"S-T": Let $1 \leq W \leq \delta^{-1}$. Let $\mathbb{T}$ be a set of $\sim W^2$ δ-tubes in $[0,1]$, s.t. each W^{-1}-tube contains ≤ 1 of them.

In $\mathbb{R}^2$

Then, $\forall r > \max(\delta^{1-\varepsilon}|\mathbb{T}|, 1)$,

$$|P_r(\mathbb{T})| \lesssim_\varepsilon \delta^{-\varepsilon} \frac{|\mathbb{T}|^2}{r^3}.$$

Guth-Katz: Let $\mathcal{L}$ a set of L lines in $\mathbb{R}^3$, s.t. every plane contains $\leq L^{1/2}$ of the lines,

then $\forall \quad 2 \leq r \lesssim L^{1/2}$:

$$|P_r(\mathcal{L})| \lesssim \frac{L^{3/2}}{r^2}.$$

"G-K": Let $1 \leq W \leq \delta^{-1}$. Let $\mathbb{T}$ be a set of $\sim W^4$ δ-tubes in $\mathbb{R}^3$, s.t. each W^{-1}-tube contains ≤ 1 of them. Then,

$$|P_r(\mathbb{T})| \lesssim_\varepsilon \delta^{-\varepsilon} \cdot \frac{|\mathbb{T}|^{3/2}}{r^2}, \quad \forall\, r > \max\{\delta^{2-\varepsilon}|\mathbb{T}|, 1\}.$$

Combining notation:

Let $n=2,3$.

Thm: Let $1 \leq W \leq \delta^{-1}$. Let $\mathbb{T}$ be a set of $\sim W^{2(n-1)}$ δ-tubes in $B^n(0,1)$, s.t. each W^{-1}-tube contains ≤ 1 of them. Then, $\forall r > \max(\delta^{n-1-\varepsilon/4}|\mathbb{T}|, 1)$,

$$|P_r(\mathbb{T})| \lesssim \delta^{-\varepsilon} \cdot \boxed{\frac{|\mathbb{T}|^{\frac{n}{n-1}}}{r^{\frac{n+1}{n-1}}}}.$$

slightly larger range than $\delta^{n-1-\varepsilon}|\mathbb{T}|$

For proof, we will be fixing δ, and show Thm $\forall$ $1 \leq W \leq \delta^{-1}$.

Thm: Let $1 \leq W \leq \delta^{-1}$, $n = 2,3$.

$\mathbb{T}$: set of $\sim W^{2(n-1)}$ δ-tubes in $B^n(0,1)$, s.t. each W^{-1}-tube contains ≤ 1 δ-tube in $\mathbb{T}$.

Then: $|P_r(\mathbb{T})| \lesssim \delta^{-\varepsilon} \dfrac{|\mathbb{T}|^{\frac{n}{n-1}}}{r^{\frac{n+1}{n-1}}}$,

$\forall\, r > \max\left(\delta^{n-1-\varepsilon/4}|\mathbb{T}|, 1\right)$

① Thm holds for $\delta \sim 1$:

Then, $W \sim 1$, so if 2 δ-tubes intersect, their angle is $\gtrsim W^{-1} \sim 1 \to$ they intersect "only once"

$\Rightarrow |P_r(\mathbb{T})| \lesssim |\mathbb{T}|^2 \sim 1 \lesssim_\varepsilon \delta^{-\varepsilon} \dfrac{|\mathbb{T}|^{\frac{n}{n-1}}}{r^{\frac{n+1}{n-1}}}$,

$\forall\, r \ldots$

So, fixing any $\delta > 0$, we may assume that Thm holds $\forall\, \tilde{\delta} > 2\delta$.

inductive hyp.

② Thm holds for $r \gtrsim |\mathbb{T}|^{1/2} \sim W^{n-1}$ (then $|P_r(\mathbb{T})| = 0$).

We may also assume that, fixing r, Thm holds $\forall\, \tilde{r} > r$.

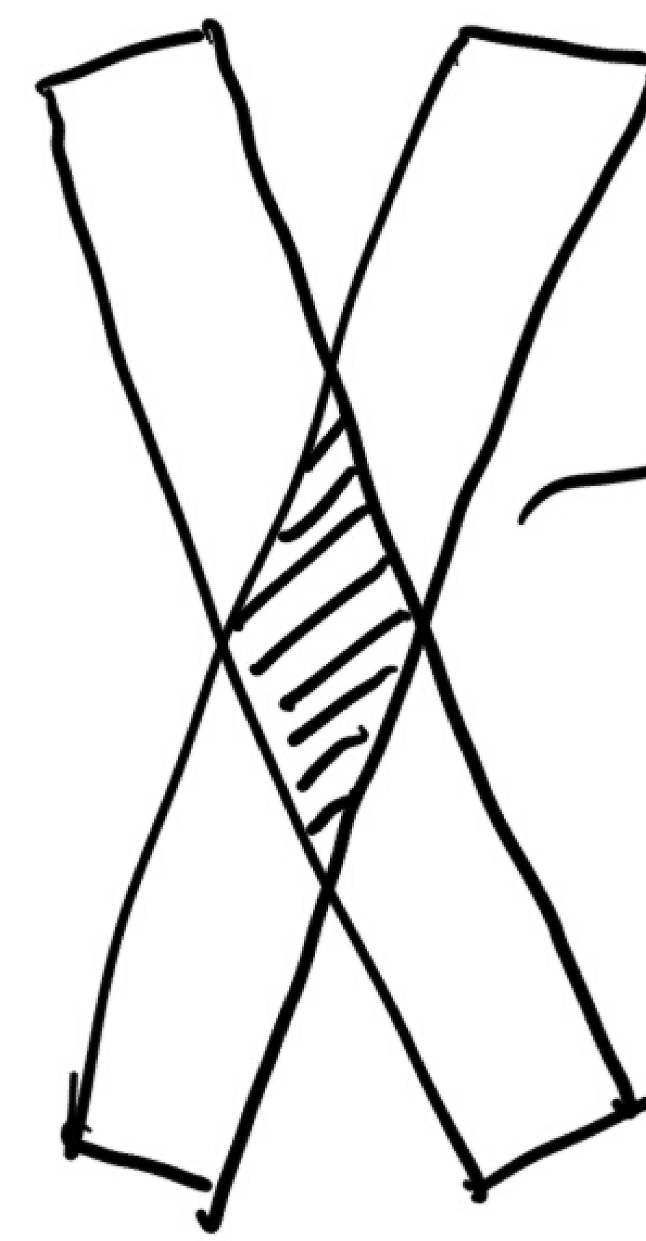
Thm: Let $1 \le W \le \delta^{-1}$, $n = 2,3$.

$\mathbb{T}$: set of $\sim W^{2(n-1)}$ δ-tubes in $B^n(0,1)$,
s.t. each W^{-1}-tube contains ≤ 1 δ-tube in $\mathbb{T}$.

Then: $|P_r(\mathbb{T})| \lesssim \delta^{-\varepsilon} \dfrac{|\mathbb{T}|^{\frac{n}{n-1}}}{r^{\frac{n+1}{n-1}}}$,

$\forall\, r > \max\left(\delta^{n-1-\varepsilon/4}|\mathbb{T}|, 1\right)$

③ Thm holds $\forall\, W \lesssim \delta^{-\varepsilon/1000n}$:



angle $\gtrsim W^{-1}$,
int. contains $\lesssim W$ δ-cubes

$$\Rightarrow |P_r(\mathbb{T})| \lesssim |\mathbb{T}|^2 \cdot W \xrightarrow[r \lesssim W^{n-1}]{} |P_r(\mathbb{T})|\, r^{\frac{n+1}{n-1}} \lesssim |\mathbb{T}|^2 W \cdot W^{n+1}$$

$\Rightarrow$ ---- ✓

Thm: Let $1 \leq W \leq \delta^{-1}$, $n = 2,3$.

$\mathbb{T}$: set of $\sim W^{2(n-1)}$ δ-tubes in $B^n(0,1)$,

s.t. each W^{-1}-tube contains ≤ 1 δ-tube in $\mathbb{T}$.

Then: $|P_r(\mathbb{T})| \lesssim \delta^{-\varepsilon} \dfrac{|\mathbb{T}|^{\frac{n}{n-1}}}{r^{\frac{n+1}{n-1}}}$,

$\forall\, r > \max\left(\delta^{n-1-\varepsilon/4}|\mathbb{T}|, 1\right)$

(4) Thm holds when

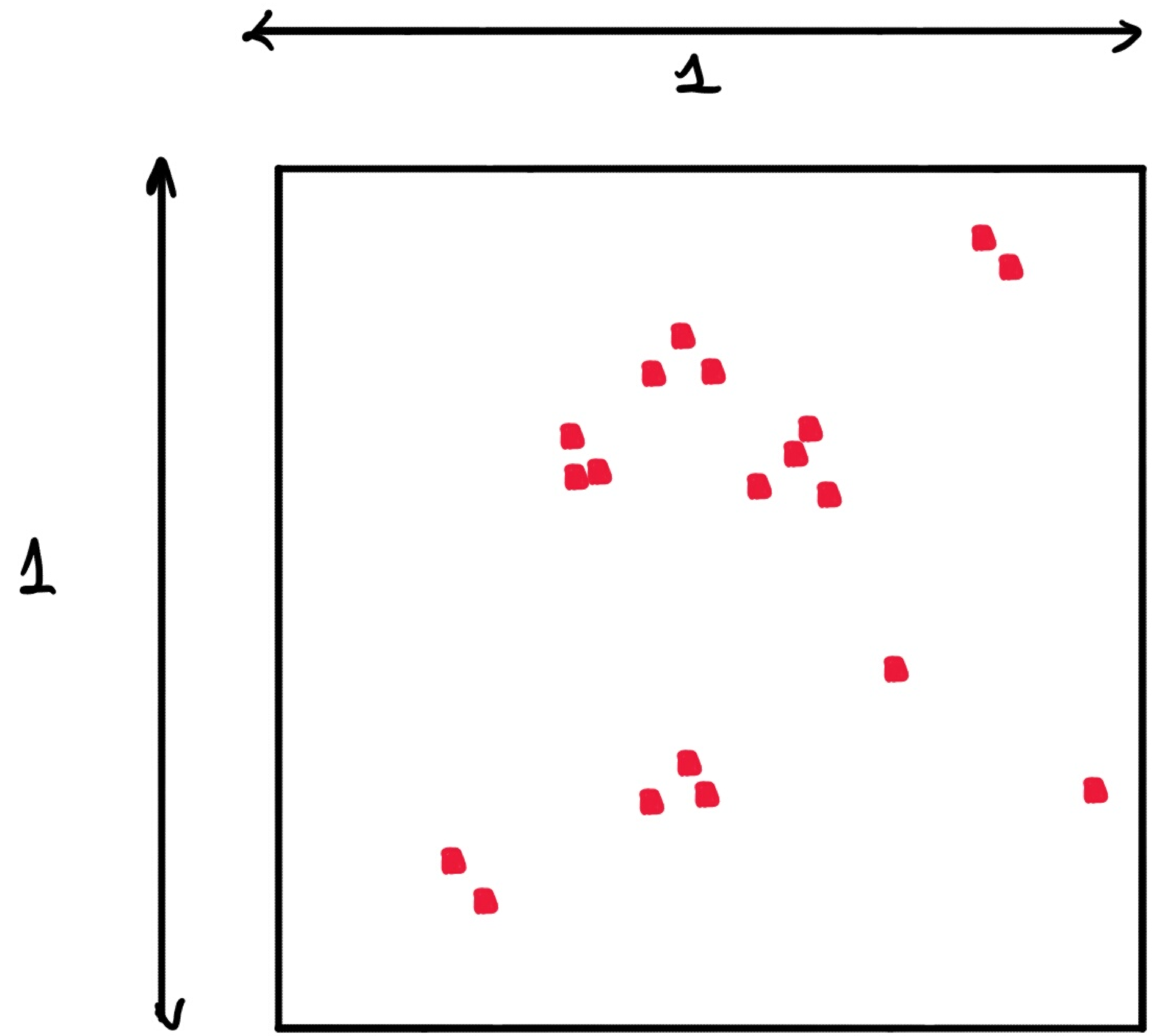
$\delta^{n-1-\varepsilon/4}|\mathbb{T}| \gtrsim |\mathbb{T}|^{1/2} \sim W^{n-1}$:

$|P_r(\mathbb{T})| = 0$.

$\Longleftrightarrow W \gtrsim \delta^{-1+\frac{\varepsilon}{4(n-1)}}$ ✓

So, we may assume that $\boxed{W \lesssim \delta^{-1+\varepsilon/8}}$.

So, let $\delta^{-\varepsilon/1000n} \lesssim W \leq \delta^{-1+\varepsilon/8}$.

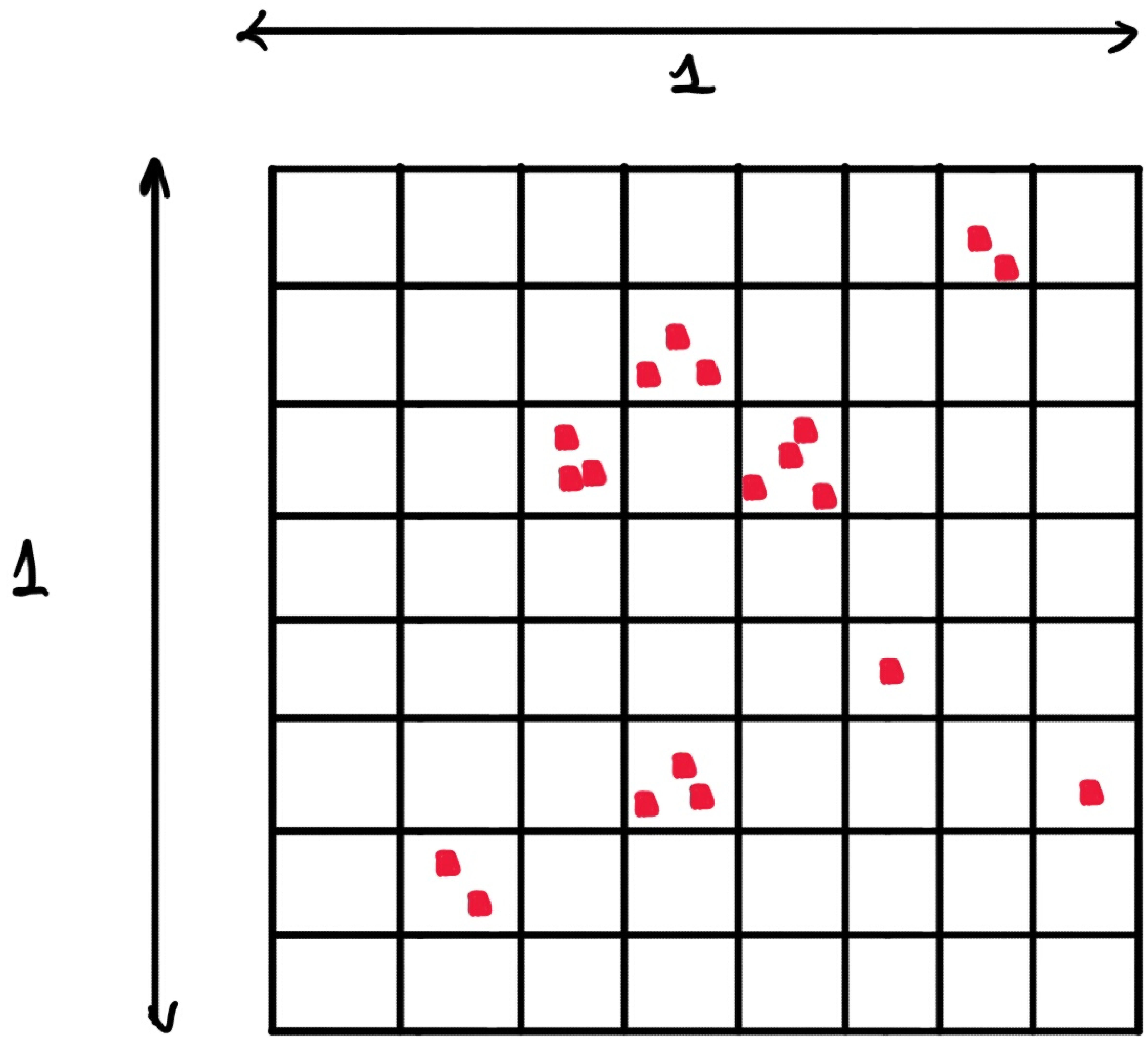


■ $\rightsquigarrow$ δ-cube in $P_r(\mathbb{T})$.

