"S-T": Let $1 \leq W \leq \delta^{-1}$. Let $\mathbb{T}$ be a set of $\sim W^2$ δ-tubes in $[0,1]$, s.t. each W^{-1}-tube contains ≤ 1 of them.

In $\mathbb{R}^2$

Then, $\forall r > \max(\delta^{1-\varepsilon}|\mathbb{T}|, 1)$,

$$|P_r(\mathbb{T})| \lesssim_\varepsilon \delta^{-\varepsilon} \frac{|\mathbb{T}|^2}{r^3}.$$

<u>Guth - Katz:</u> Let $\mathcal{L}$ a set of L lines in $\mathbb{R}^3$, s.t.
every plane contains $\leq L^{1/2}$ of the lines,
then $\forall \quad 2 \leq r \lesssim L^{1/2}$:

$$|P_r(\mathcal{L})| \lesssim \frac{L^{3/2}}{r^2}.$$

"G - K": Let $1 \leq W \leq \delta^{-1}$. Let $\mathbb{T}$ be a set of $\sim W^4$ δ-tubes in $\mathbb{R}^3$, s.t. each W^{-1}-tube contains ≤ 1 of them. Then,

$$|P_r(\mathbb{T})| \lesssim_\varepsilon \delta^{-\varepsilon} \cdot \frac{|\mathbb{T}|^{3/2}}{r^2}, \quad \forall\, r > \max\{\delta^{2-\varepsilon}|\mathbb{T}|, 1\}.$$

Combining notation:

Thm: Let $1 \le W \le \delta^{-1}$. Let $n=2,3$. Let $\mathbb{T}$ be a set of $\sim W^{2(n-1)}$ δ-tubes in $B^n(0,1)$, s.t. each W^{-1}-tube contains ≤ 1 of them. Then, $\forall r > \max(\delta^{n-1-\varepsilon/4}|\mathbb{T}|, 1)$,

$$|P_r(\mathbb{T})| \lesssim \delta^{-\varepsilon} \cdot \boxed{\frac{|\mathbb{T}|^{\frac{n}{n-1}}}{r^{\frac{n+1}{n-1}}}}.$$

slightly larger range than $\delta^{n-1-\varepsilon}|\mathbb{T}|$

For proof, we will be fixing δ, and show Thm $\forall$ $1 \le W \le \delta^{-1}$.

<u>Thm:</u> Let $1 \leq W \leq \delta^{-1}$, $n = 2,3$.

$\mathbb{T}$: set of $\sim W^{2(n-1)}$ δ-tubes in $B^n(0,1)$,

s.t. each W^{-1}-tube contains ≤ 1 δ-tube in $\mathbb{T}$.

Then: $$|P_r(\mathbb{T})| \lesssim \delta^{-\varepsilon} \frac{|\mathbb{T}|^{\frac{n}{n-1}}}{r^{\frac{n+1}{n-1}}},$$

$\forall\, r > \max\left(\delta^{n-1-\varepsilon/4}|\mathbb{T}|, 1\right)$

① Thm holds for $\delta \sim 1$:

Then, $W \sim 1$, so if 2 δ-tubes intersect, their angle is $\gtrsim W^{-1} \sim 1 \rightarrow$ they intersect "only once"

$$\Rightarrow |P_r(\mathbb{T})| \lesssim |\mathbb{T}|^2 \sim 1 \lesssim_\varepsilon \delta^{-\varepsilon} \frac{|\mathbb{T}|^{\frac{n}{n-1}}}{r^{\frac{n+1}{n-1}}},$$

$\forall\, r \ldots$

$\boxed{\text{So, fixing any } \delta > 0, \text{ we may assume that Thm holds } \forall\, \tilde{\delta} > 2\delta.}$

inductive hyp.

② Thm holds for $r \gtrsim |\mathbb{T}|^{1/2} \sim \underline{W^{n-1}}$ (then $|P_r(\mathbb{T})| = 0$).

We may also assume that fixing r, Thm holds $\forall\, \tilde{r} > r$.

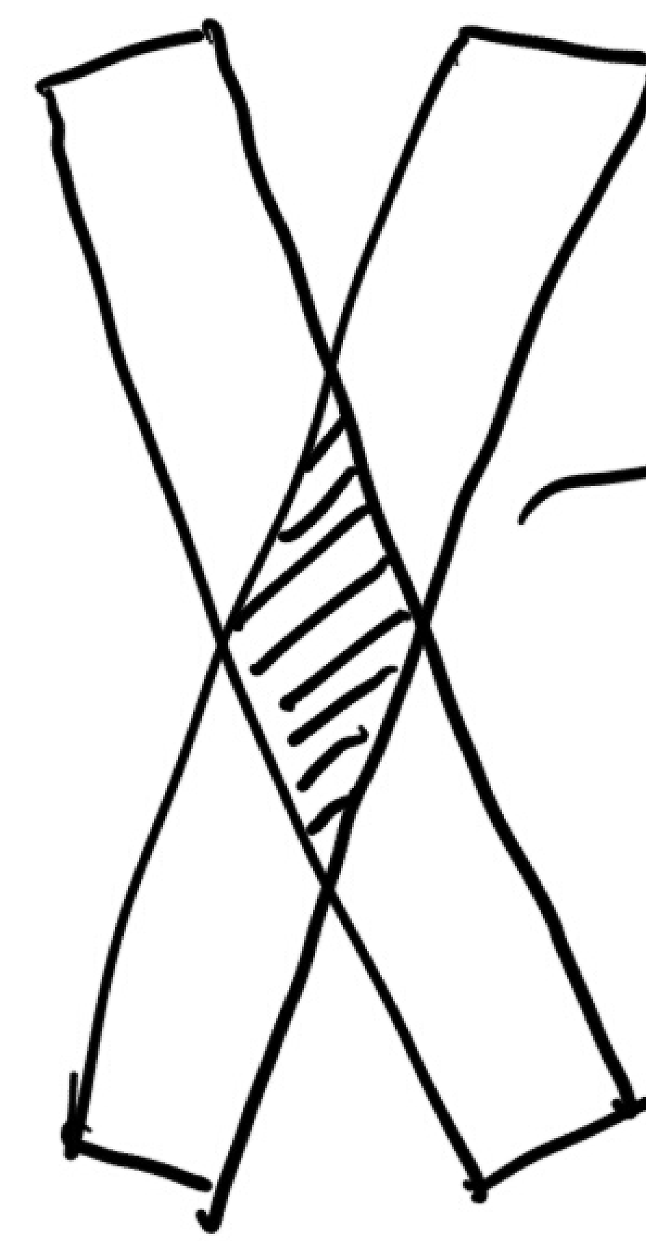
Thm: Let $1 \le W \le \delta^{-1}$, $n = 2,3$.

$\mathbb{T}$: set of $\sim W^{2(n-1)}$ δ-tubes in $B^n(0,1)$, s.t. each W^{-1}-tube contains ≤ 1 δ-tube in $\mathbb{T}$.

Then: $|P_r(\mathbb{T})| \lesssim \delta^{-\varepsilon} \dfrac{|\mathbb{T}|^{\frac{n}{n-1}}}{r^{\frac{n+1}{n-1}}}$,

$\forall\, r > \max\left(\delta^{n-1-\varepsilon/4}|\mathbb{T}|, 1\right)$

③ Thm holds $\forall\, \underline{W \lesssim \delta^{-\varepsilon/1000n}}$:



angle $\gtrsim W^{-1}$,
int. contains $\lesssim W$ δ-cubes

$$\Leftarrow \quad |P_r(\mathbb{T})| \lesssim |\mathbb{T}|^2 \cdot W \xrightarrow[r \lesssim W^{n-1}]{} |P_r(\mathbb{T})|\, r^{\frac{n+1}{n-1}} \lesssim |\mathbb{T}|^2 W \cdot W^{n+1}$$

$\Rightarrow$ - - - - ✓

Thm: Let $1 \le W \le \delta^{-1}$, $n = 2,3$.

$\mathbb{T}$: set of $\sim W^{2(n-1)}$ δ-tubes in $B^n(0,1)$,

s.t. each W^{-1}-tube contains ≤ 1 δ-tube in $\mathbb{T}$.

