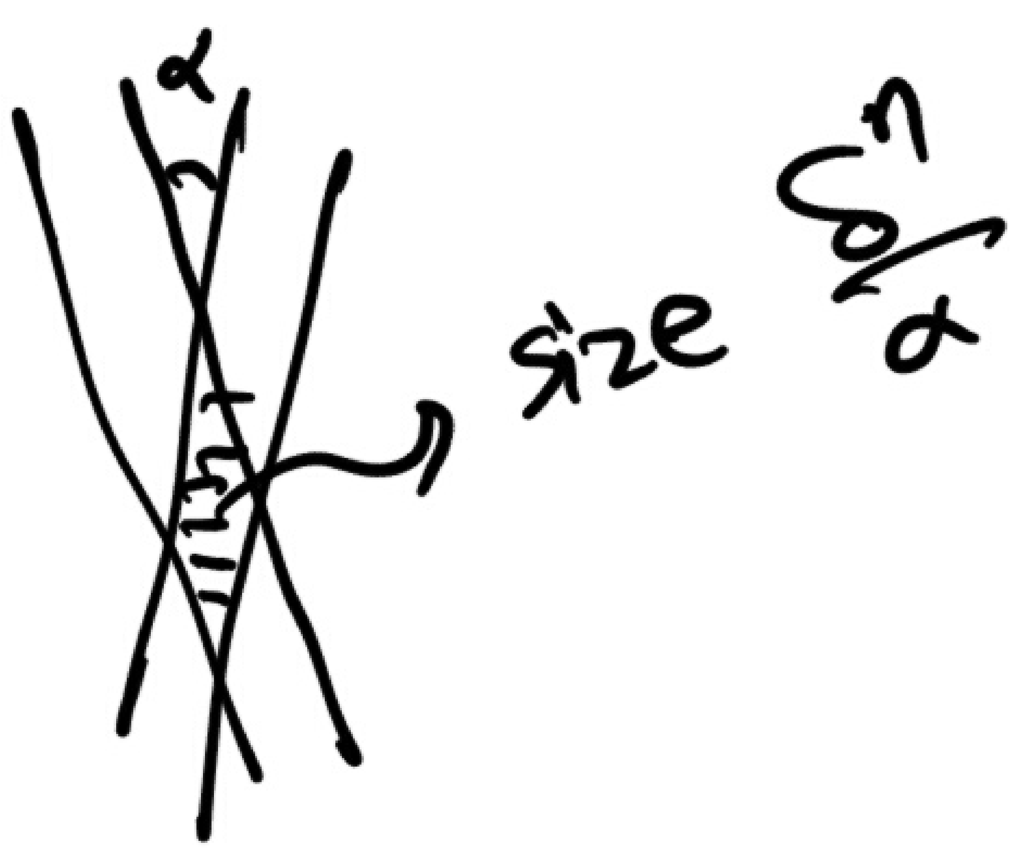
$|P_Q| = ?$



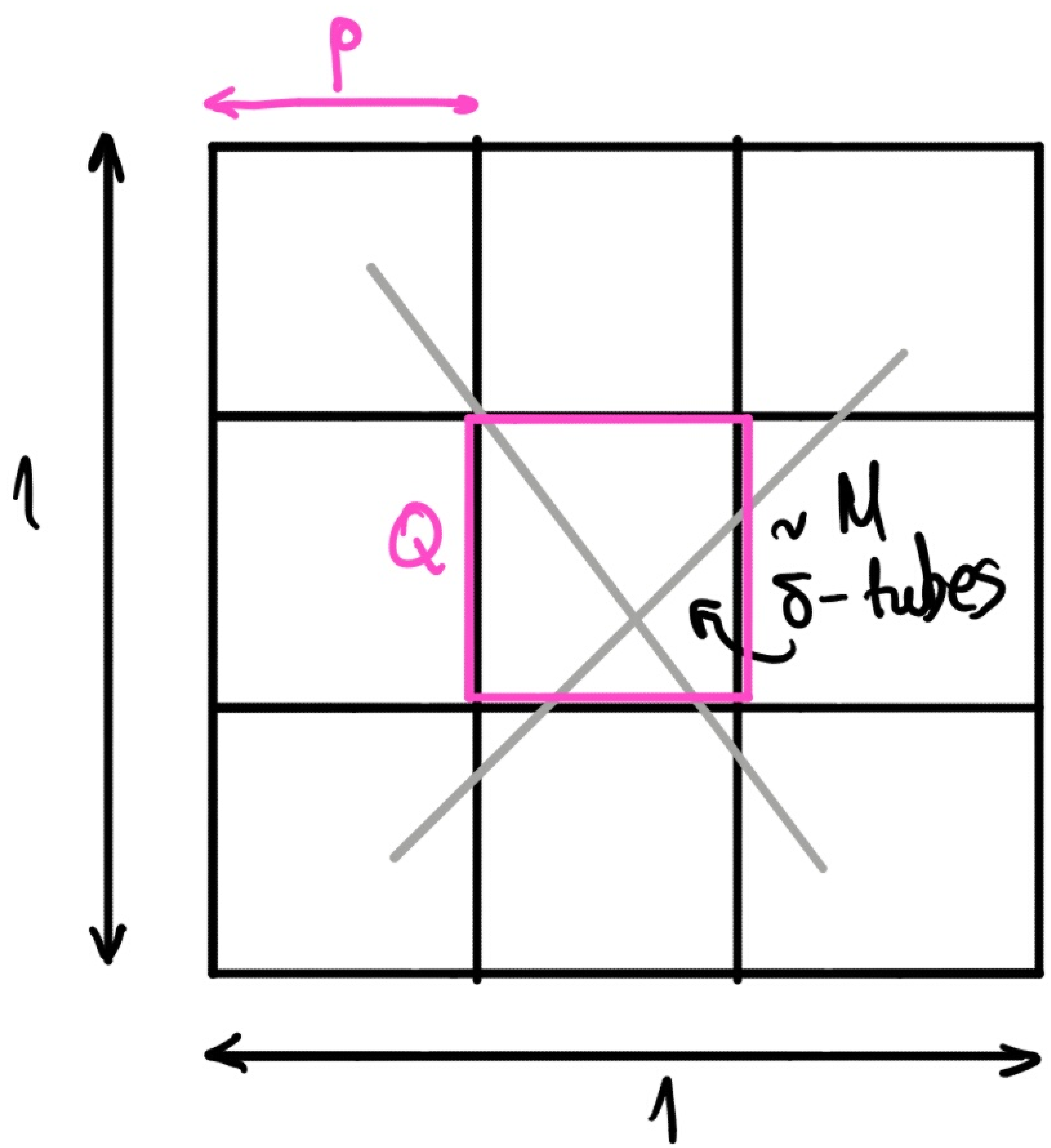
Every $q \in P_Q$ lies in the intersection of $\gtrsim 2$ (of the $\sim M$) δ-tubes in $\mathbb{T}$ that intersect at angle $\alpha \gtrsim \delta^{10\varepsilon^3}$.

So: $|P_Q| \lesssim \underbrace{\left(\text{\# pairs of tubes that can intersect at angle } \alpha\right)}_{\lesssim M^2} \cdot \underbrace{\left(\text{max \# of } \delta\text{-cubes in each such intersection}\right)}_{\lesssim \alpha^{-1}}$