So, let $\delta^{-\varepsilon/1000n} \lesssim W \leq \delta^{-1/8}$.

Let $1 < D \leq W$
$\sim \delta^{-\varepsilon^4}$ in the end...

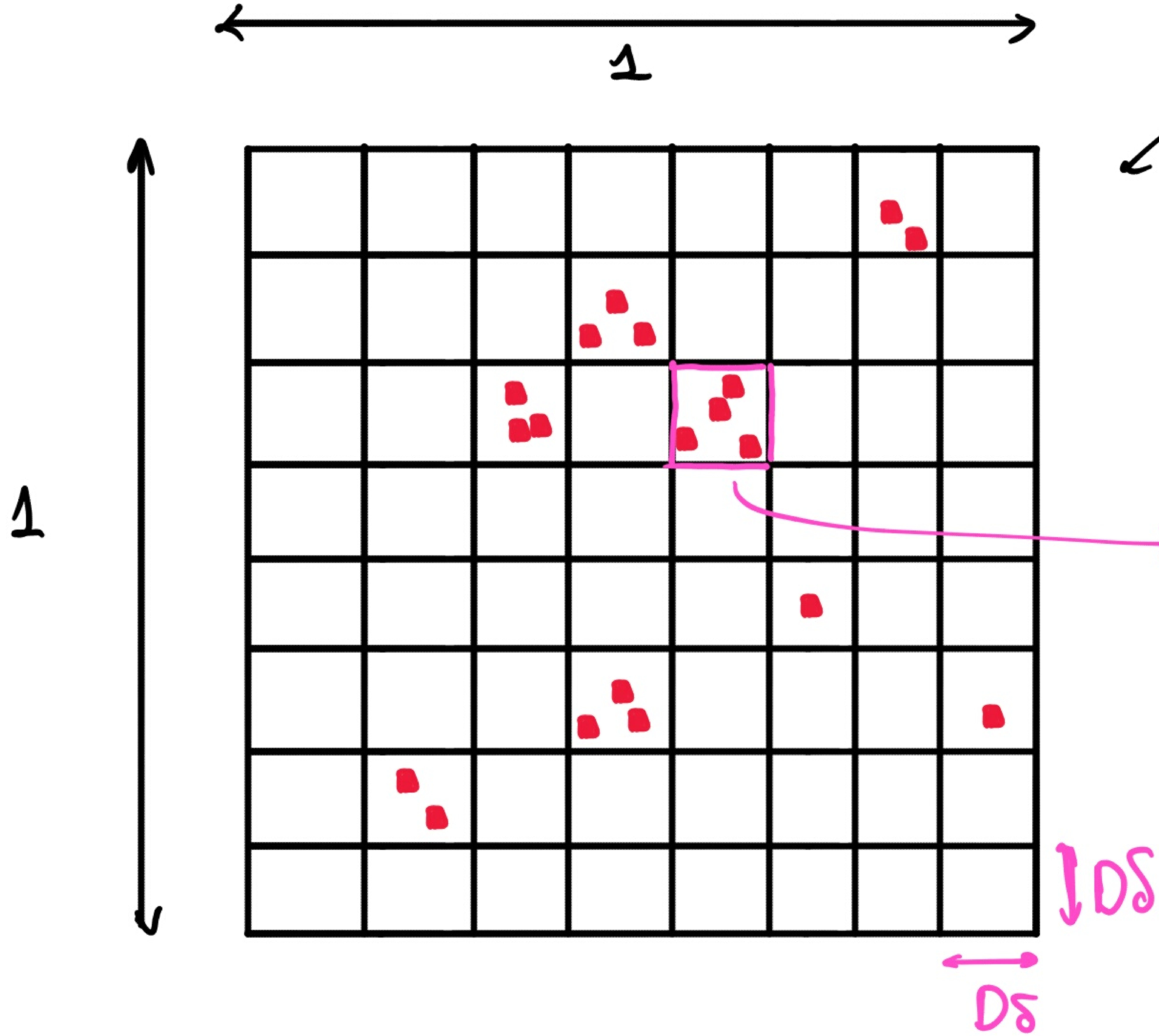
split in $D\delta$-cubes



▪ $\rightsquigarrow$ δ-cube in $P_r(T)$.

So, let $\delta^{-\varepsilon/1000n} \lesssim W \leq \delta^{-1/8}$. Let $1 < D \leq W$

$\sim \delta^{-\varepsilon^4}$ in the end...

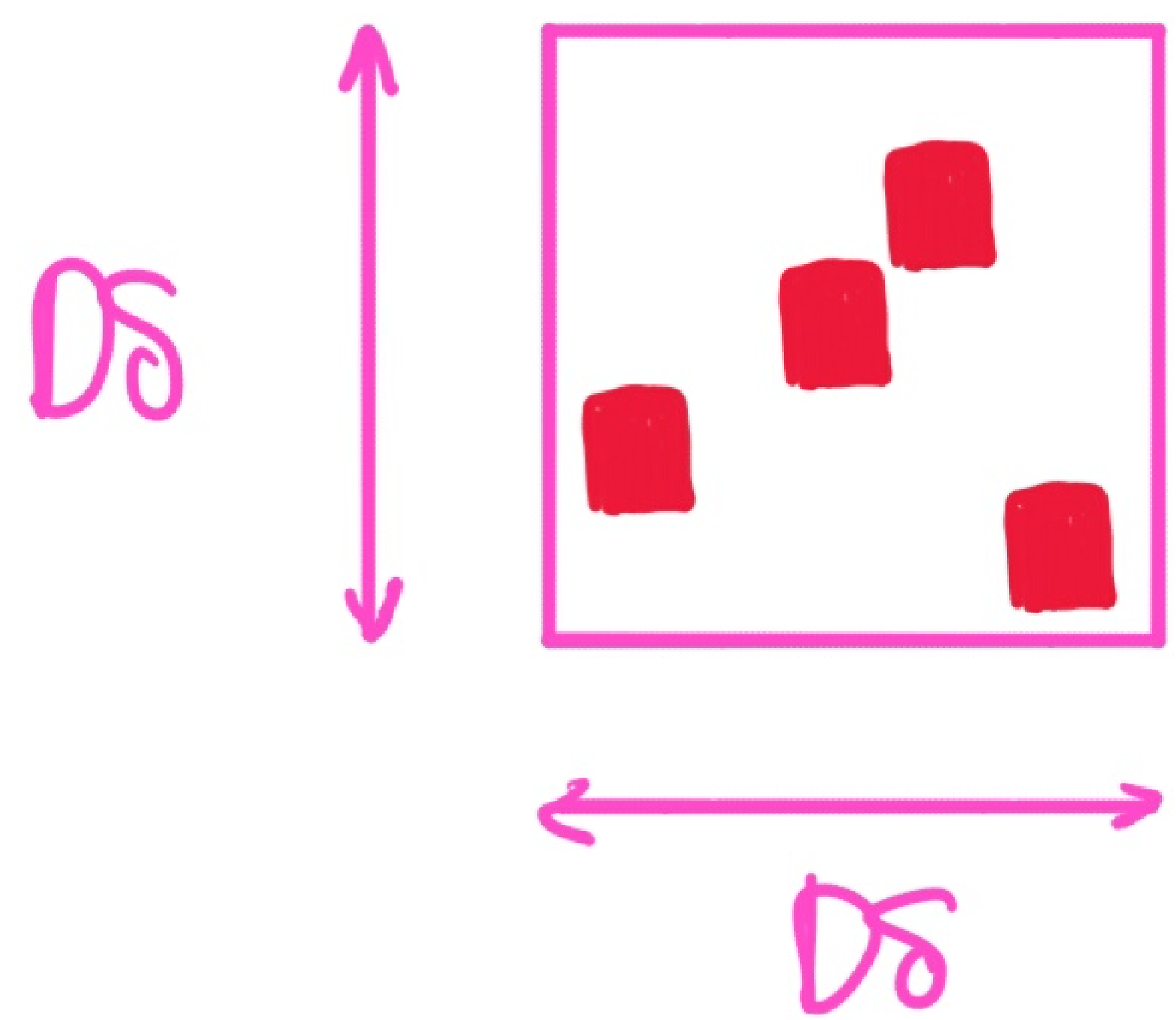


split in $D\delta$-cubes

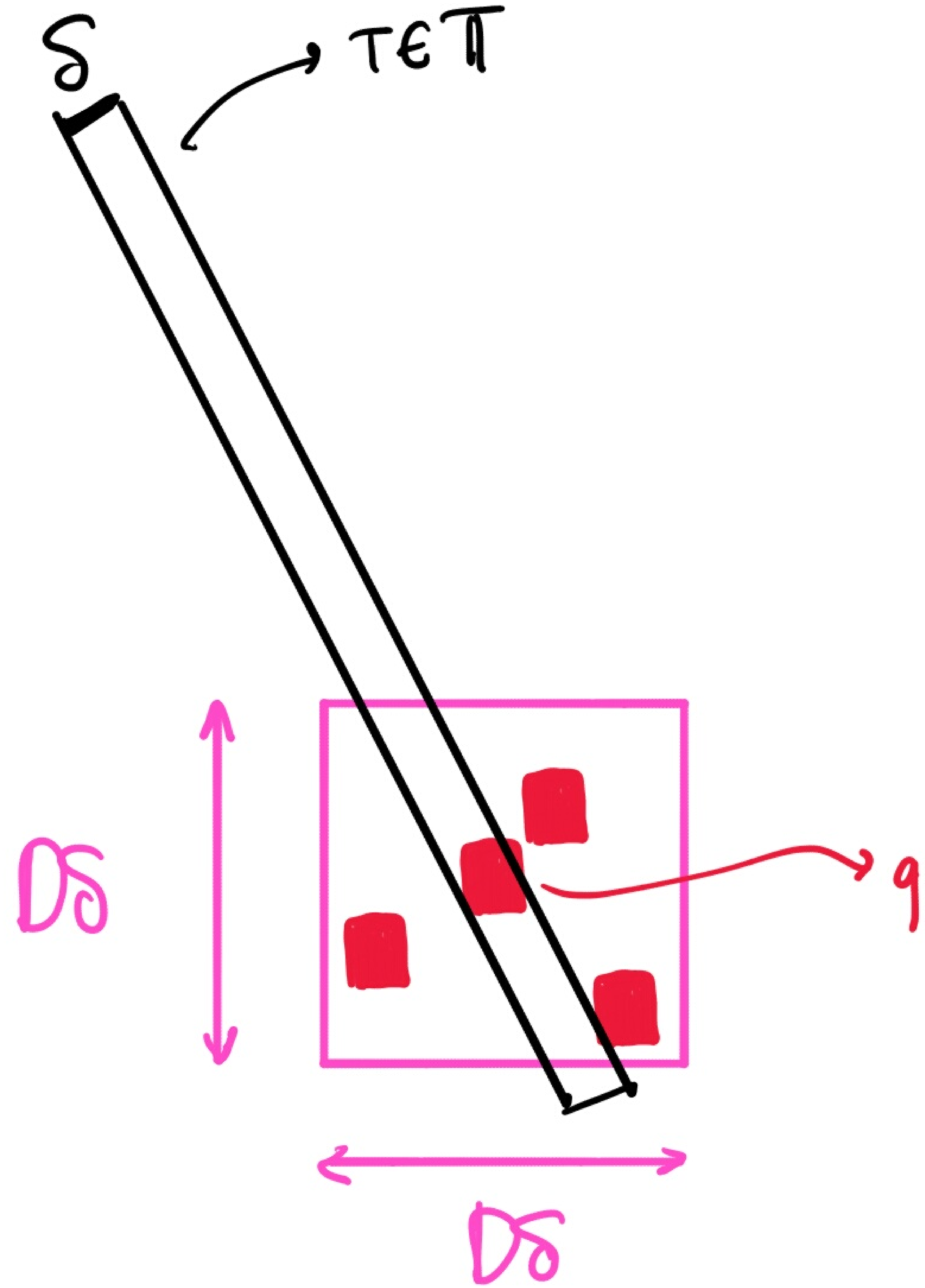
$\blacksquare \rightsquigarrow \delta$-cube in $P_r(T)$.

Q, a $D\delta$-cube.

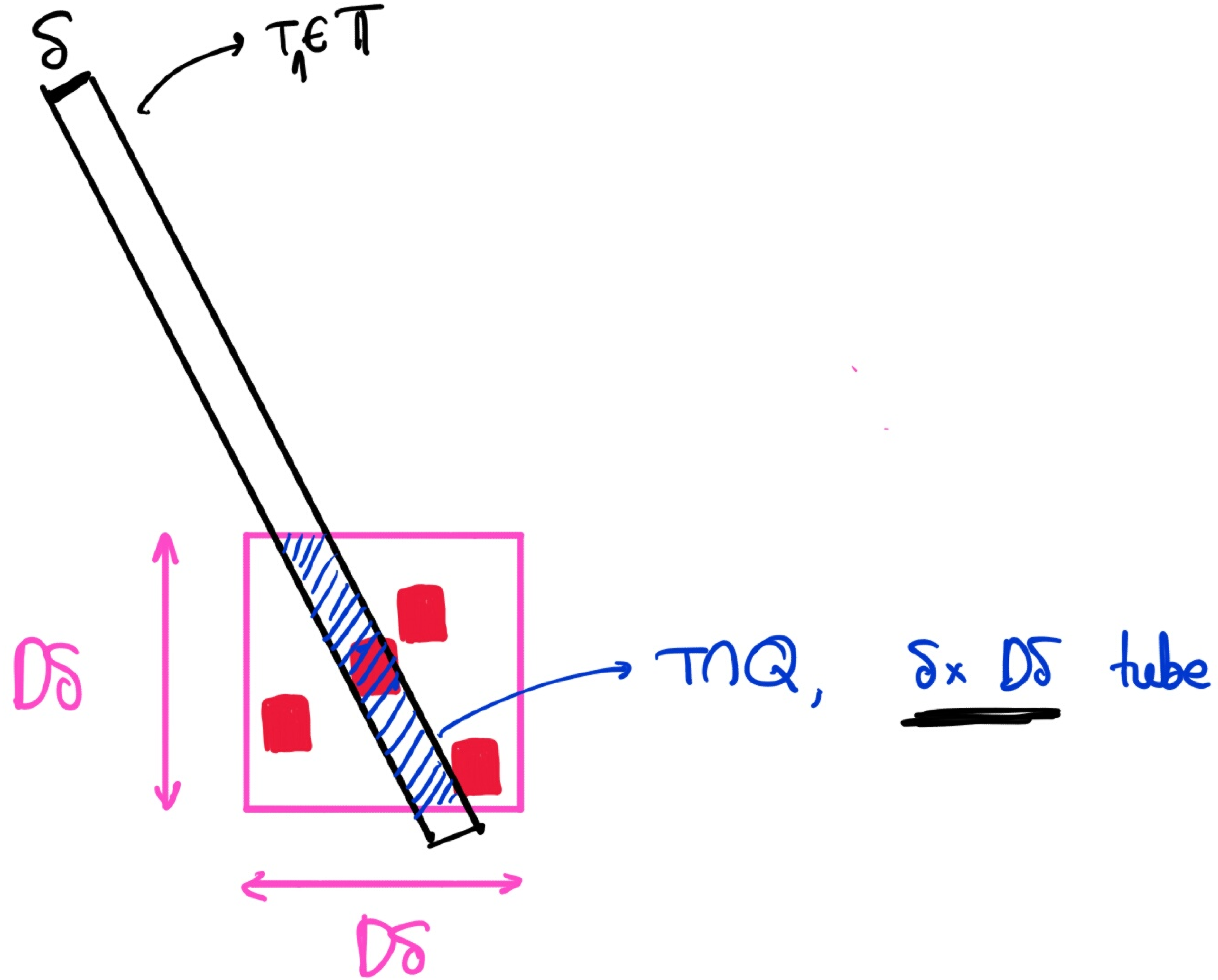
Inside Q :