Then: $|P_r(\mathbb{T})| \lesssim \delta^{-\varepsilon} \frac{|\mathbb{T}|^{\frac{n}{n-1}}}{r^{\frac{n+1}{n-1}}}$,

$\forall\, r > \max\left(\delta^{n-1-\varepsilon/4}|\mathbb{T}|, 1\right)$

④ Thm holds when

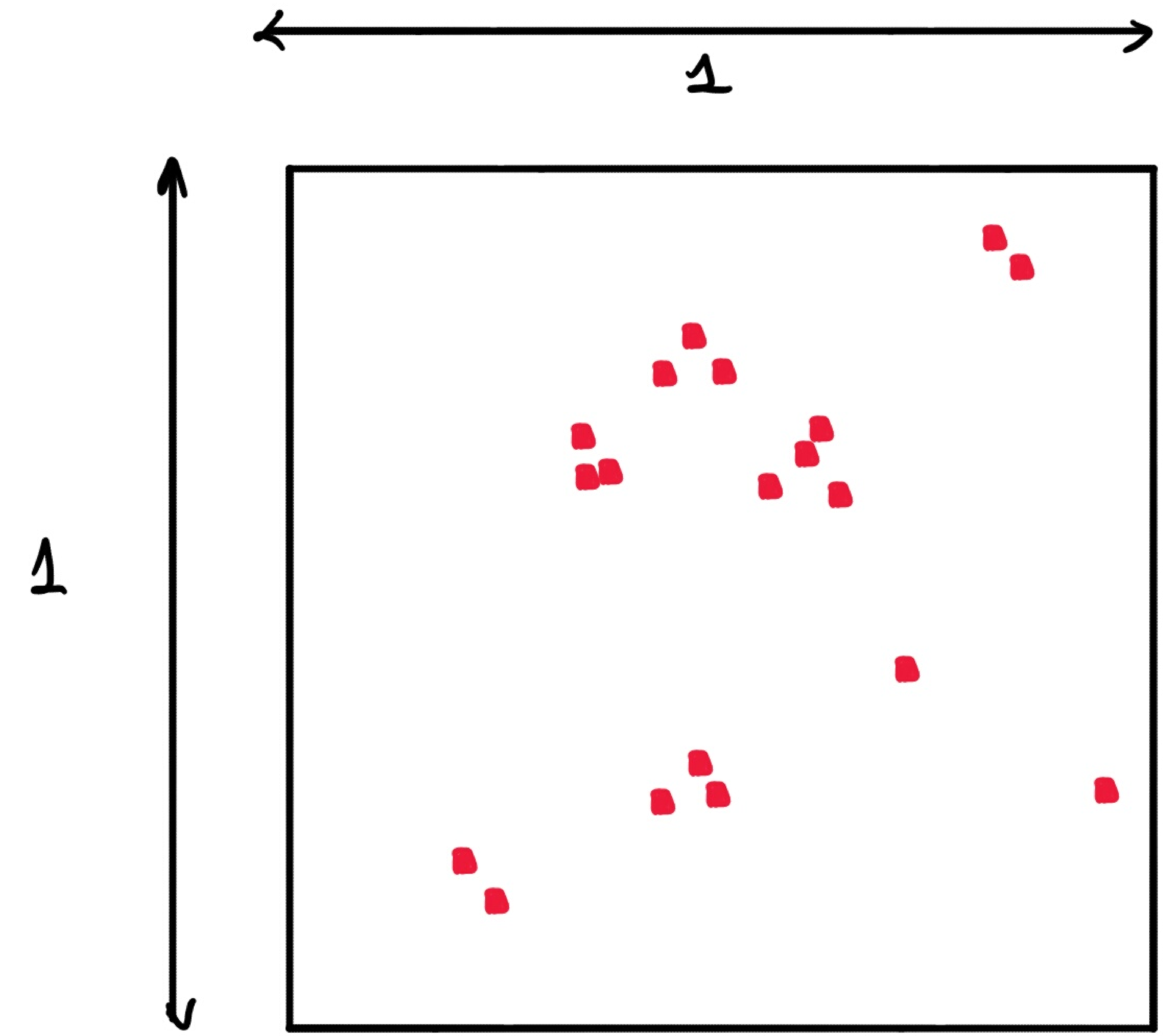
$\underline{\delta^{n-1-\varepsilon/4}|\mathbb{T}| \gtrsim |\mathbb{T}|^{1/2} \sim W^{n-1}}$:

$|P_r(\mathbb{T})| = 0.$

$\Longleftrightarrow W \gtrsim \delta^{-1+\frac{\varepsilon}{4(n-1)}}$ ✓

So, we may assume that $\boxed{W \lesssim \delta^{-1+\varepsilon/8}}$.

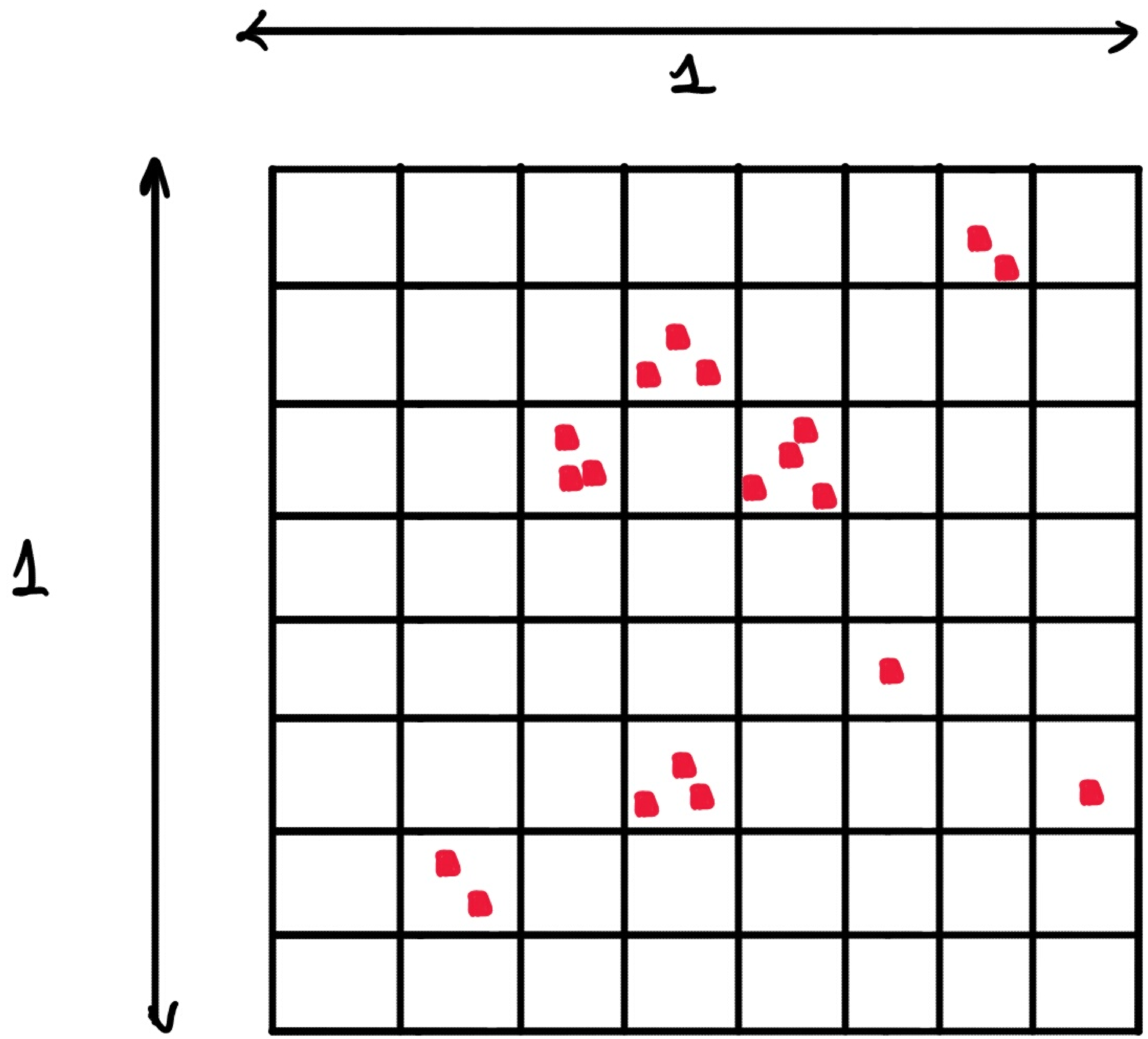
So, let $\delta^{-\varepsilon/1000n} \lesssim W \leq \delta^{-1+\varepsilon/8}$.