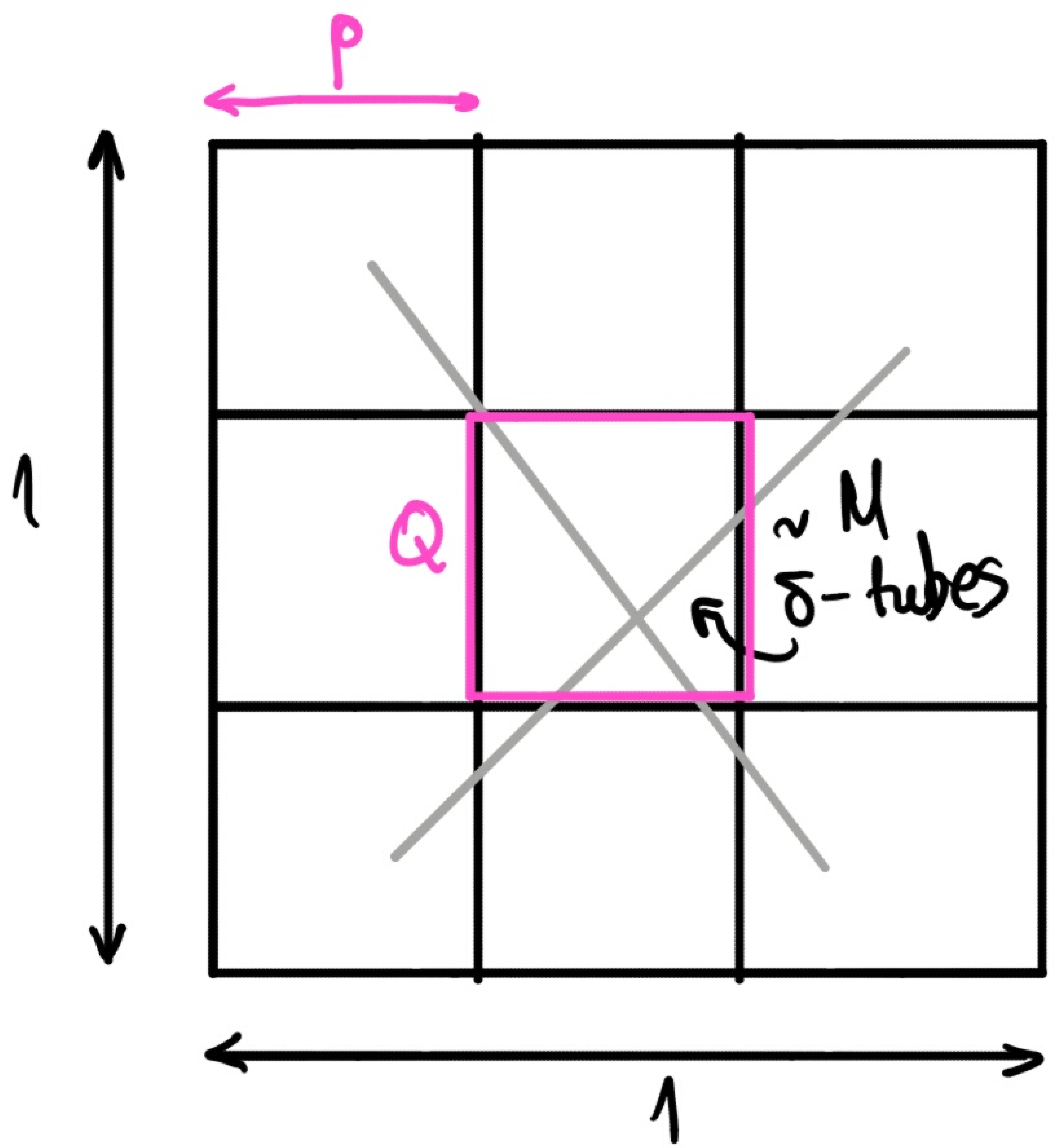
$$\sim M^2 \alpha^{-1} \lesssim M^2 \delta^{-10\varepsilon^3}.$$



So, $|P_r(\mathbb{T})| \lesssim |P| = \sum_{Q \in Q_M} |P_Q| \lesssim |Q_M| \cdot (M^2 \delta^{-10\varepsilon^3})$.