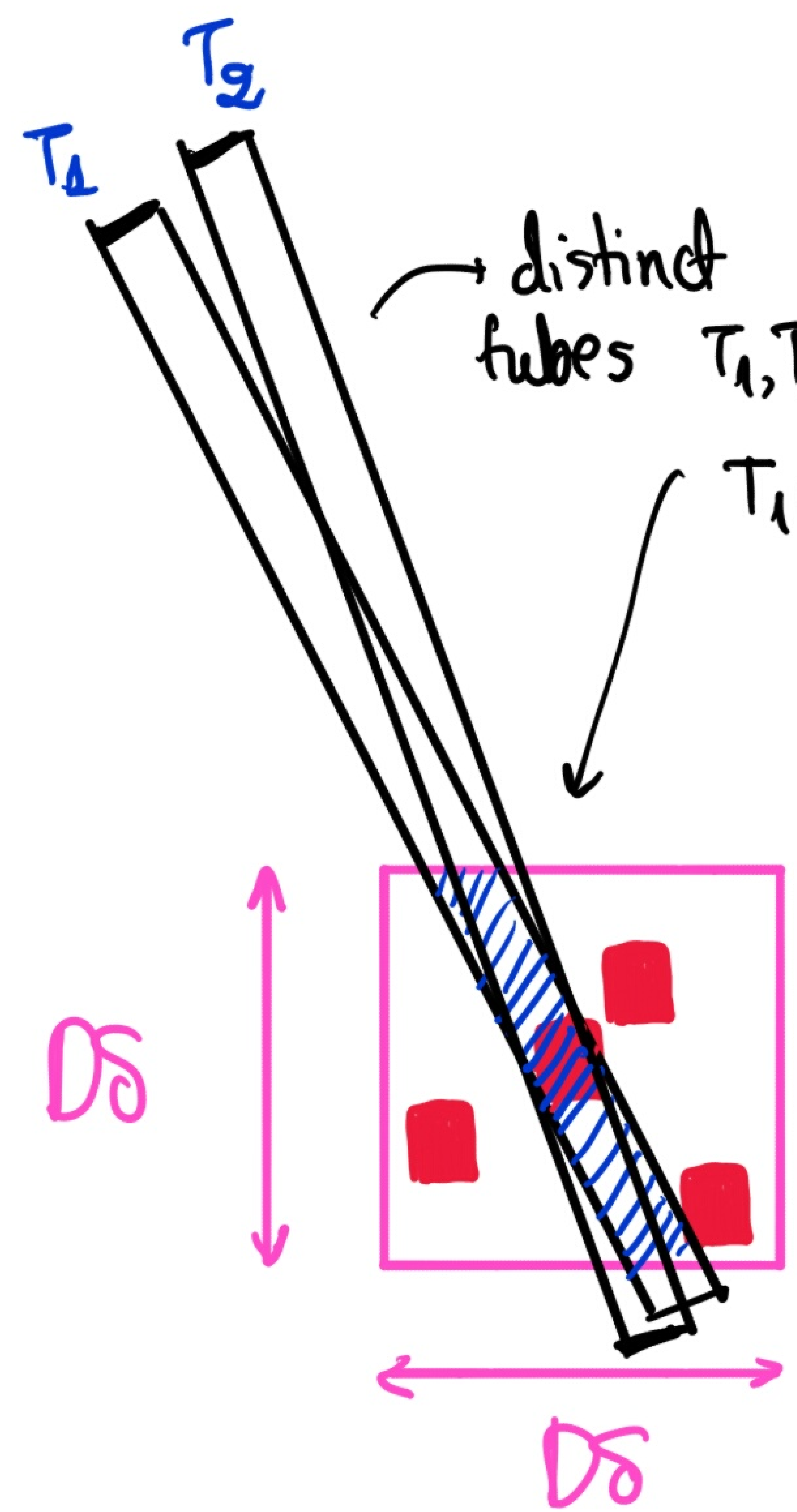
Inside Q :



Inside Q :



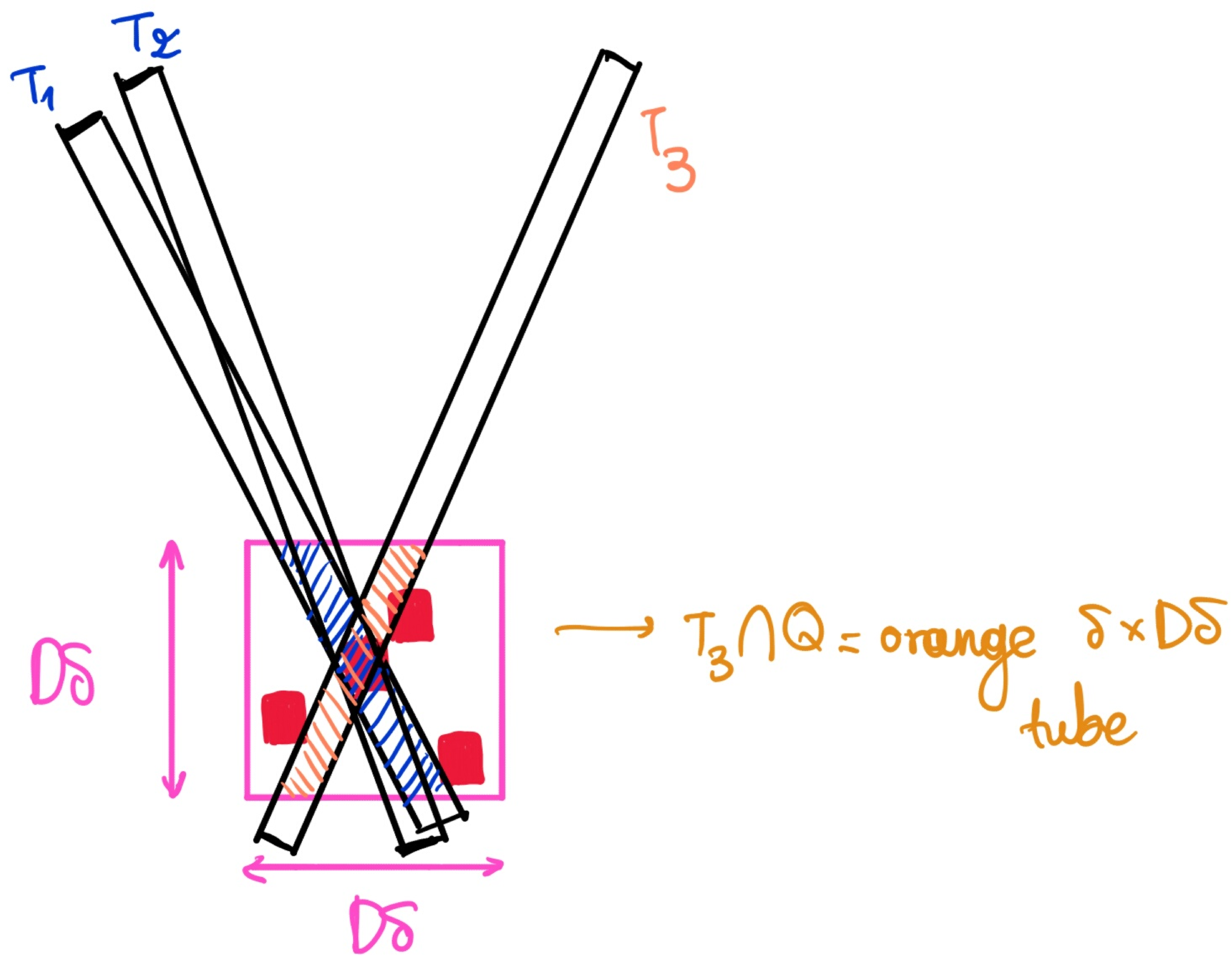
Inside Q :



distinct tubes $T_1, T_2 \in \mathbb{T}$

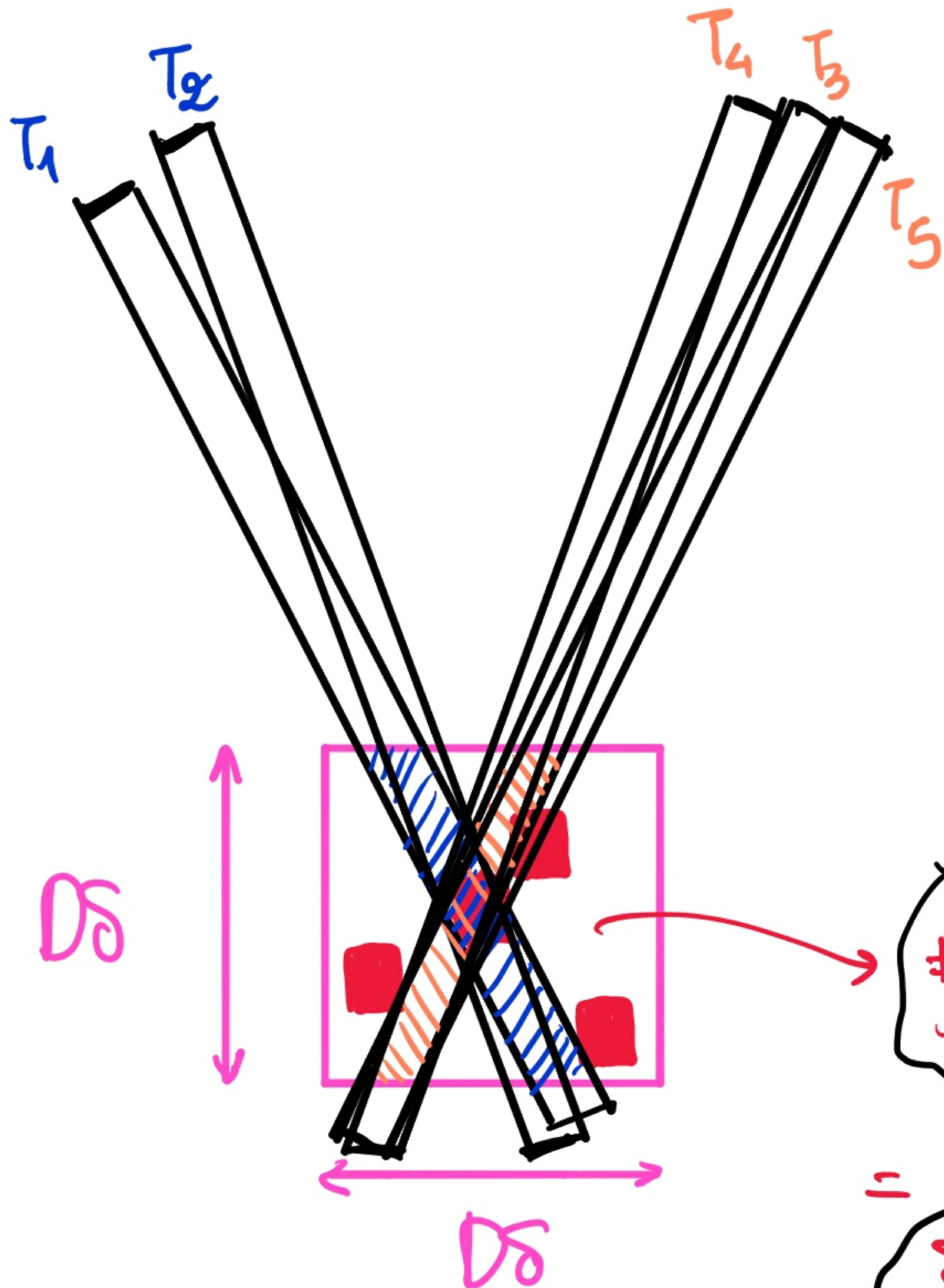
$T_1 \cap Q = T_2 \cap Q =$ the blue $\delta \times D\delta$ tube

Inside Q :



T_1
T_2
T_3
$D\delta$
$D\delta$
$T_3 \cap Q$ = orange $\delta \times D\delta$ tube

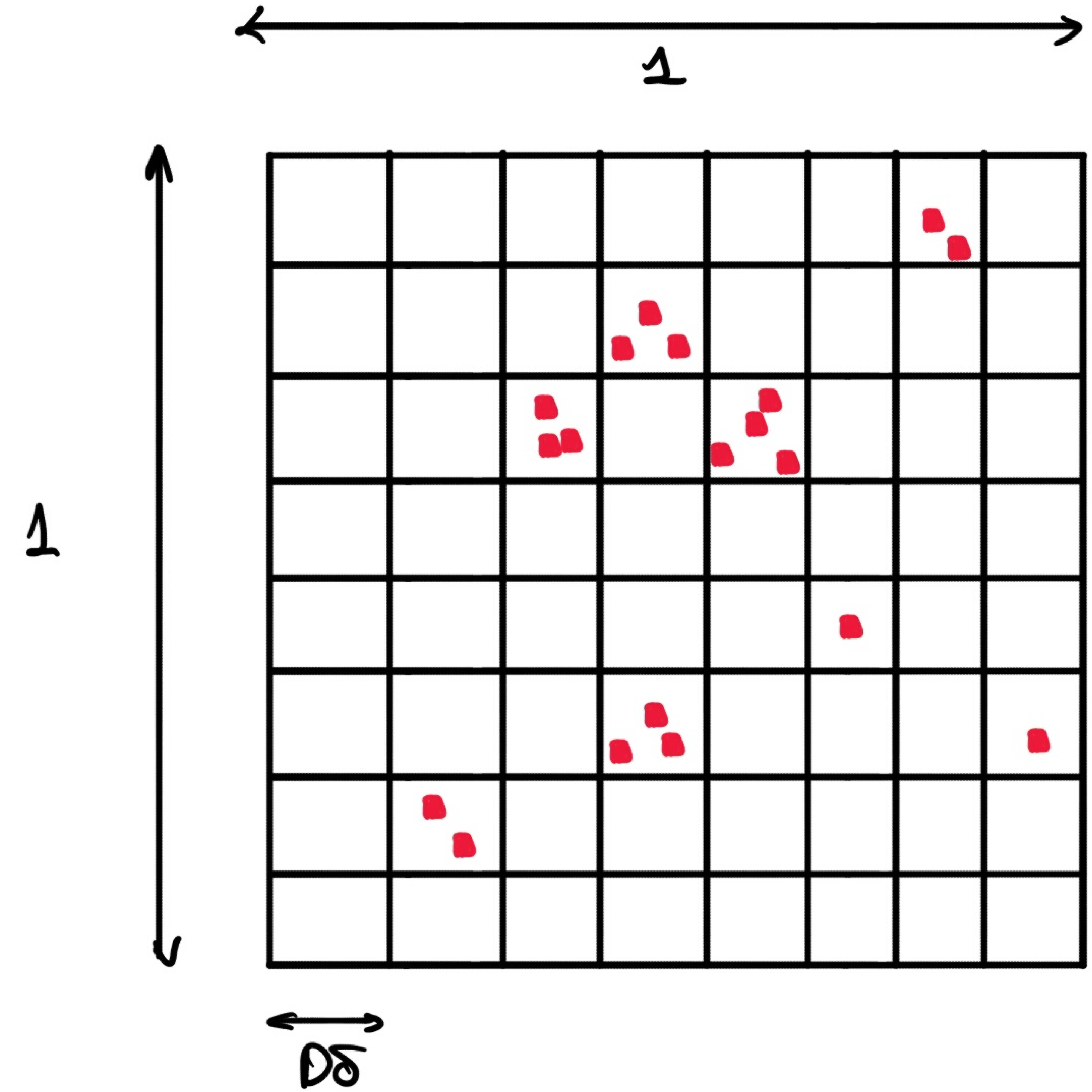
Inside Q :



all T_3, T_4, T_5 have <u>same</u> intersection with Q: the orange $\delta \times D\delta$ - tube.