■ $\rightsquigarrow$ δ-cube in $P_r(\mathbb{T})$.

So, let $\delta^{-\varepsilon/1000n} \lesssim W \leq \delta^{-1/8}$. Let $1 < D \leq W$

$\sim \delta^{-\varepsilon^4}$ in the end...

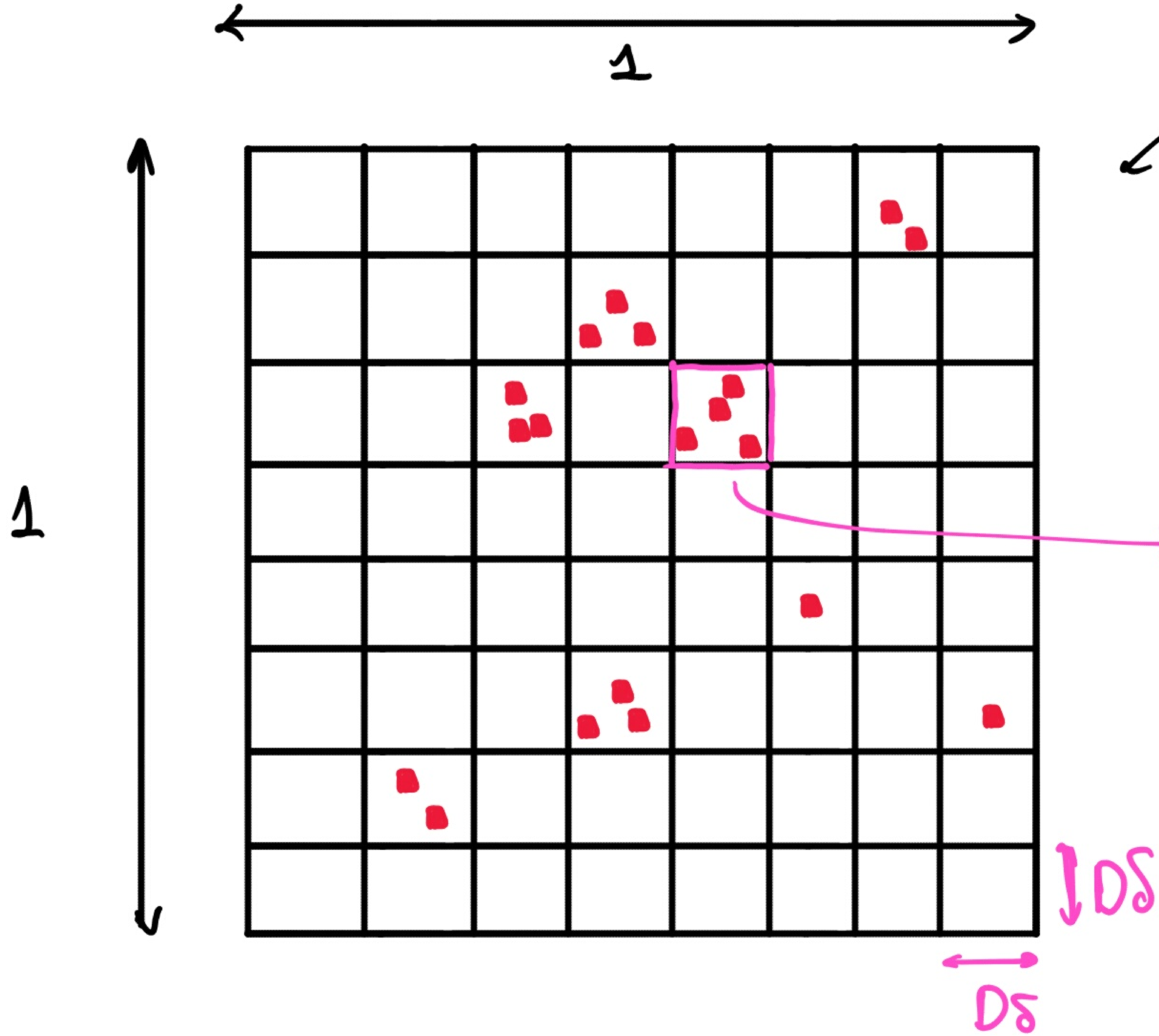


split in $D\delta$-cubes

$\blacksquare \rightsquigarrow \delta$-cube in $P_r(T)$.

So, let $\delta^{-\varepsilon/1000n} \lesssim W \le \delta^{-1/8}$. Let $1 < D \le W$

$\sim \delta^{-\varepsilon^4}$ in the end...