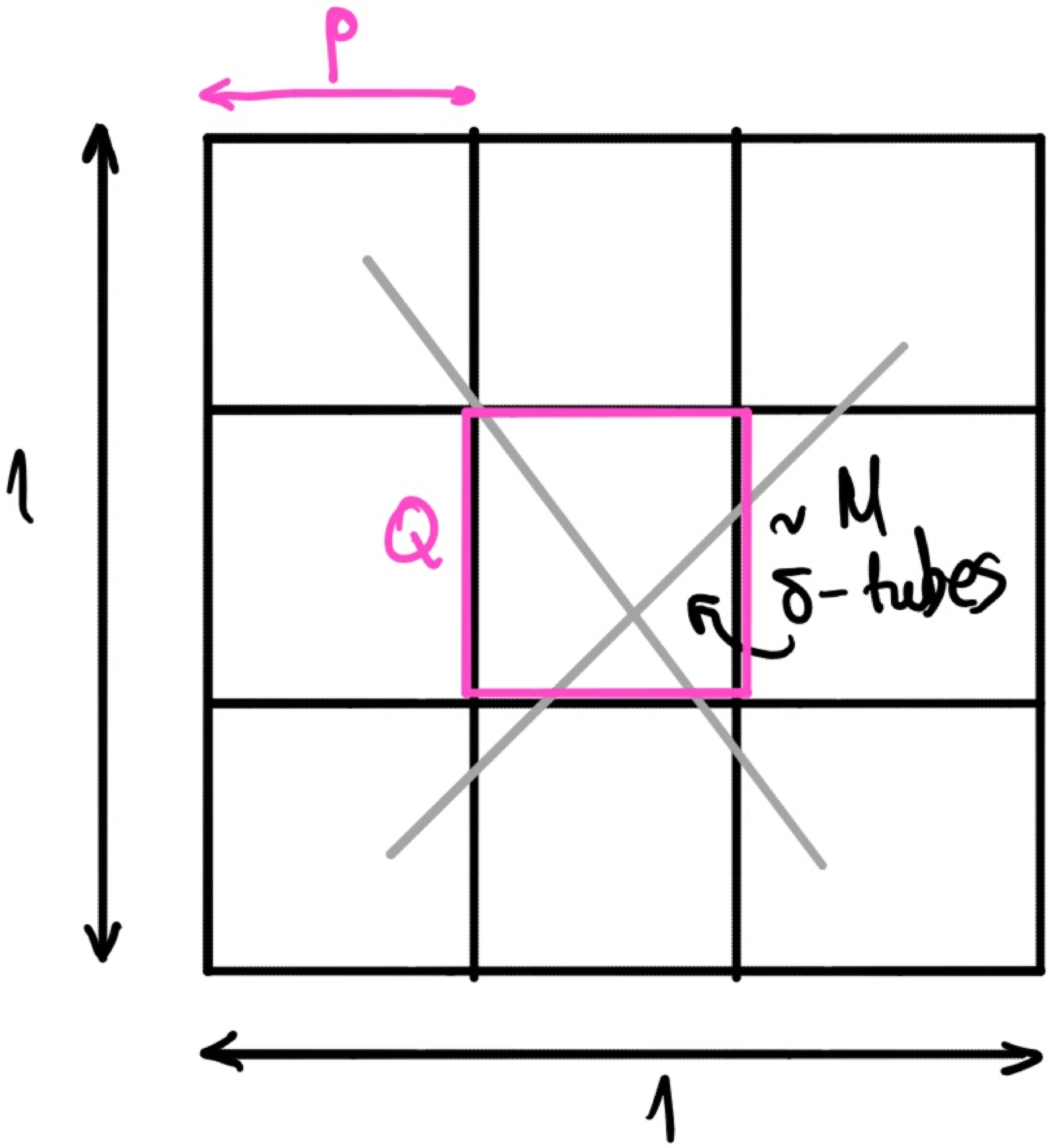
ρ
1
Q
~ M
δ-tubes
1



fatten each δ-tube around its core line, so that

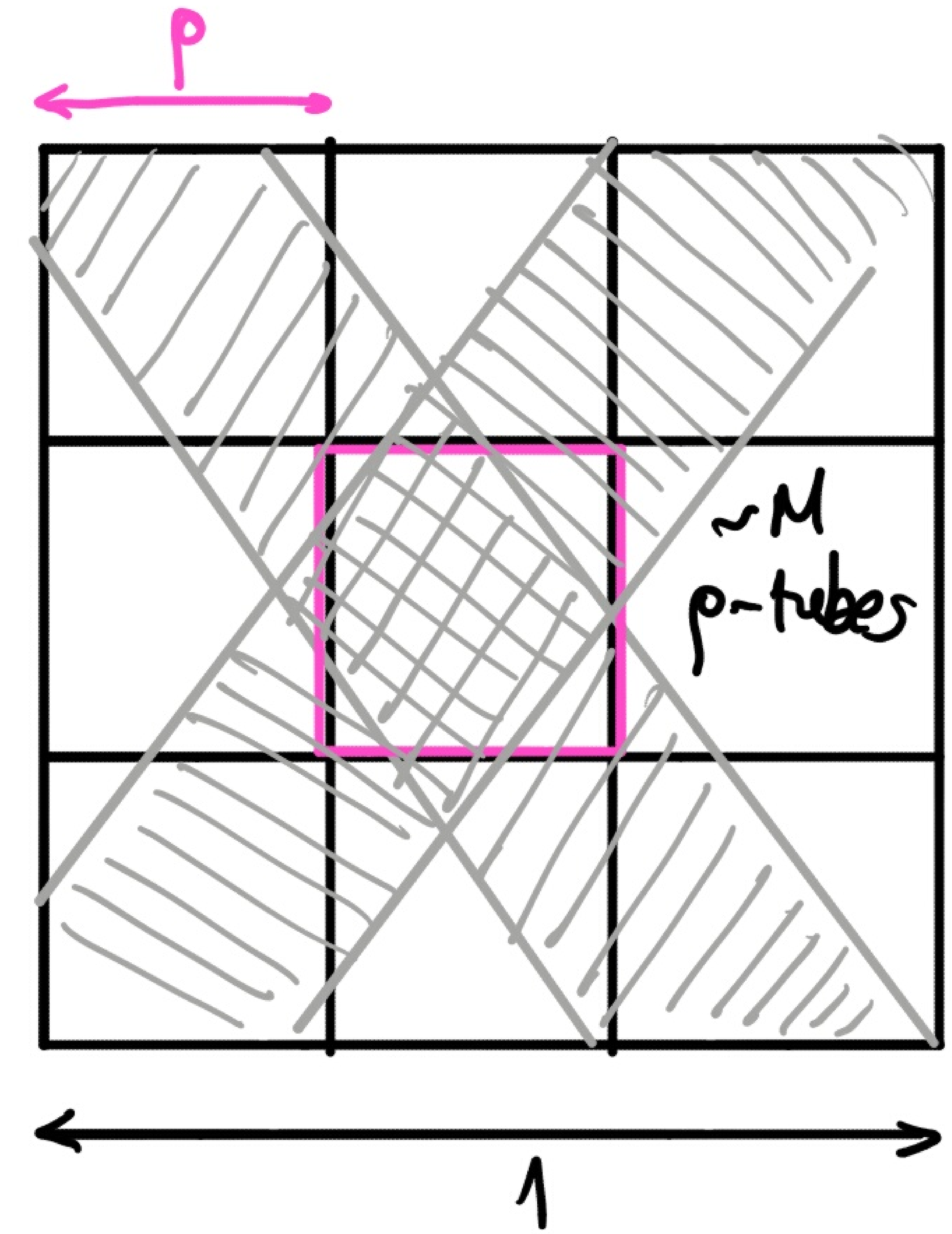
it becomes a ρ-tube


ρ
1
1
Q
~M
δ-tubes
fatten each δ-tube
around its core
line, so that
it becomes
a ρ-tube
ρ < W^-1
~M
ρ-tubes
1
1



fatten each δ-tube around its core line, so that it becomes a ρ-tube

$\rho < W^{-1}$, so the ρ-tubes are still distinct and well-spaced.