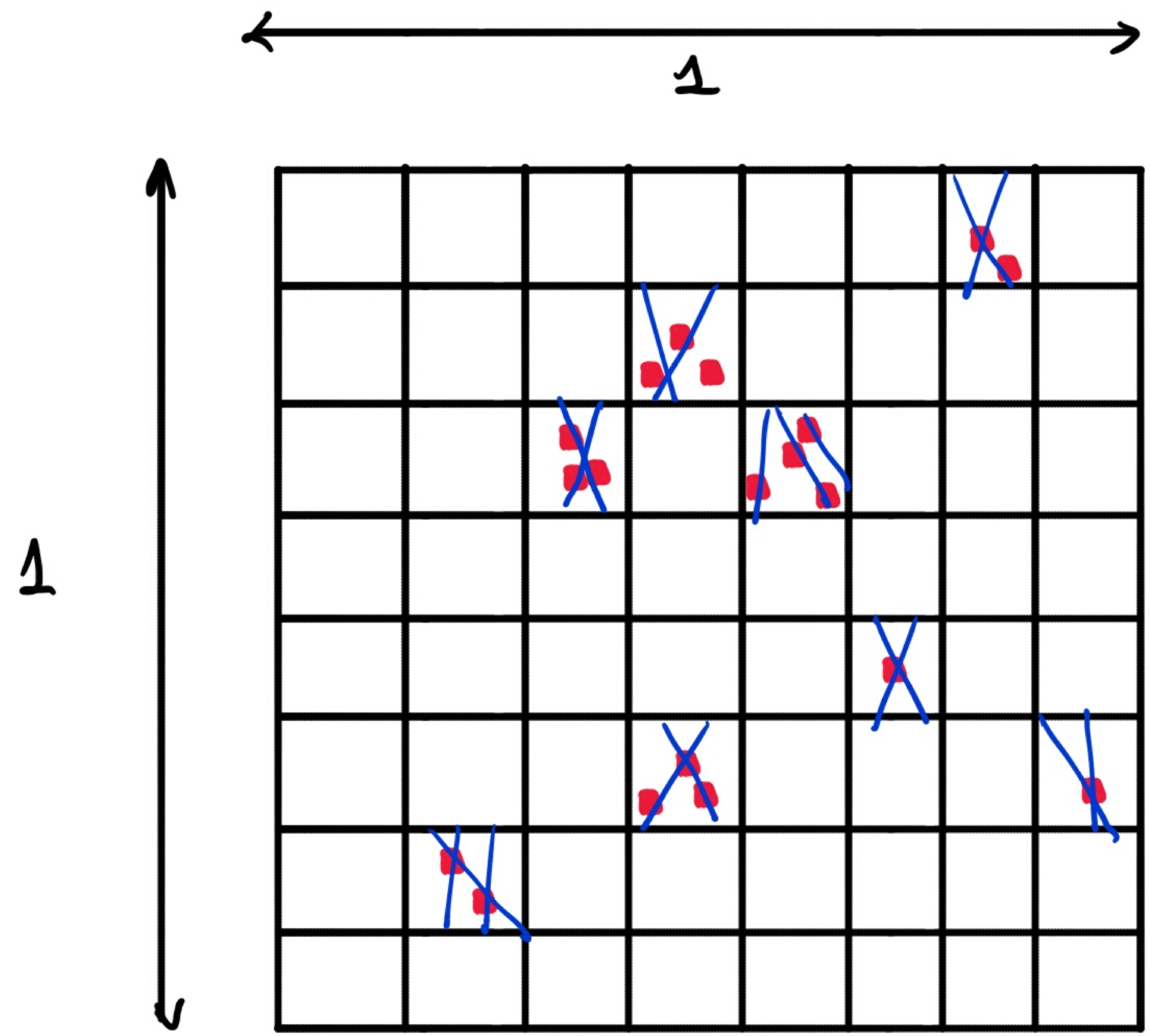
$$\underbrace{\#\text{ of tubes through } q}_{=r} = \sum_{\substack{\delta\times D\delta \\ \text{tubes through} \\ q}} \#\{\delta\text{-tubes that fully contain the } \delta\times D\delta \text{ tube}\}.$$

Rough plan:



Rough plan:

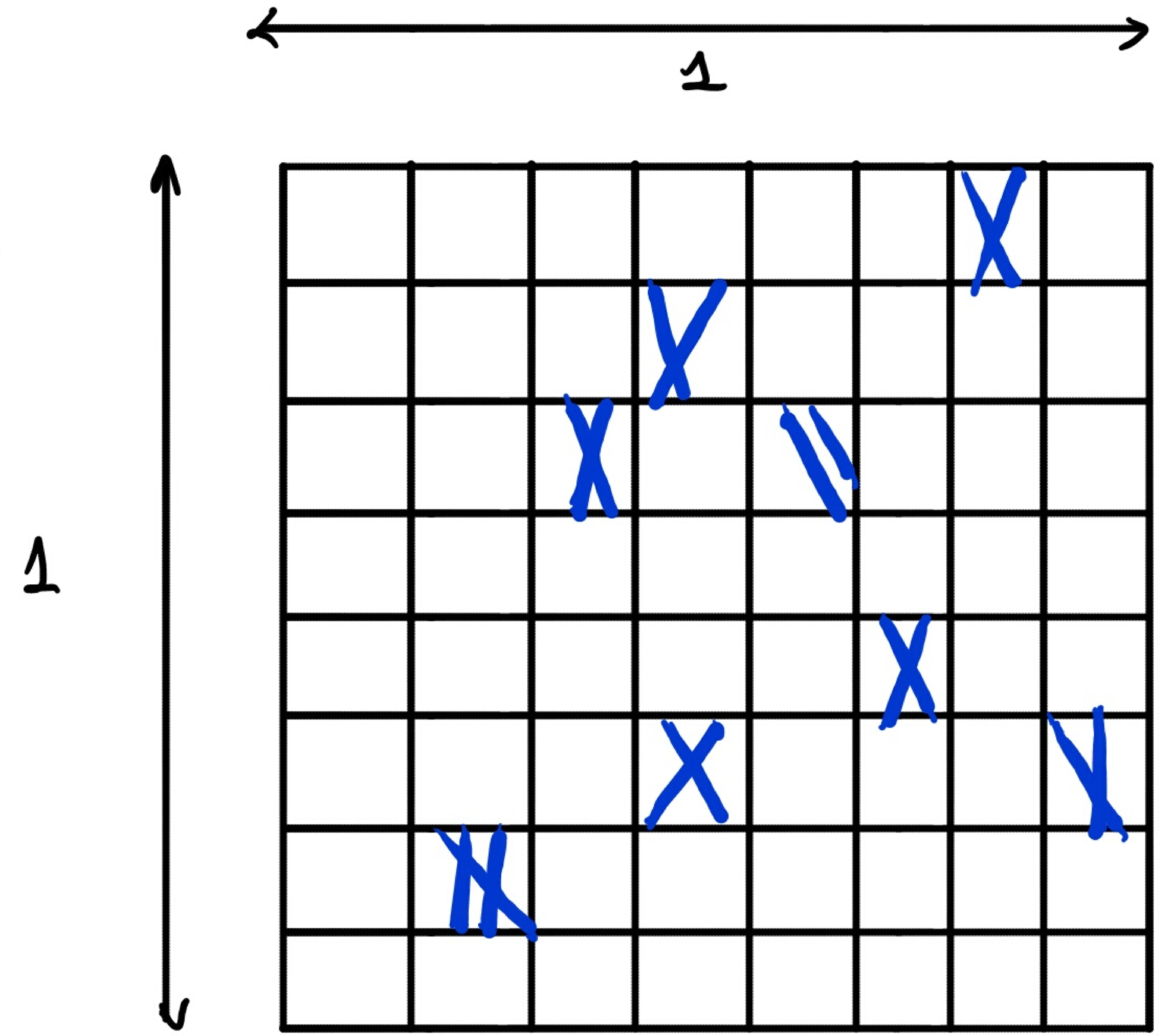
In each $D\delta \times D\delta$ cube Q, control # of δ-cubes via # short tubes through them.



Rough plan:

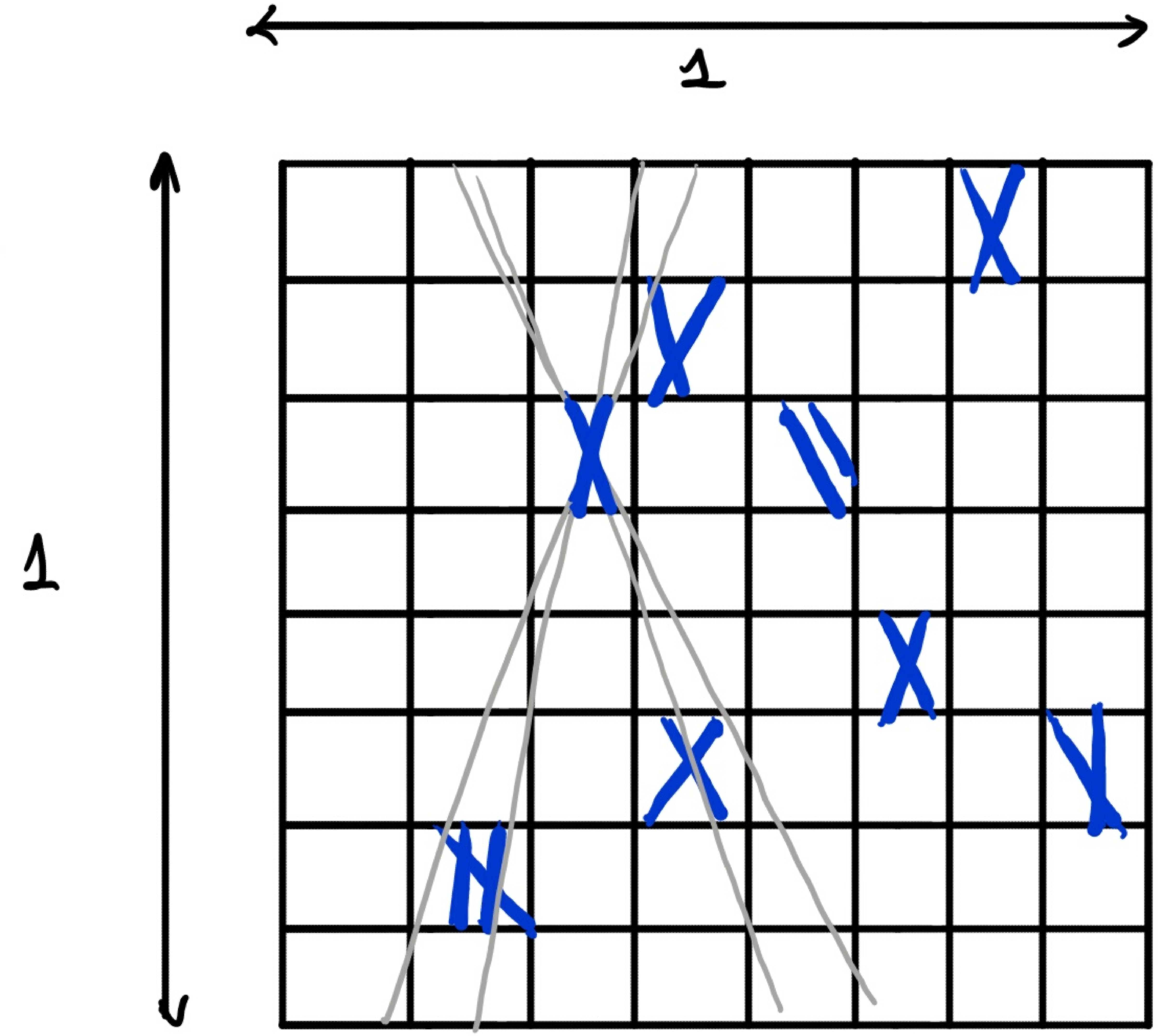
In each Q, control # of δ-cubes by # of short tubes through them.

We control # of short tubes via # of long tubes containing them.

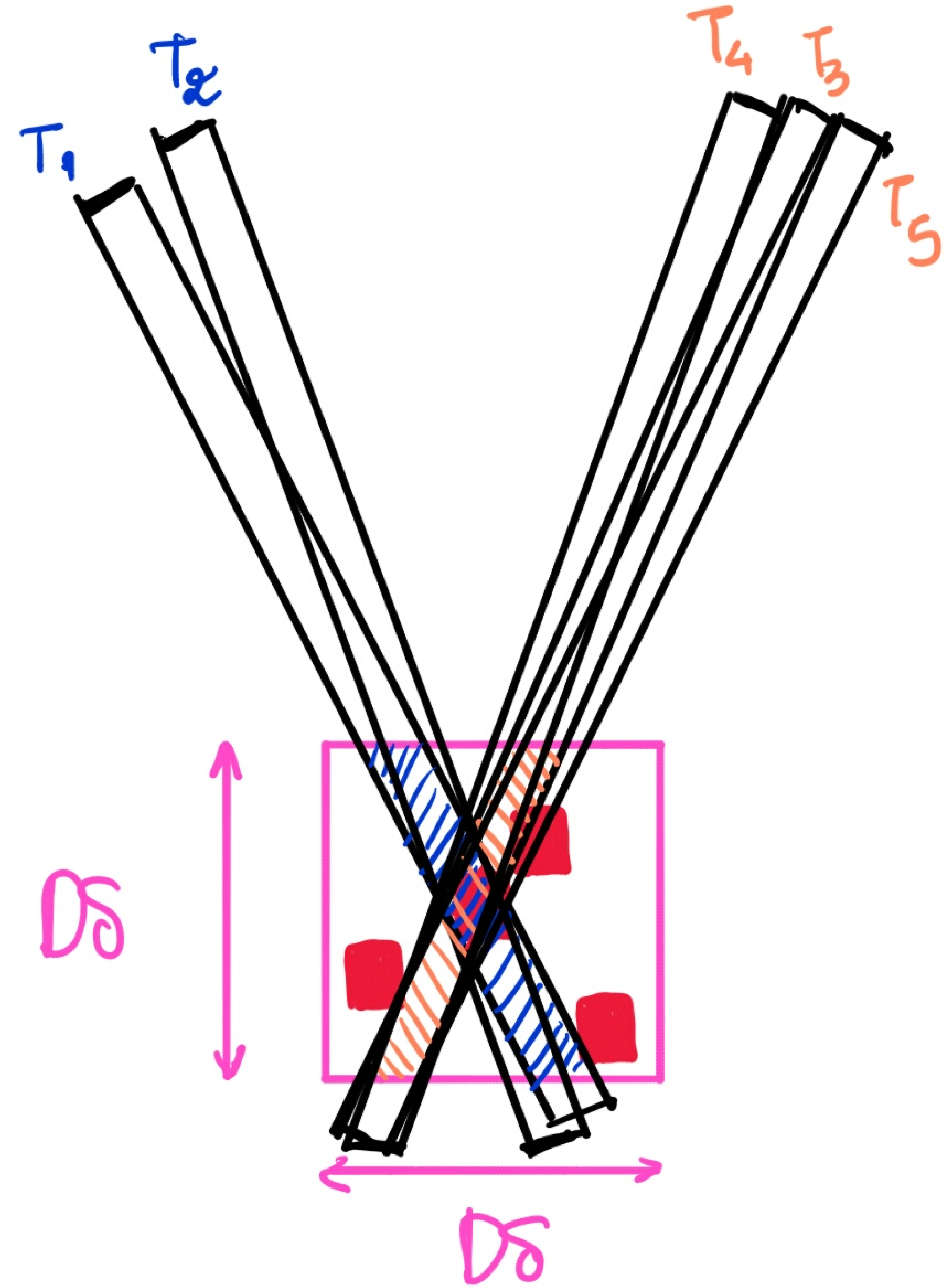


Rough plan:

In each Q, control # of δ- cubes by # of short tubes through them.