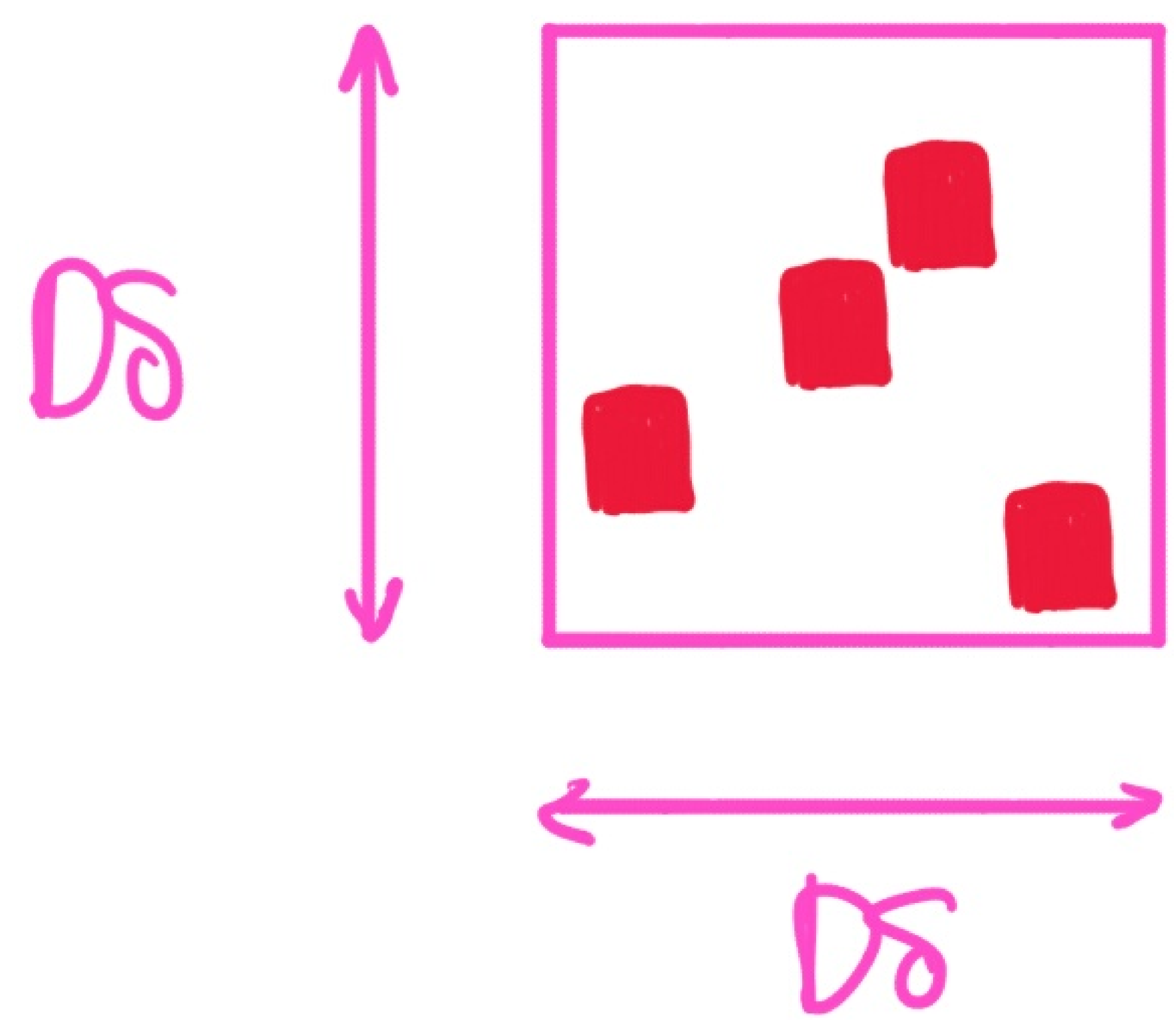


split in $D\delta$-cubes

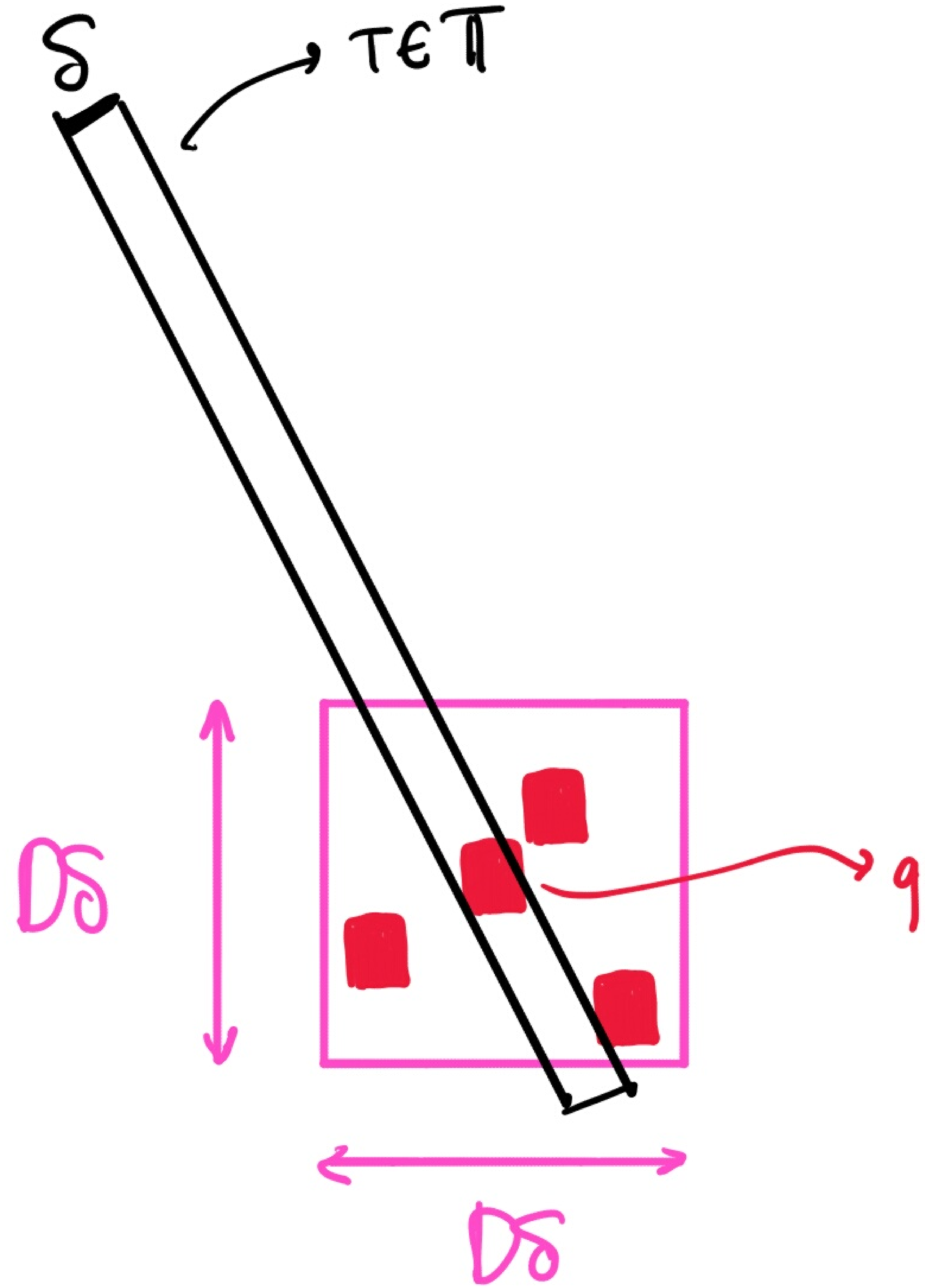
■ $\rightsquigarrow$ δ-cube in $P_r(T)$.

Q, a $D\delta$-cube.

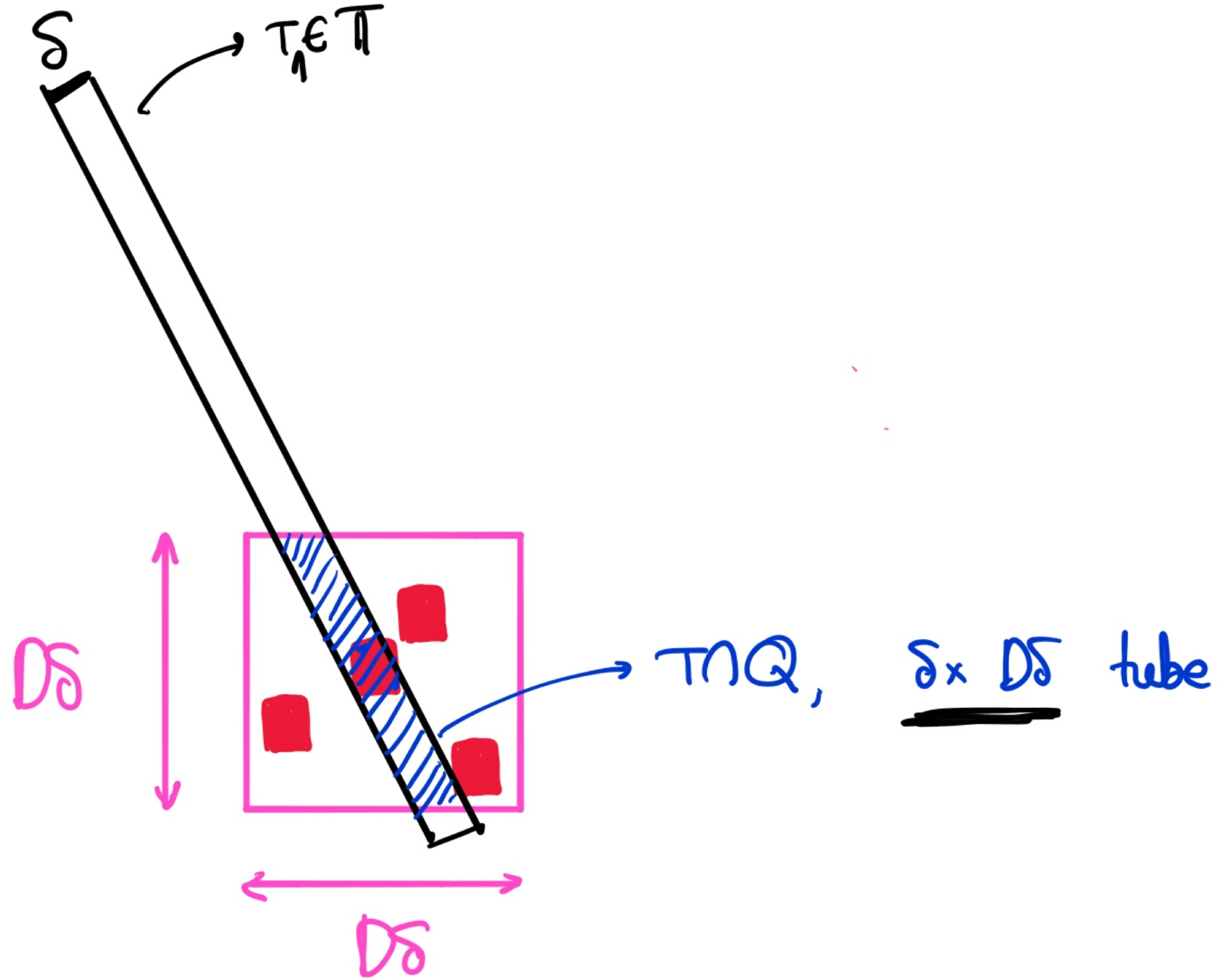
Inside Q :