δ-tubes in $\mathbb{T}$ $\rightsquigarrow$ ρ-tubes in $\tilde{\mathbb{T}}$, each W^{-1}-tube contains ≤ 1 ρ-tube in $\tilde{\mathbb{T}}$, $|\tilde{\mathbb{T}}| = |\mathbb{T}| = W^{2(n-1)}$

And: every $Q \in \underline{Q_M}$ lives in $\underline{P_M(\tilde{\mathbb{T}})}$.

So: $|Q_M| \le |P_M(\tilde{T})| \lesssim \rho^{-\varepsilon} \dfrac{|\tilde{T}|^{n/n-1}}{M^{\frac{n+1}{n-1}}}$ $\quad M \ge r$.

!

$$\rightarrow |P_r(T)| \lesssim \rho^{-\varepsilon} \frac{|T|^{n/n-1}}{M^{\frac{n+1}{n-1}}} \cdot M^2 \delta^{-10\varepsilon^3} \qquad \le 1$$

$$\rho = W^{-2} \ge \delta \, \delta^{-\varepsilon/10}$$

$$\lesssim \delta^{-\varepsilon} \cdot \delta^{\varepsilon^2/10} \cdot |T|^{\frac{n}{n-1}} \delta^{-10\varepsilon^3} \cdot \frac{\left(\delta^{-\varepsilon^3}\right)^{\frac{n+1}{n-1}}}{r^{\frac{n+1}{n-1}}}$$

$$\lesssim \delta^{-\varepsilon} \frac{|T|^{n/n-1}}{r^{\frac{n+1}{n-1}}} \cdot \left(\delta^{\varepsilon^2/10 - 20\varepsilon^3}\right)$$

$\ll 1$ ✓

$M \geq 2$ ✓ $(M \geq r)$

$\rho = W^{-2}$

$$M > \underbrace{\rho^{n-1-\varepsilon/4} |\Pi|}_{\parallel} \overset{?}{>} \delta^{n-1-\varepsilon/4} |\Pi|$$

$$\cancel{W^{-2(n-1)}} W^{\varepsilon/2} \cdot \cancel{W^{2(n-1)}} = W^{\varepsilon/2}$$

What if $M \leq W^{\varepsilon/2}$??

For $M \lesssim W^{\varepsilon/2}$:

$\rho = W^{-2}$

$$|P_r(\mathbb{T})| \lesssim \left(M^2 \delta^{-10\varepsilon^3}\right) \cdot \underline{|P_M(\tilde{\mathbb{T}})|}$$

$\rightarrow$ ρ-cubes

$$|P_M(\tilde{\mathbb{T}})| \lesssim |\tilde{\mathbb{T}}| \cdot \#\{\rho\text{-cubes in each } \rho\text{-tube}\}$$

$$\sim |\mathbb{T}| \cdot \frac{1}{\rho} \sim |\mathbb{T}| W^2 = \begin{cases} |\mathbb{T}|^2, & n=2 \\ |\mathbb{T}|^{3/2}, & n=3 \end{cases} = |\mathbb{T}|^{\frac{n}{n-1}}.$$

$$\Rightarrow |P_r(\mathbb{T})| \lesssim M^2 \delta^{-10\varepsilon^3} \cdot \frac{|\mathbb{T}|^{\frac{n}{n-1}}}{r^{\frac{n+1}{n-1}}} \cdot \left(\delta^{-\varepsilon^3}\right)^{\frac{n+1}{n-1}} \quad \ll \delta^{-\varepsilon}$$

$$\lesssim \left(W^{\varepsilon} \delta^{-10\varepsilon^3 - 10\varepsilon^3}\right) \frac{|\mathbb{T}|^{n/n-1}}{r^{\frac{n+1}{n-1}}} \lesssim \delta^{-\varepsilon/2 + \frac{\varepsilon^2}{20} - 20\varepsilon^3} \cdot \frac{|\mathbb{T}|^{n/n-1}}{r^{\frac{n+1}{n-1}}}.$$

$\rightarrow W \lesssim \delta^{-1/2 + \varepsilon/20}$