Ex: blue short tube is "2-rich" for $\mathbb{T}$, short orange tube is "3-rich".



Arrange short tubes according to # of long tubes containing each,

arrange δ-cubes according to # of short tubes containing each.

$\forall$ M dyadic, let

$\mathbb{T}_{Q,M} := \{ \delta \times D\delta$ tubes T_Q, s.t. T_Q lies in $\sim M$ δ-tubes in $\mathbb{T}\}$.

$$P_Q := \{ q \in \underbrace{P_r(\mathbb{T})}_{=P} : q \in Q \}$$

$\forall$ E dyadic, let

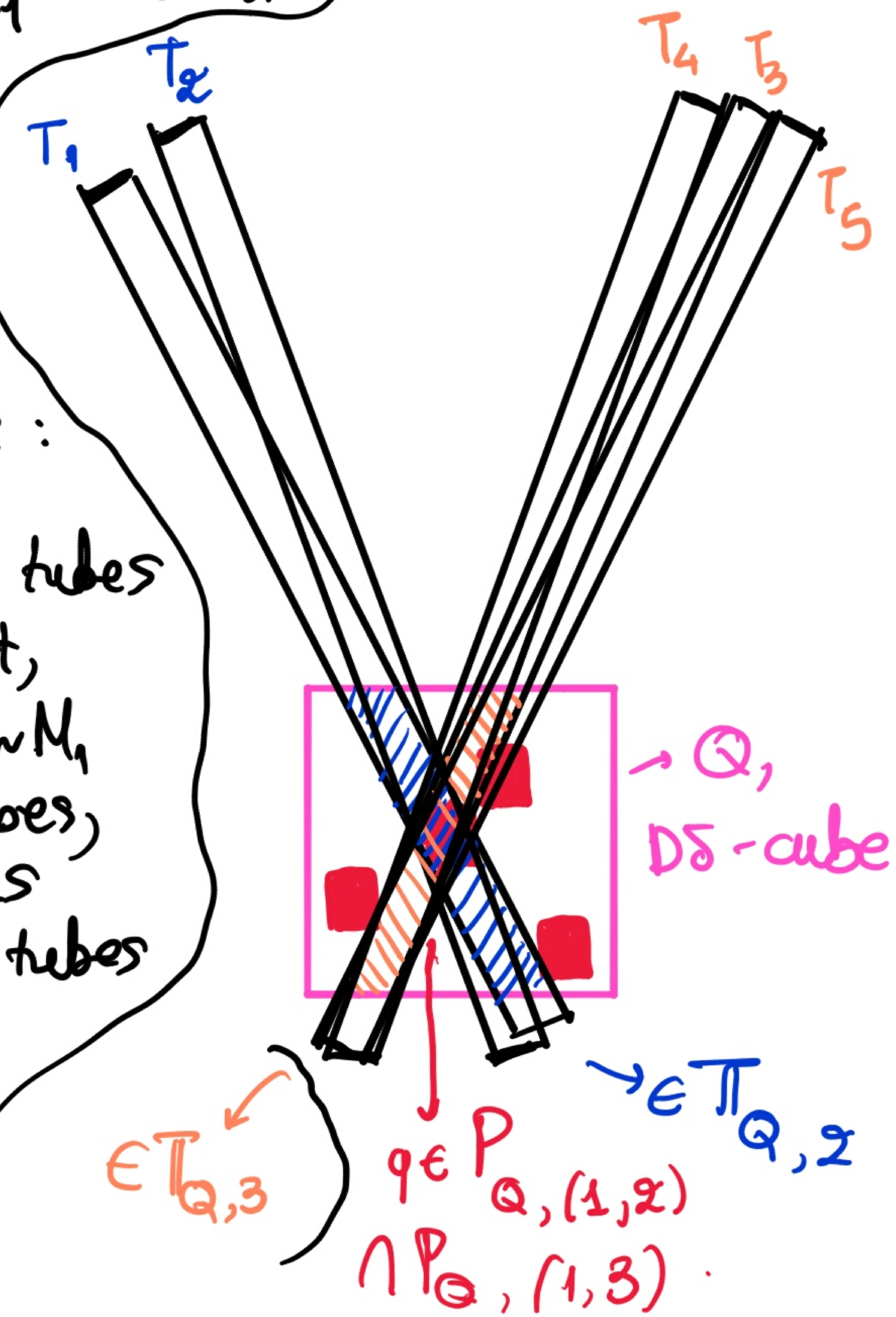
$P_{Q,(E,M)} := \{ q \in P_Q : q$ lies in $\sim E$ (short) tubes in $\mathbb{T}_{Q,M} \}$.

$r = E_1 M_1 + E_2 M_2$

$E_i M_i \le r$

$\Uparrow$

a red cube: say has $\sim E_1$ short tubes through it, each in $\sim M_1$ long tubes, also has $\sim E_2$ short tubes through it, each in $\sim M_2$ long tubes.



$$|P|\cdot r \sim \sum_Q |P_Q|\cdot r \sim \sum_Q \sum_{(E,M)} |P_{Q,(E,M)}|\cdot EM$$

$$\sim \sum_{(E,M)} \sum_Q |P_{Q,(E,M)}|\cdot EM$$

Pick a dominant (E,M): $|P|\cdot r \approx \sum_Q \underbrace{|P_{Q,(E,M)}|}_{=P_Q} \cdot \underbrace{EM}_{\leq r} \leq$

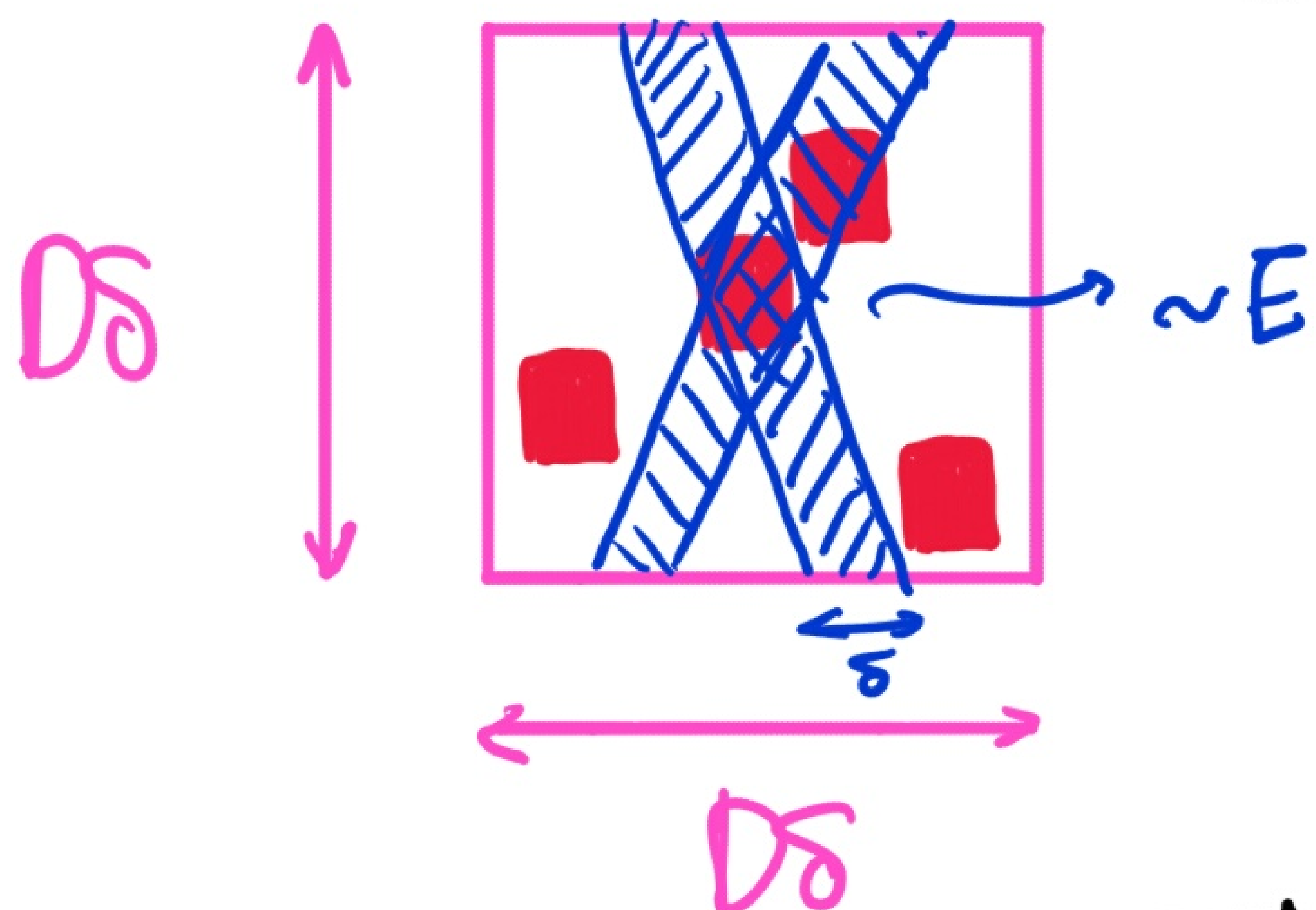
we may assume that each of our δ-cubes lies in $\sim E$ short tubes, and each short tube lives in $\sim M$ long tubes

$$\leq \underbrace{\left(\sum_Q |P_Q|\right)}_{\leq |P|} \cdot r$$

$\rightarrow$ $|P| \lessapprox \sum_Q |P_Q|$, $EM \approx r$. Also: $E \lessapprox D^{n-1}$.

$$|P| \lessapprox \sum_Q \underbrace{|P_Q|}_{\delta\text{-cubes},\ \in P_E(\underbrace{\Pi_{Q,M}}_{=:\ \Pi_Q})}$$

the short tubes are not necessarily well-spaced. $\Rightarrow$ we will use the unconditional incidence thm to bound $|P_Q|$ in each Q

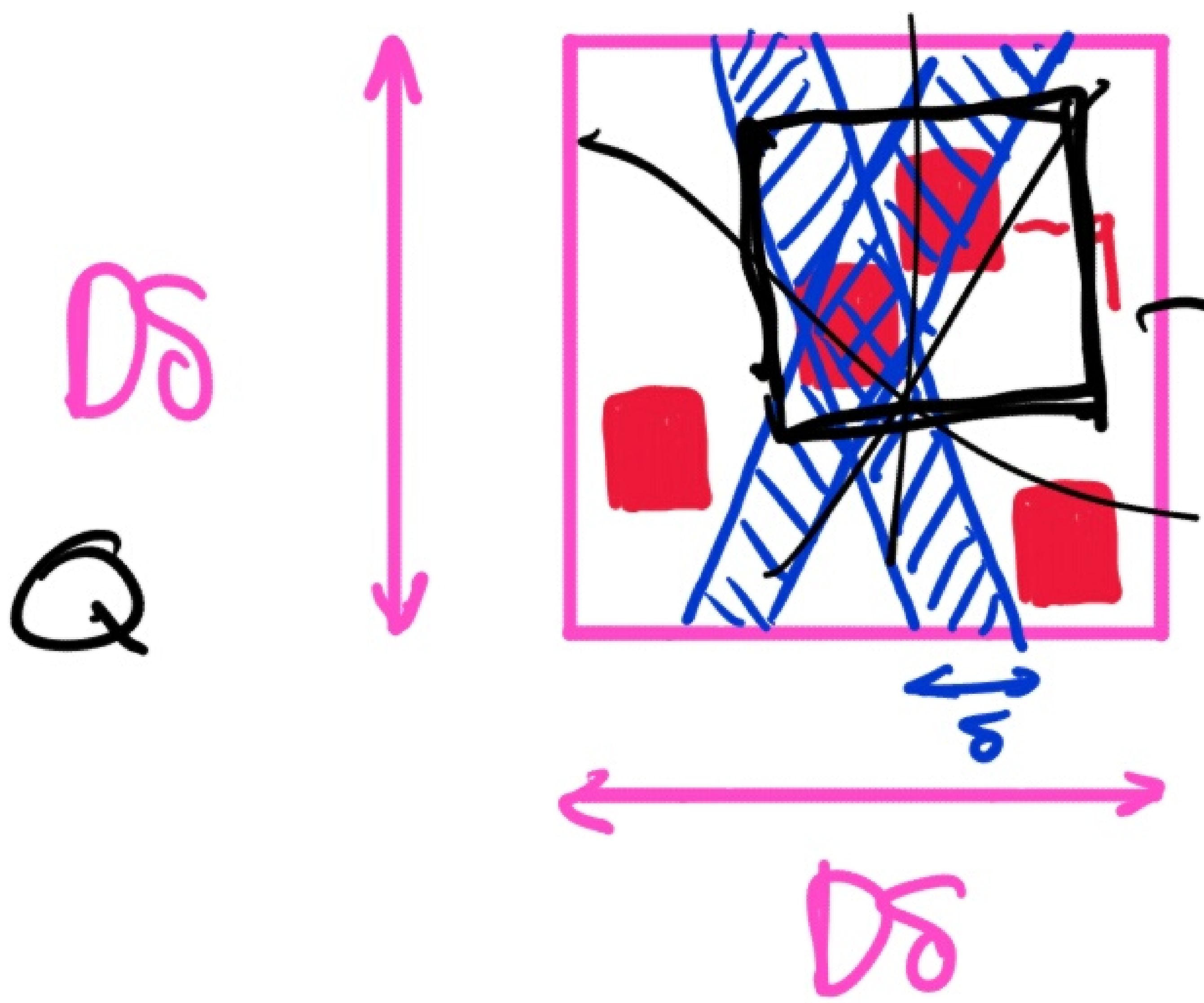


thin case in Q — thick case in Q

$$\rightarrow |P| \lessapprox \underbrace{\sum_{\text{thin case in } Q} |P_Q|}_{=: A} + \underbrace{\sum_{\text{thick case in } Q} |P_Q|}_{=: B}$$

If $|P| \lessapprox \sum_{Q:\text{ thick case holds}} |P_Q|$: In each such $D\delta$- cube Q,

$\exists$ fin. overlapping $2S\delta$- cubes Q_j, s.t. $\bigcup Q_j$ contains $\gtrapprox |P_Q|$ of the δ- cubes in P_Q, each Q_j is intersected by $\underline{\gtrsim D^{n\varepsilon^3} S^{n-1} \cdot E}$ $\delta \times D\delta$ - tubes in $\mathbb{T}_Q$ $\longrightarrow$ each Q_j intersected by $\gtrapprox D^{-n\cdot\varepsilon^3} S^{n-1} EM$ δ- tubes in $\mathbb{T}$.