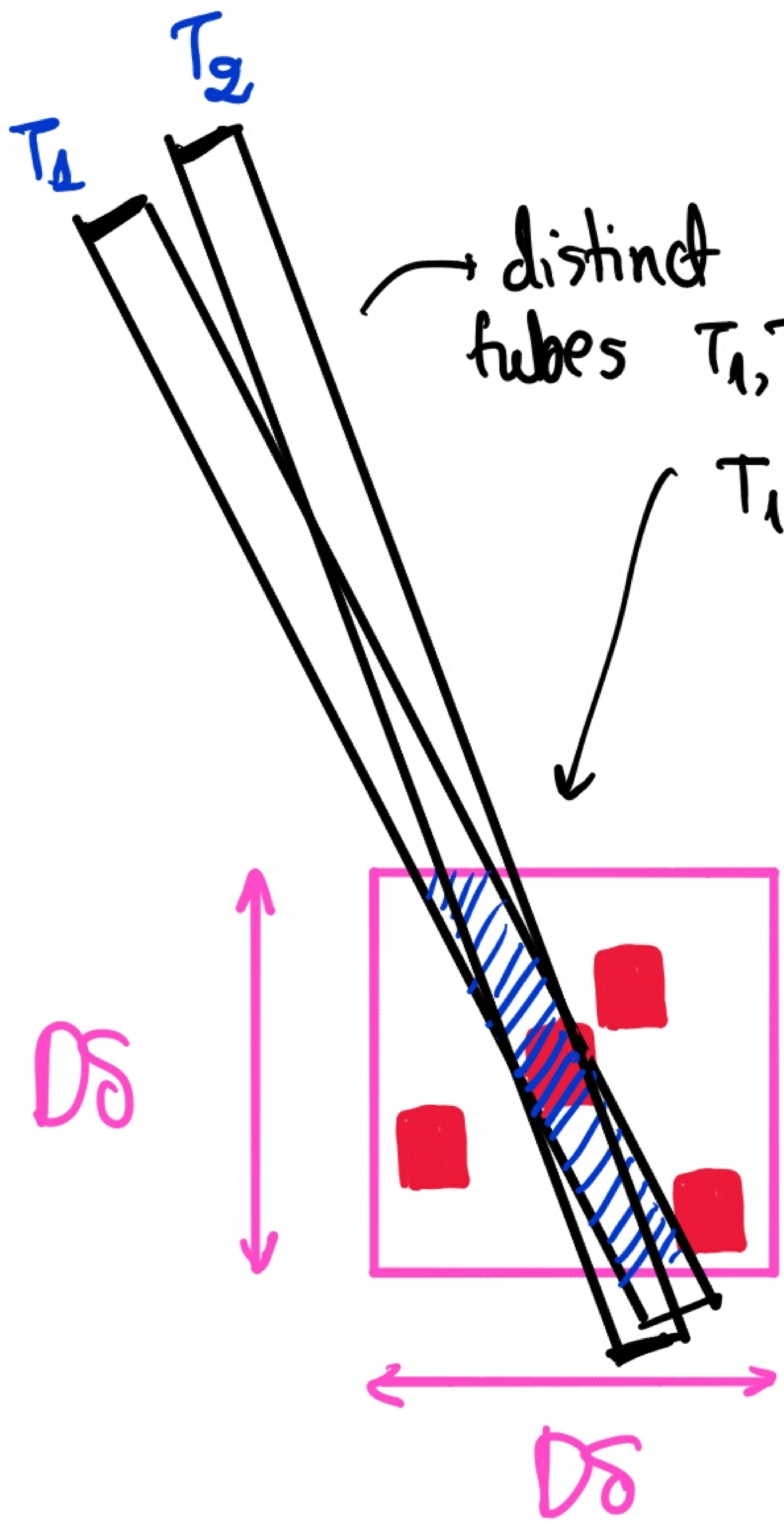
Inside Q :



Inside Q :



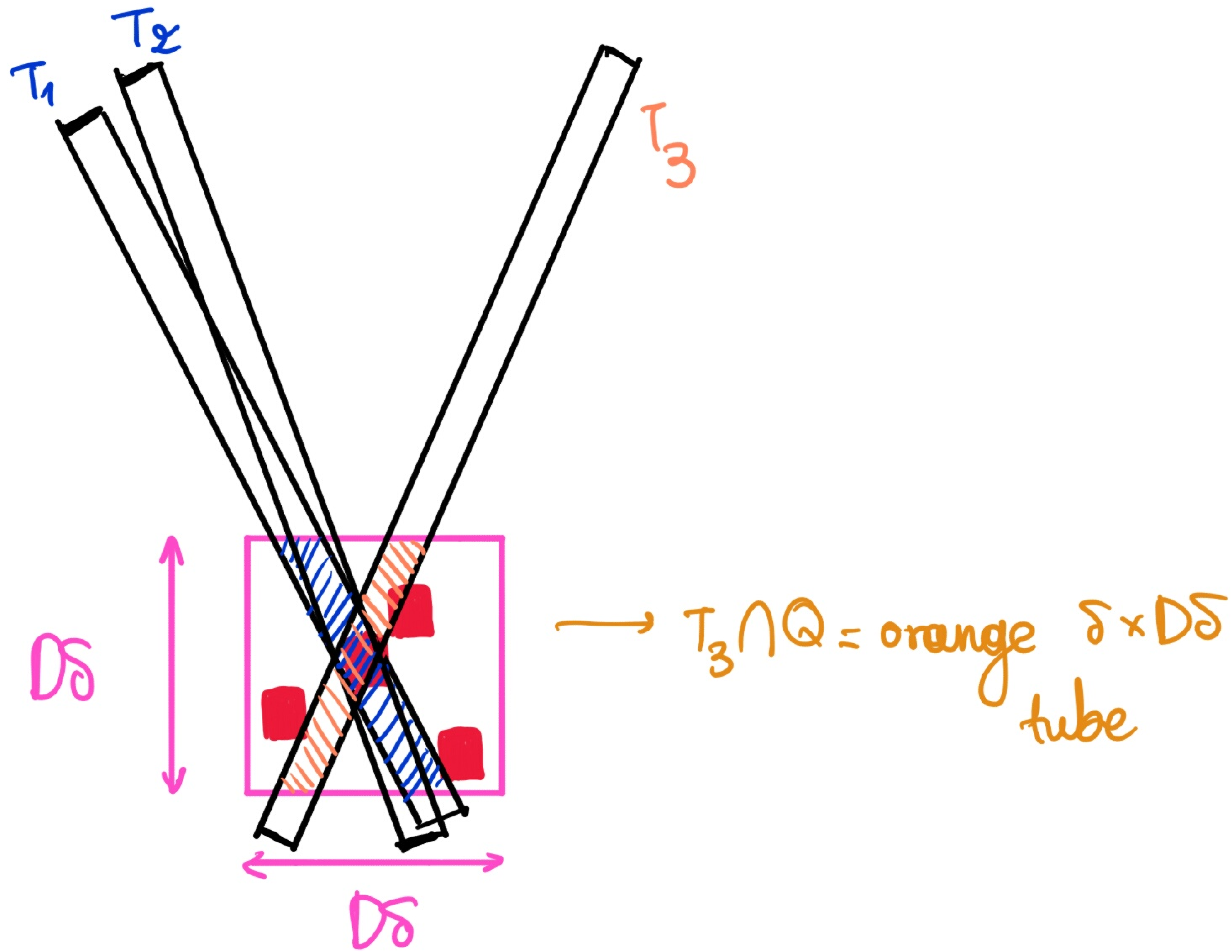
Inside Q :