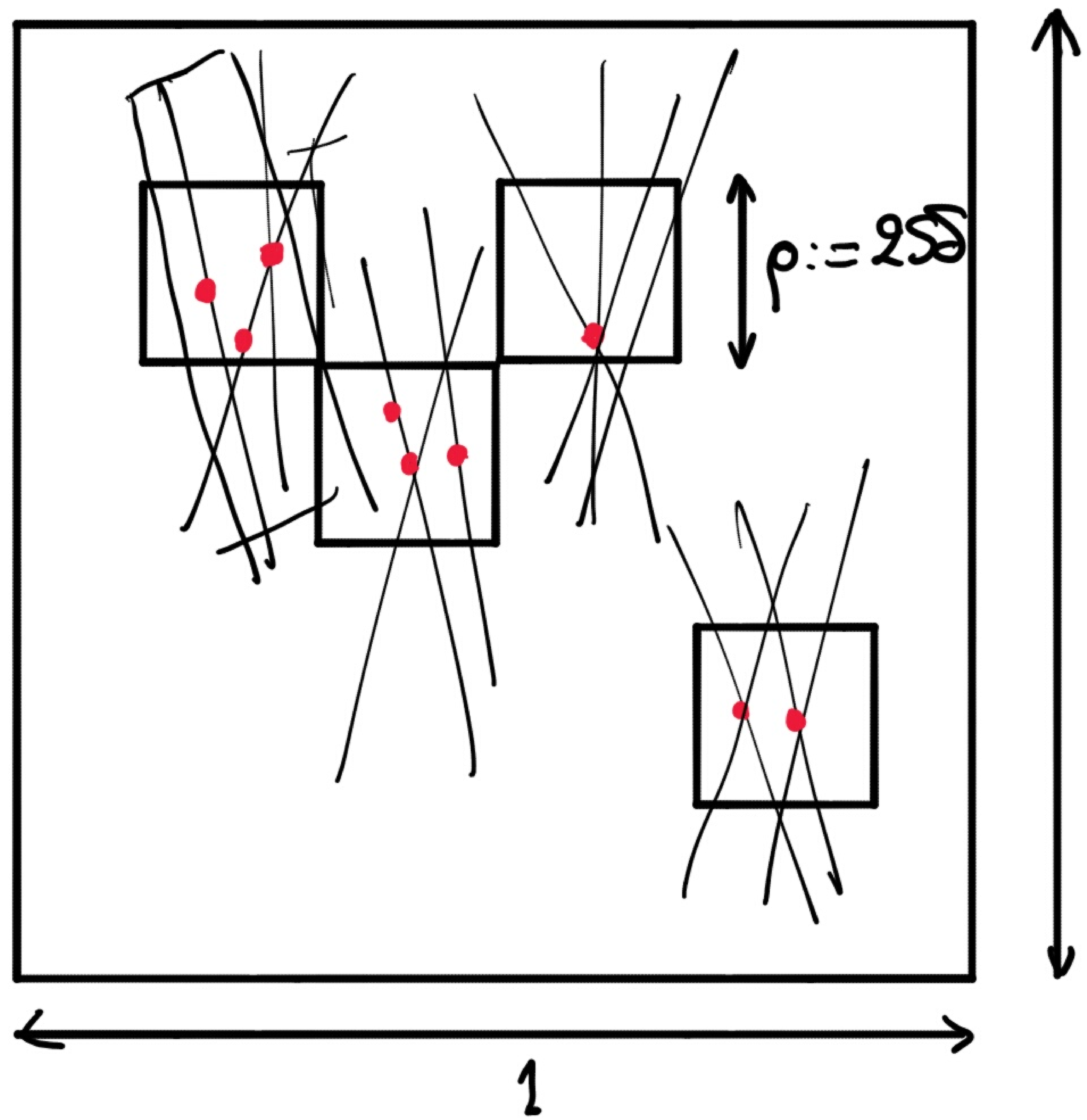


$2S\delta$- cubes, $\gtrapprox S^{n-1}\epsilon$-rich.

$S = D^{\varepsilon/10n}$

(each is intersected by $\gtrapprox D^{-n\varepsilon^3} S^{n-1} \cdot E$ $\delta \times D\delta$ tubes).

So, $\exists$ fin. overlapping $2S\delta$- cubes $\bar{Q}_j$ s.t. $\bigcup \bar{Q}_j$ contains $\gtrapprox |P|$ of the δ- cubes in P, and each is intersected by $\gtrsim D^{-n\cdot\varepsilon^3} S^{n-1} (\varepsilon M) \gtrapprox \underline{D^{-n\varepsilon^3}} \underline{S^{n-1} r}$ δ-tubes in $\mathbb{T}$



$S = D^{\varepsilon/10n}$

$\Rightarrow \underline{S^{n-1}} = D^{\varepsilon\cdot\left(\frac{n-1}{10n}\right)}$

$\rho := 2S\delta$

Each W^{-1}- tube contains ≤ 1 δ- tube $\rightarrow$ also contains ≤ 1 $\underline{\rho\text{- tube}}$ $(\rho < W^{-1})$.

So, each $\rho \underset{\rho}{\boxed{}} \in P^{\parallel k}_{\gtrsim D^{-10n\varepsilon^3} \delta^{n-1} \cdot r}(\widetilde{\mathbb{T}})$, where $\widetilde{\mathbb{T}}$ is the family of ρ-tubes we get if we thicken every δ-tube around its core line, so that it becomes a ρ-tube.

$\rho = 2\delta$

So, $|P| \lesssim \left| P_{\gtrsim D^{-10n\varepsilon^3} \delta^{n-1} \cdot r}(\widetilde{\mathbb{T}}) \right| \cdot \frac{\rho^n}{\delta^n}$

$$\overset{!}{\lessapprox_\varepsilon} \rho^{-\varepsilon} \cdot \frac{|\mathbb{T}|^{\cancel{\frac{n}{n-1}}}}{\left(D^{-10n\varepsilon^3} \delta^{n-1} \cdot r\right)^{\frac{n+1}{n-1}}} \cdot \delta^n \approx_\varepsilon \frac{\delta^{-\varepsilon}\delta^{-\varepsilon}}{D^{-10n\varepsilon^3}\delta} \frac{|\mathbb{T}|^{\frac{n}{n-1}}}{r^{\frac{n+1}{n-1}}}$$

$\lesssim \delta^{-\varepsilon}$?

i.e. : $\frac{D^{10n\varepsilon^3}}{\delta^{1+\varepsilon}} \ll 1$? $\delta = D^{\varepsilon/10n}$ ✓, OK, as long as $D \gtrsim O(1)$.

... As long as $\kappa = D^{-10n\varepsilon^3} \underset{\overset{\shortparallel}{D^{\frac{\varepsilon(n-1)}{10n}}}}{S^{n-1}} r = D^{\overbrace{\left(\frac{\varepsilon(n-1)}{10n} - 10n\varepsilon^3\right)}^{}} \cdot r$

$\underbrace{}_{\exp > 0}$

$$> \max\left(\underline{\rho^{n-1-\varepsilon/4} |\tilde{\Pi}|}, \underline{1} \right).$$

- $r > 1 \Rightarrow \kappa (> r) > 1$. ✓
- $\rho^{n-1-\varepsilon/4} |\tilde{\Pi}| = (2S\delta)^{n-1-\varepsilon/4} \cdot |\Pi|$, so we want

$$D^{-10n\varepsilon^3} \cdot S^{n-1} \cdot r > (2S\delta)^{n-1-\varepsilon/4} |\Pi|.$$

$r > \delta^{n-1-\varepsilon/4} |\Pi|$, so suffices:

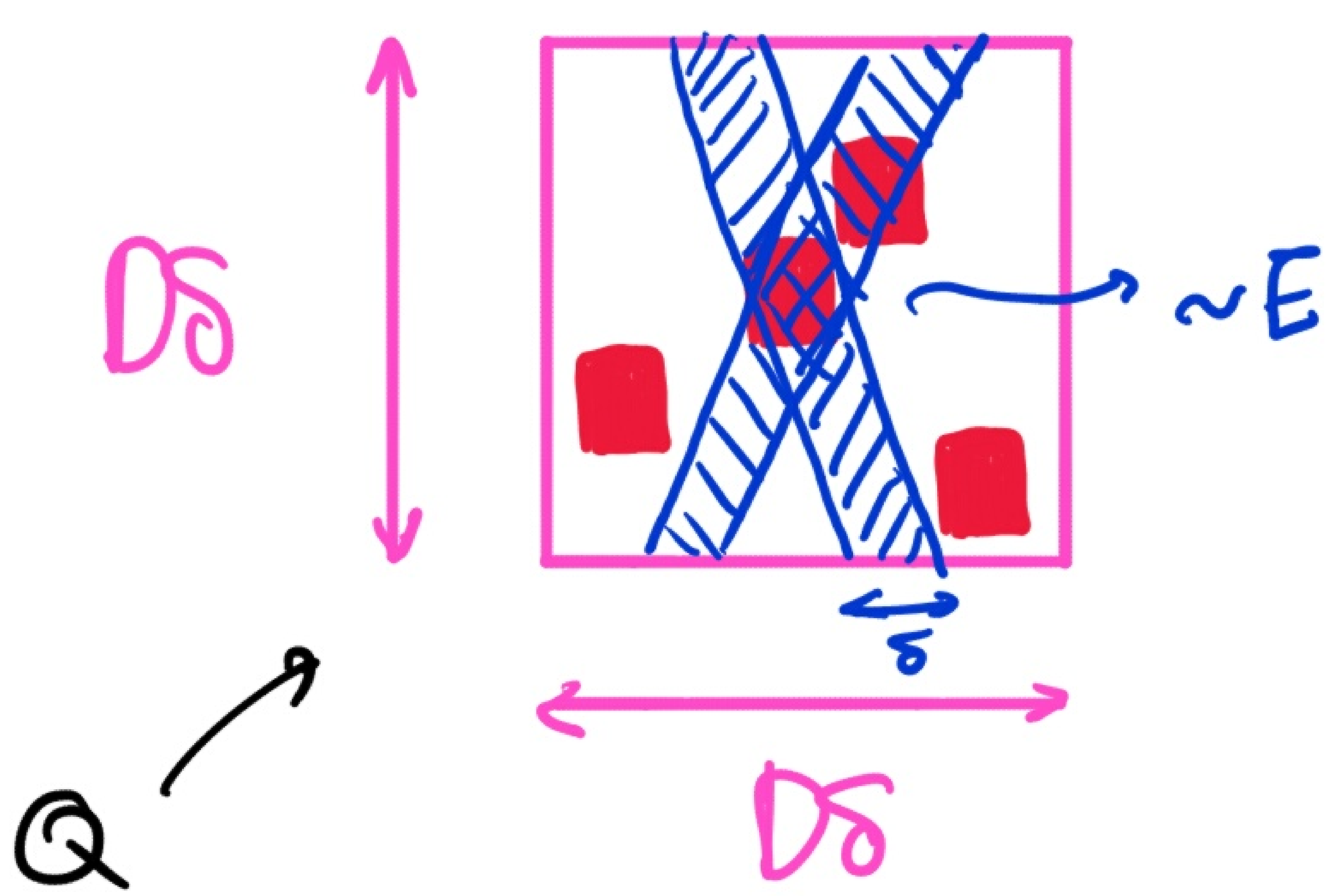
$D^{-10n\varepsilon^3} \cancel{S^{n-1}} \gtrsim S^{\cancel{n-1}-\varepsilon/4} \iff S^{\varepsilon/4} D^{-10n\varepsilon^3} \gtrsim 1 \iff D^{\varepsilon/40n - 10n\varepsilon^3} \gtrsim 1$

✓.

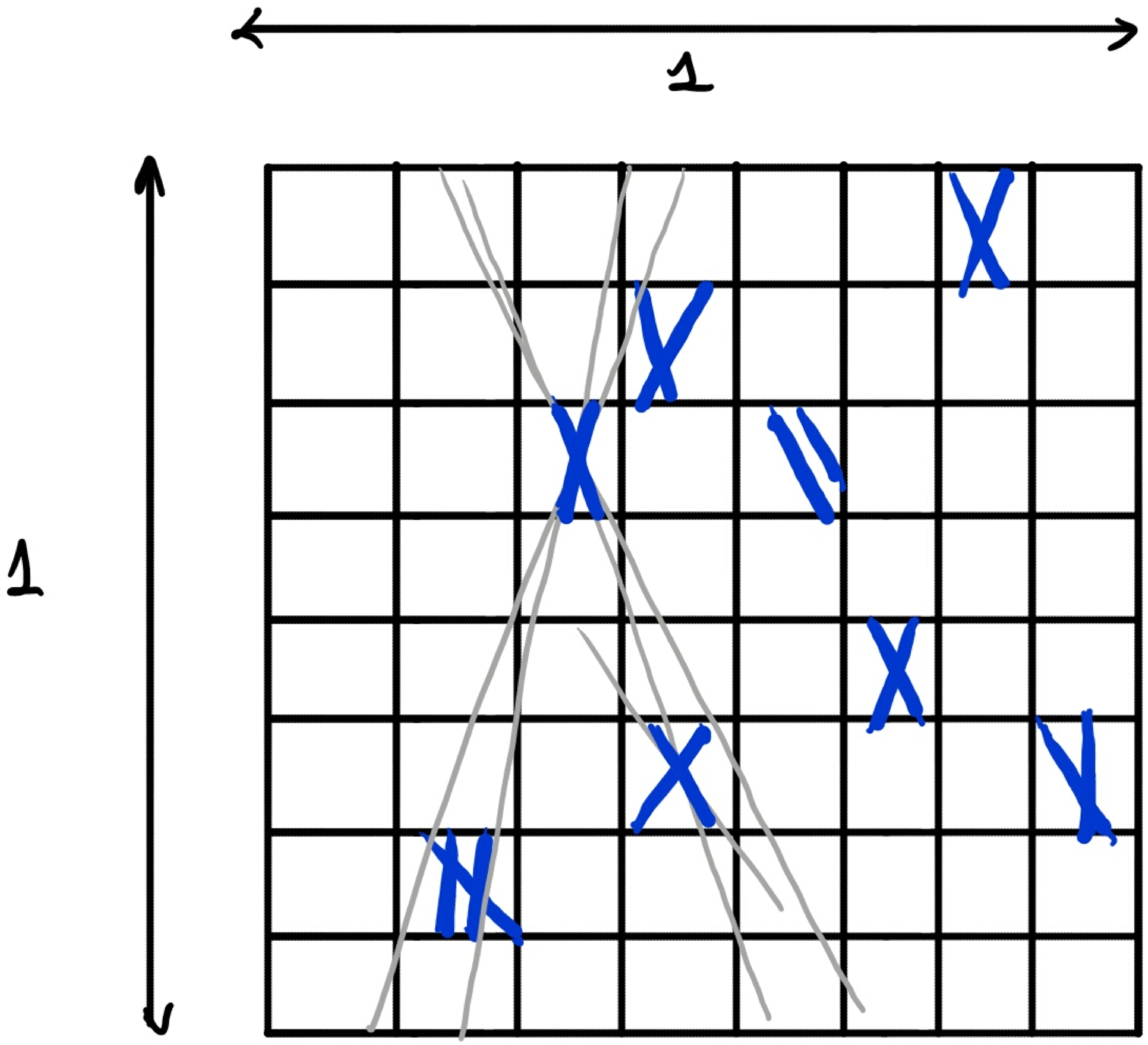
$$\boxed{\text{If } |P| \lessapprox A:} \quad \underbrace{|P| \lessapprox}_{\text{log loss}} \sum_{Q:\text{ thin case holds}} \underline{\underline{|P_Q|}}.$$