distinct tubes $T_1, T_2 \in \mathbb{T}$

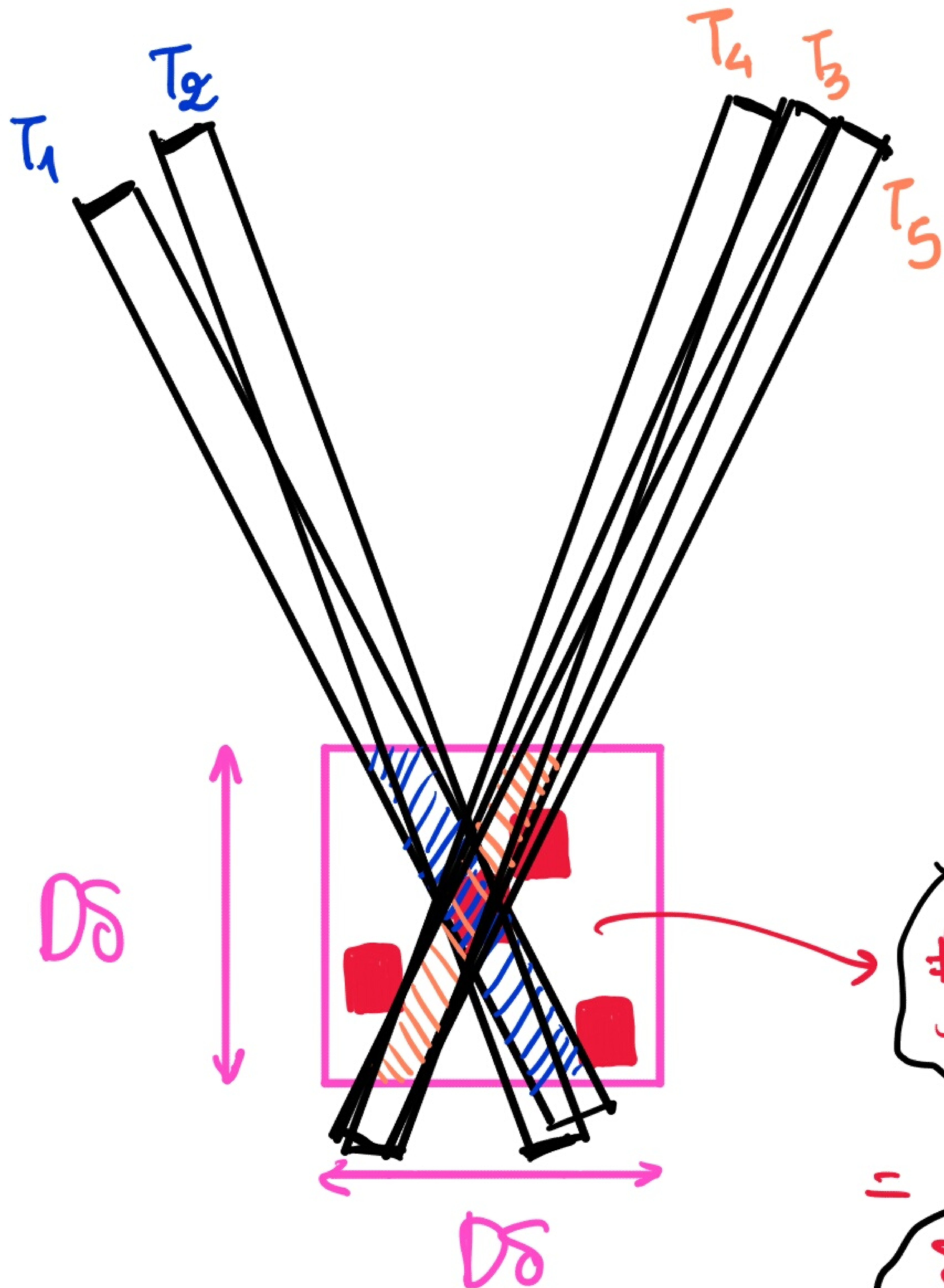
$T_1 \cap Q = T_2 \cap Q =$ the blue $\delta \times D\delta$ tube

Inside Q :



T_1
T_2
T_3
$D\delta$
$D\delta$
$T_3 \cap Q$ = orange $\delta \times D\delta$ tube

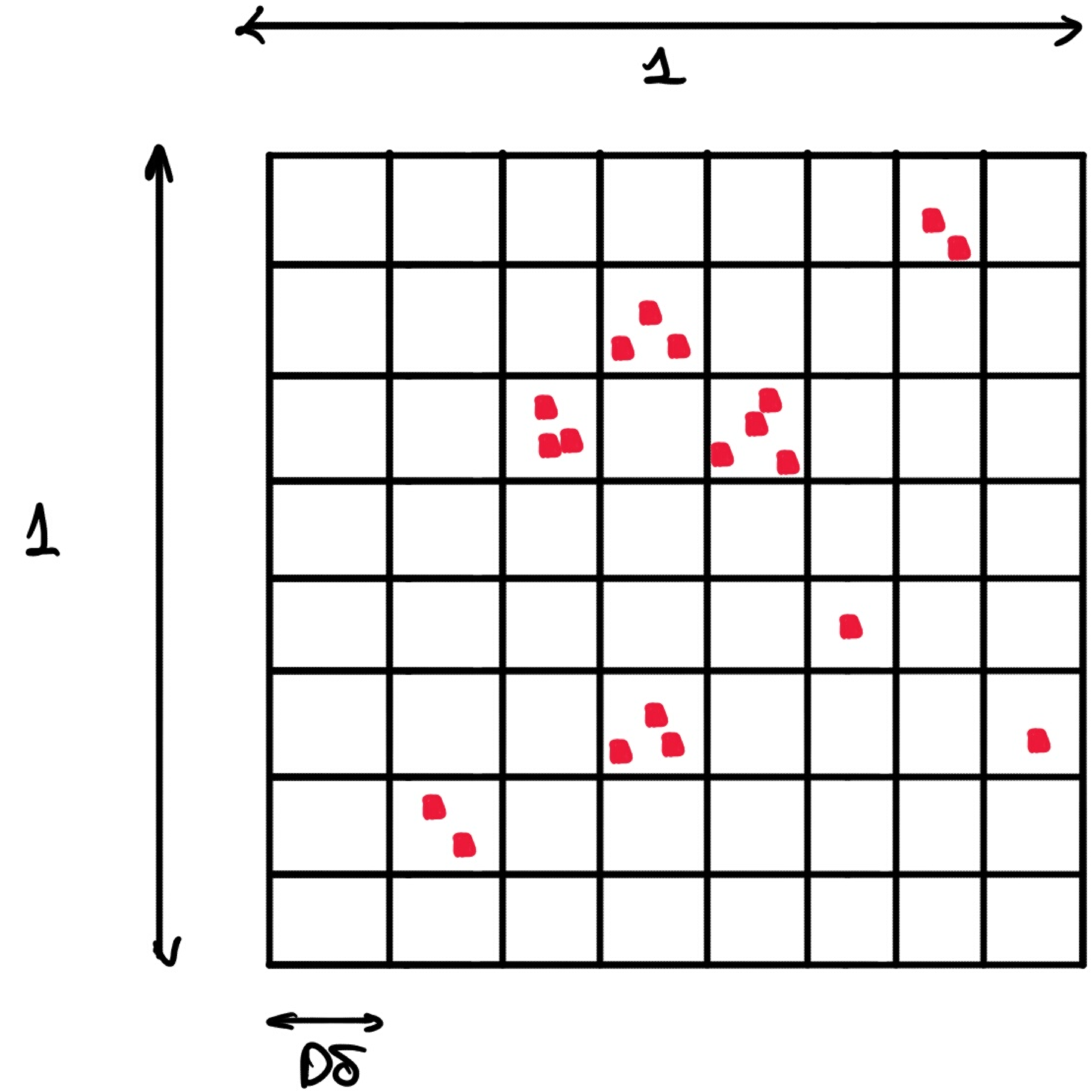
Inside Q:



all T_3, T_4, T_5 have <u>same</u> intersection with Q: the orange $\delta \times D\delta$ - tube.

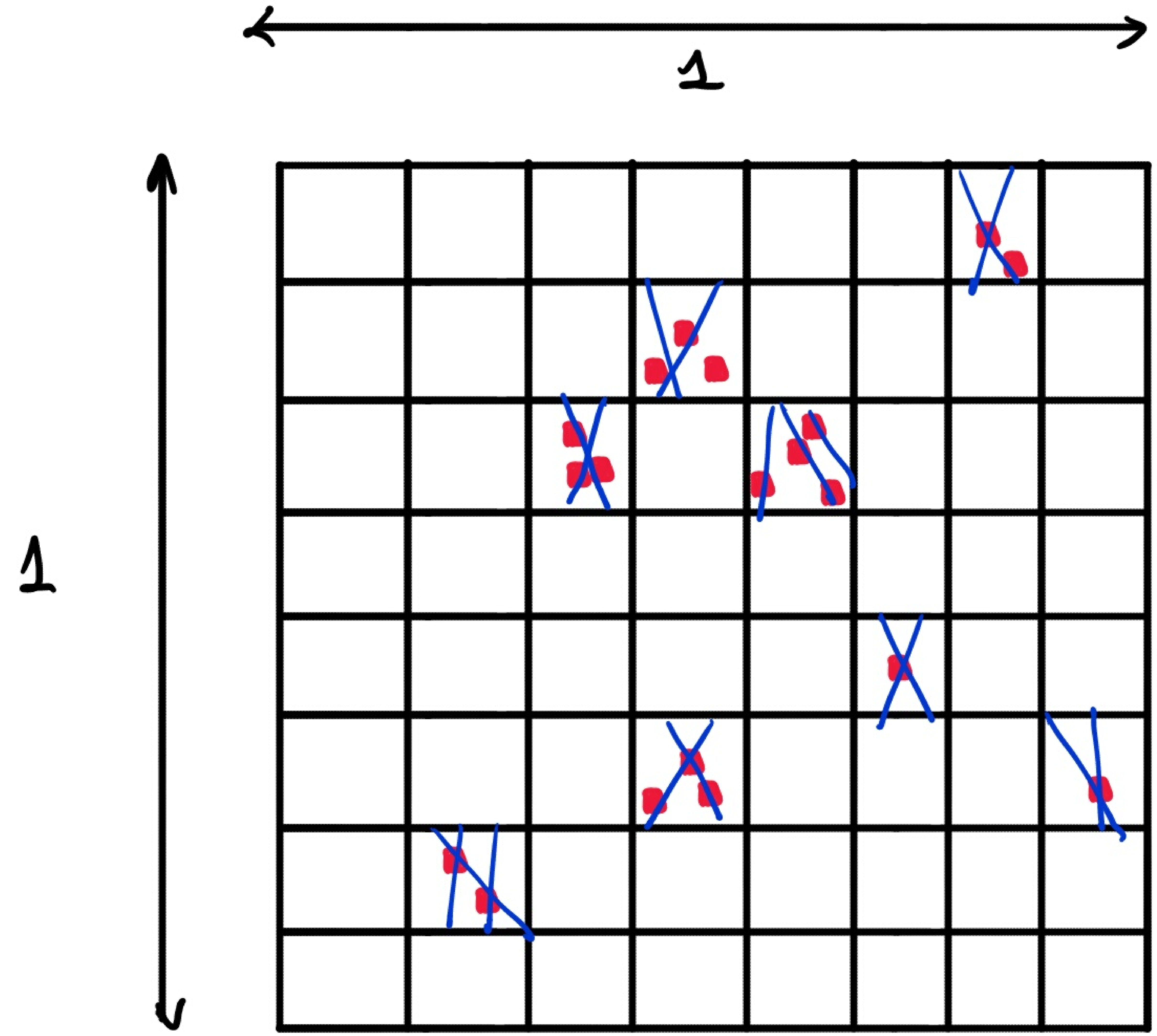
$$\underbrace{\#\text{ of tubes through } q}_{=r} = \sum_{\substack{\delta\times D\delta \\ \text{tubes through} \\ q}} \#\{\delta\text{-tubes that fully contain the } \delta\times D\delta \text{ tube}\}.$$

Rough plan:



Rough plan:

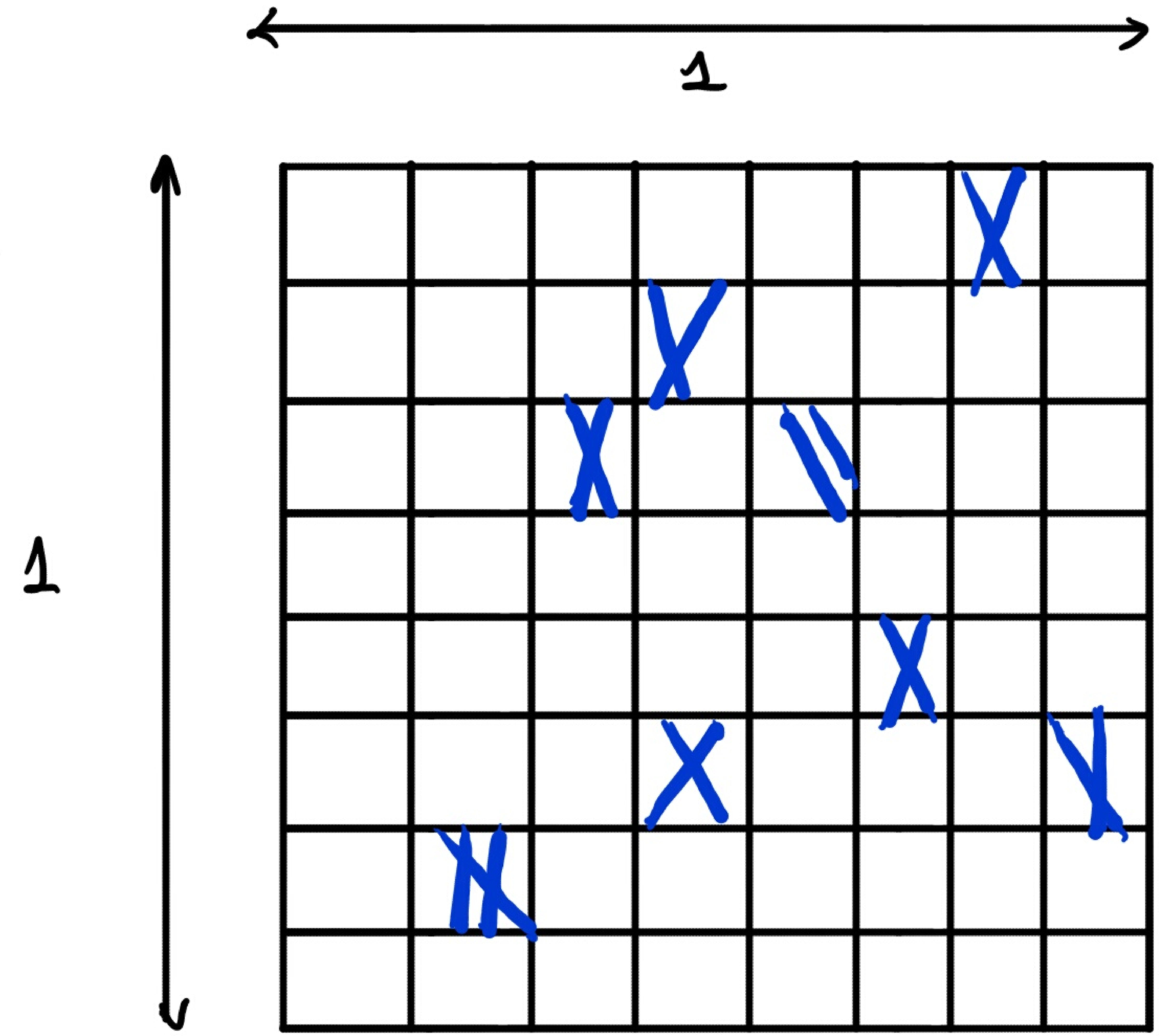
In each $D\delta \times D\delta$ cube Q, control # of δ-cubes via # short tubes through them.