$: \forall$ such Q,

$$|P_Q| \lessapprox \delta^n \, D^{n-1} \cdot \frac{|\mathbb{T}_Q|}{E^2} \to$$



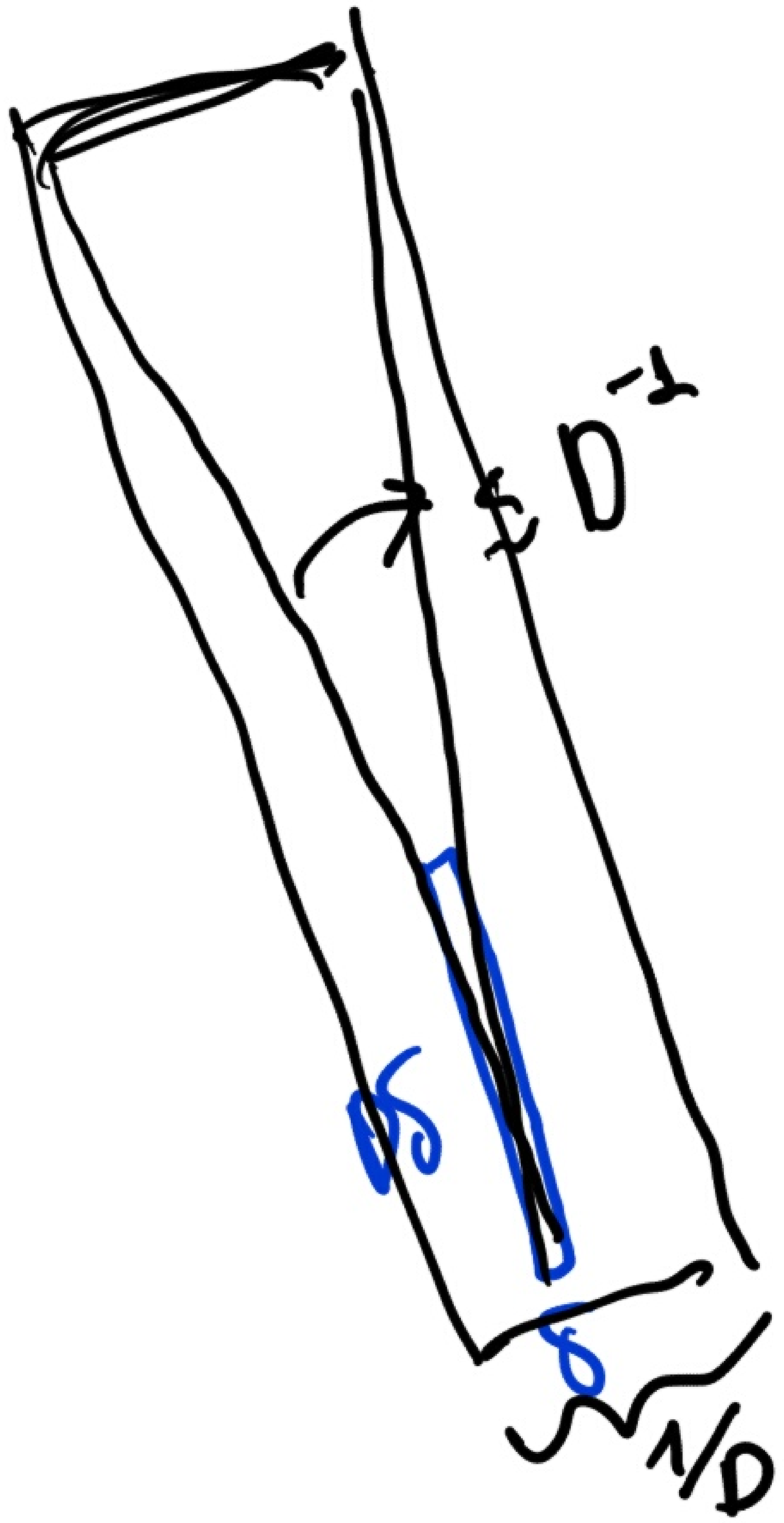
$$|P| \lessapprox \sum_Q |P_Q| \lessapprox D^{\varepsilon/10} \cdot \frac{D^{n-1}}{E^2} \cdot \underbrace{\sum_Q |\mathbb{T}_Q|}_{\text{all the short blue tubes in } B^n(0,1).}$$



Let $\overline{\mathbb{T}}$ be the set of (short) $\delta \times D\delta$ tubes $\left(\bigcup_Q \mathbb{T}_Q \right)$.

Each is $\sim M$-rich for $\mathbb{T}$ (i.e. is contained in $\sim M$ δ-tubes in $\mathbb{T}$).

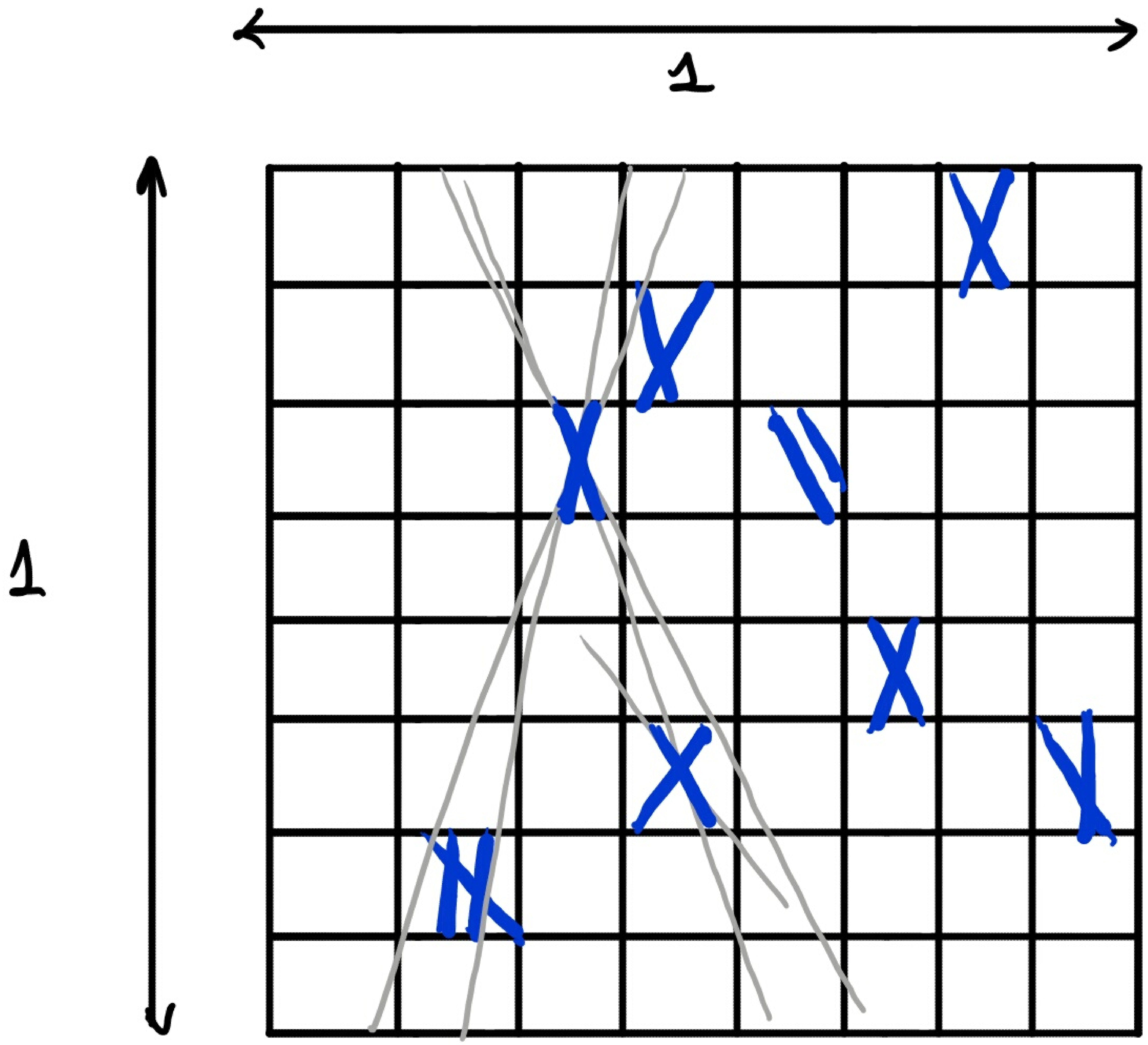
To control $|\overline{\mathbb{T}}|$ efficiently, we need to understand, $\forall \overline{T} \in \overline{\mathbb{T}}$, what the δ-tubes in $\mathbb{T}$ that contain it look like.



All the $(\sim N)$ δ-tubes in $\mathbb{T}$ containing $\bar{T}$ are contained in the <u>same</u> $1/D$-tube.

We associate $\bar{T}$ to this (unique) $1/D$-tube

So, consider $\underbrace{\text{all } 1/D\text{-tubes}}_{\sim D^{2(n-1)}}$ $\square$ in $B^n(0,1)$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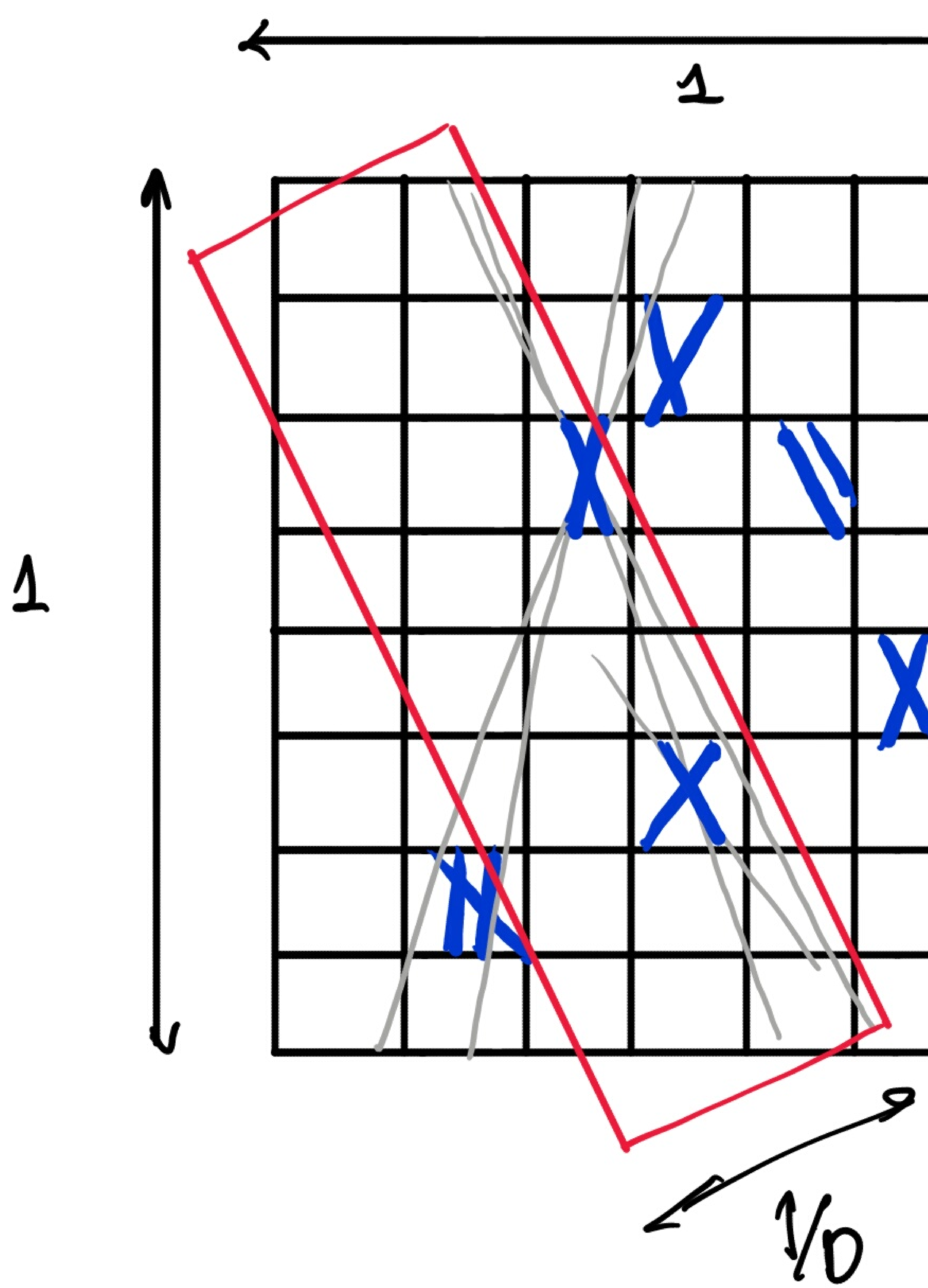


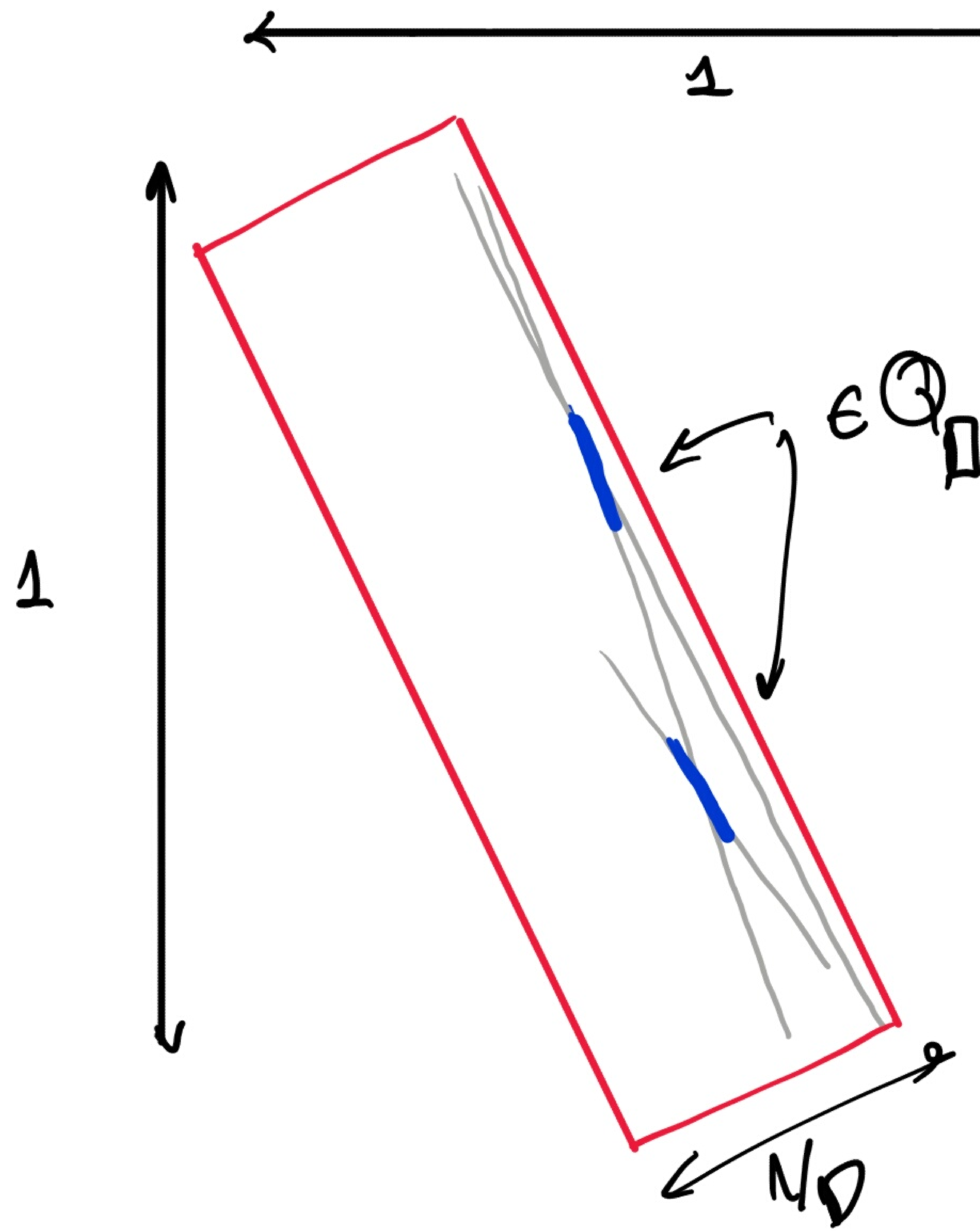
$\forall \square$, let $\mathbb{T}_\square$ be the set of δ-tubes in $\mathbb{T}$ that live in $\square$, $Q_\square$ the set of $\delta \times D\delta$-tubes in $\overline{\mathbb{T}}$ that are contained in $\mathbb{T}_\square$. Then:

$$\overline{\mathbb{T}} = \coprod_\square Q_\square, \quad \mathbb{T} = \coprod_\square \mathbb{T}_\square.$$

And: $\forall \square$, $Q_\square$ is M-rich for tubes in $\mathbb{T}_\square$.

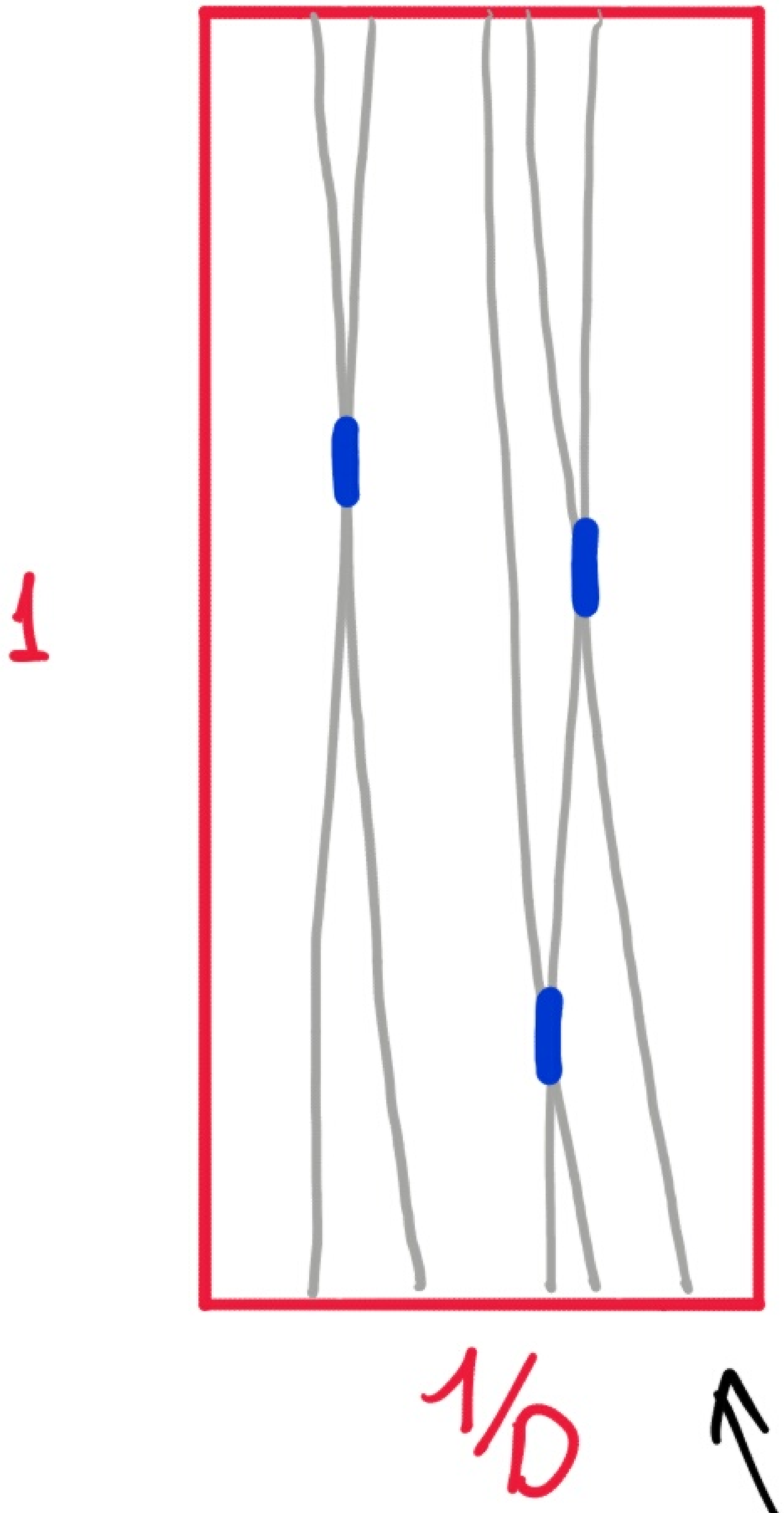
So: $\underline{\sum_Q |\mathbb{T}_Q|} = |\overline{\mathbb{T}}| = \sum_\square |Q_\square| \lesssim$

$\lesssim D^{2(n-1)} \cdot \max_\square |Q_\square|$



$\in \mathcal{Q}_{\square}$, each has $\sim M$ δ-tubes in $\mathbb{T}_{\square}$ containing it.

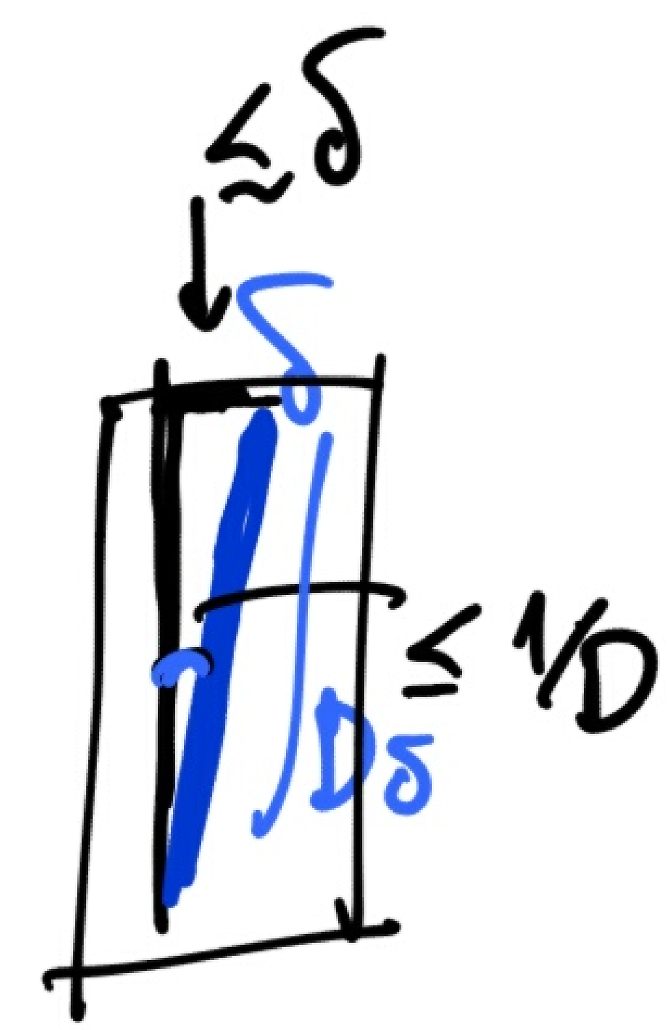
Consider the arbitrary $1/D$-tube $\square$.



$\times D$ in

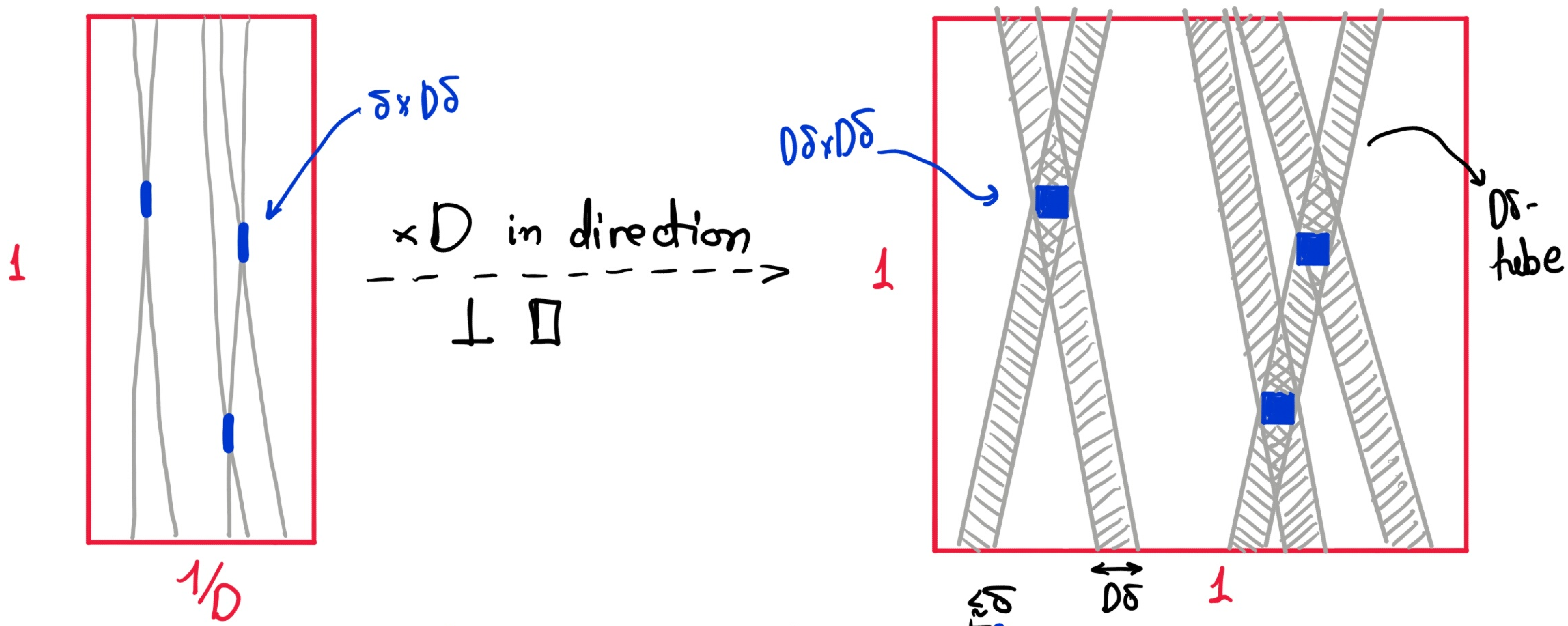
direction $\perp$ □

the core line of each | are ess. $\parallel$ □ :

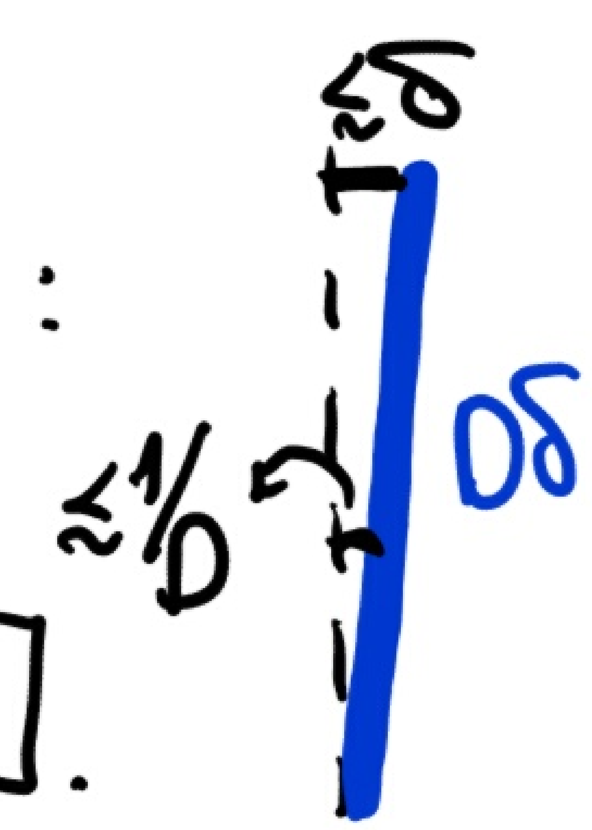


Consider the arbitrary $1/D$-tube □.

Consider the $\delta \times D\delta$ tubes in $Q_{□}$ and the tubes in $\mathbb{T}_{□}$.



↳ the | are ess. // core line :



Consider the arbitrary $1/D$-tube □.

Consider the $\delta \times D\delta$ tubes in $Q_{\square}$ and the tubes in $\mathbb{T}_{\square}$.

$Q_\square \in \bar{P}_M(\mathbb{T}_\square)$ - - - - - -> $\tilde{Q}_\square \in P_M(\tilde{\mathbb{T}}_\square)$

Every w^{-1}-tube contains ≤ 1 δ-tube in $\mathbb{T}_\square$

Every $\underline{Dw^{-1}}$-tube contains ≤ 1 $D\delta$-tube in $\tilde{\mathbb{T}}_\square$.

$\Rightarrow$ add as many $D\delta$-tubes in $\tilde{\mathbb{T}}_\square$ s.t. $|\tilde{\mathbb{T}}_\square| \sim \left(\frac{w}{D}\right)^{2(n-1)}$

So, $|Q_\square| \lesssim |P_M(\tilde{\Pi}_\square)|$. We can control this, as long as

$$M > \max\{ \underbrace{\tilde{\delta}^{\,n-1-\varepsilon/4} |\tilde{\Pi}_\square|}_{\tilde{\delta} = D\delta \ ||}, 1 \}. \quad ME \approx r \to M \approx E^{-1} \cdot r \gtrsim \frac{r}{D^{n-1}}.$$

$$E \lesssim D^{n-1}$$

- $$D^{n-1-\varepsilon/4} \cdot \cancel{\delta^{n-1-\varepsilon/4}} \cdot \left(\frac{W}{D}\right)^{2(n-1)} =$$

$$= \frac{1}{D^{n-1+\varepsilon/4}} \cdot \left(\delta^{n-1-\varepsilon/4} |\Pi|\right)$$

- $$M \gtrsim \frac{r}{D^{n-1}} \overset{D \sim \delta^{-\varepsilon^4}}{\gtrsim} \delta^{\varepsilon^4 (n-1)} \cdot r \gtrsim \delta^{2\varepsilon^4} \cdot r, \quad \text{so } \geq 2$$

if $\boxed{r > \delta^{-\varepsilon^3}}$

⚠ So, whatever we've been doing so far works for $r > \delta^{-\varepsilon^3}$.
We need sth else for $r \leq \delta^{-\varepsilon^3}$.

So, $|\overline{\mathbb{T}}| \lesssim D^{2(n-1)} \cdot \max_{\square} |Q_{\square}| \lesssim D^{2(n-1)} \cdot \max_{\square} |P_M(\widetilde{\mathbb{T}}_{\square})|$

$\widetilde{\mathbb{T}}_{\square} \to D\delta$-tubes

$$\lesssim D^{2(n-1)} (D\delta)^{-\varepsilon} \cdot \frac{|\widetilde{\mathbb{T}}_{\square}|^{\frac{n}{n-1}}}{M^{\frac{n+1}{n-1}}} \sim$$

$$\sim D^{2(n-1)} (D\delta)^{-\varepsilon} \cdot \frac{(W/D)^{2(n-1)\cdot\frac{n}{n-1}}}{M^{\frac{n+1}{n-1}}} \sim$$

$E \cdot M \approx r$

$$\sim \frac{1}{D^2} D^{-\varepsilon} \delta^{-\varepsilon} \cdot \frac{|\mathbb{T}|^{\cancel{\frac{n}{n-1}}}}{M^{\frac{n+1}{n-1}}}$$

$$\delta^{-\varepsilon} \frac{|\mathbb{T}|^{\cancel{\frac{n}{n-1}}}}{M^{\frac{n+1}{n-1}}}$$

And: $|P| \lessapprox S^n \cdot \frac{D^{n-1}}{E^2} \cdot \underline{|\overline{\mathbb{T}}|} \lesssim D^{\varepsilon/10} \cdot \frac{D^{n-3}}{E^2} \cdot D^{-\varepsilon}$

$$|P| \lesssim D^{\varepsilon/10} \cdot \frac{D^{n-3}}{E^2} \cdot D^{-\varepsilon} \quad \delta^{-\varepsilon} \frac{|\pi|^{n/(n-1)}}{M^{\frac{n+1}{n-1}}}$$

$$\rightsquigarrow n=3: \quad |P| \lesssim D^{\varepsilon/10} \cdot D^{-\varepsilon} \cdot \delta^{-\varepsilon} \cdot \frac{|\pi|^{\frac{n}{n-1}}}{(EM)^2}$$

$$\lesssim D^{-\varepsilon+\varepsilon/10} \, \delta^{-\varepsilon} \cdot \frac{|\pi|^{3/2}}{r^2} \quad \checkmark$$

$$\rightsquigarrow n=2: \quad |P| \lesssim D^{\varepsilon/10} \cdot \cancel{\frac{E \le D}{D}} \, D^{-\varepsilon} \cdot \delta^{-\varepsilon} \frac{|\pi|^2}{\underbrace{(EM)^3}_{\sim r}} \quad \checkmark$$