Rough plan:

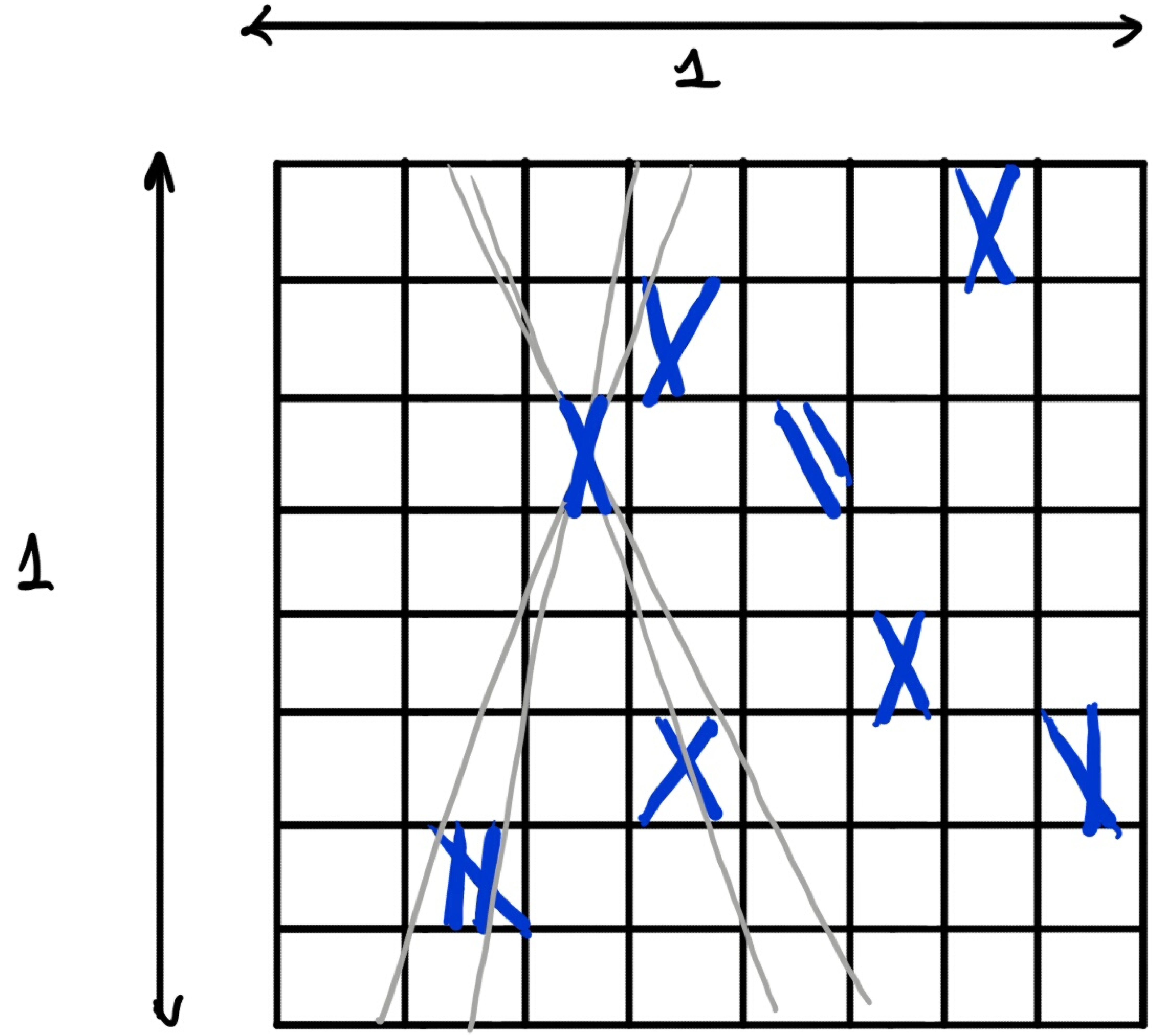
In each Q, control # of δ-cubes by # of short tubes through them.

We control # of short tubes via # of long tubes containing them.

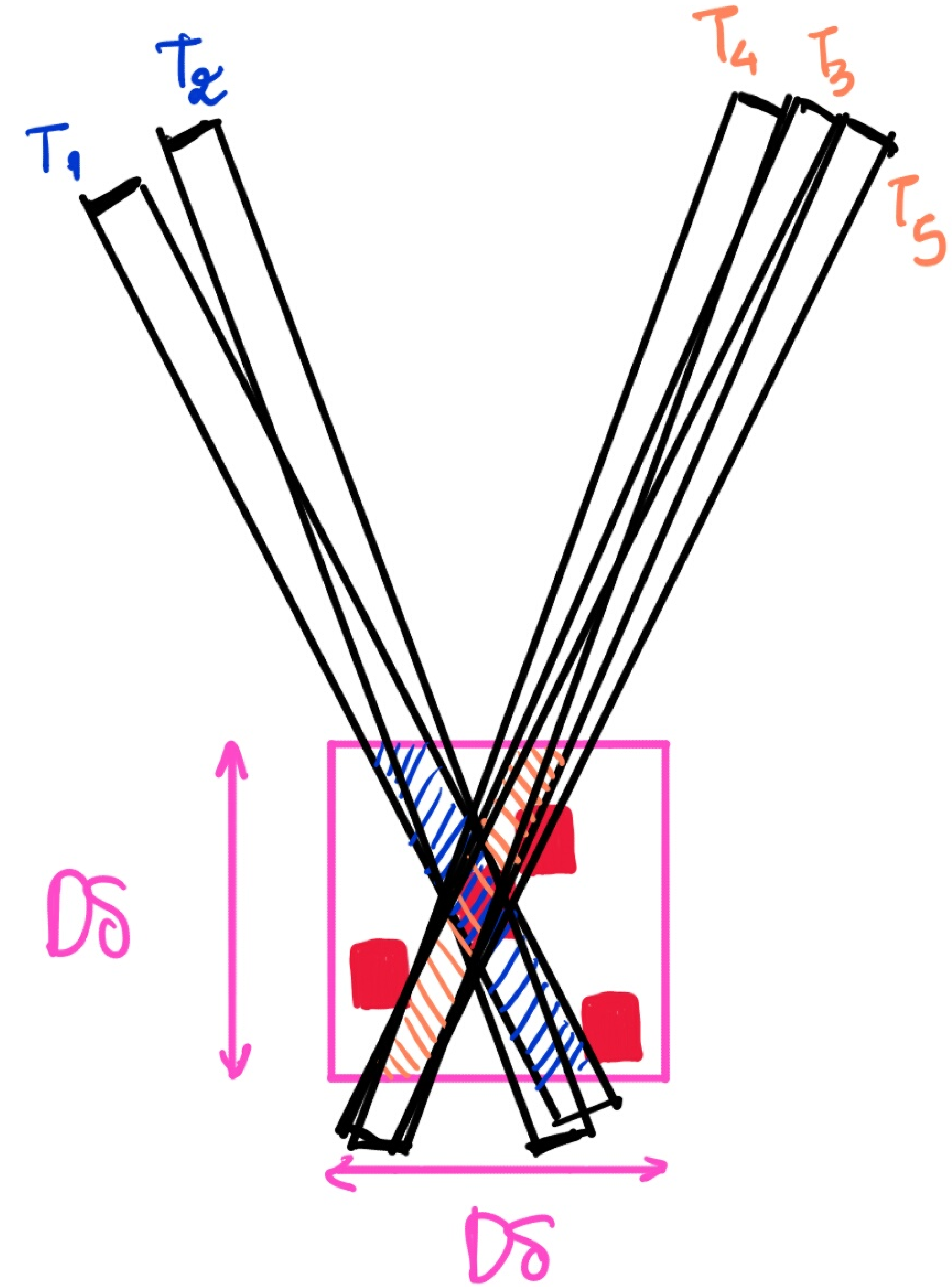


Rough plan:

In each Q, control # of δ-cubes by # of short tubes through them.



Ex: blue short tube is "2-rich" for $\mathbb{T}$, short orange tube is "3-rich".



Arrange short tubes according to # of long tubes containing each,

arrange δ-cubes according to # of short tubes containing each.

$\forall$ M dyadic, let

$\mathbb{T}_{Q,M} := \{ \delta \times D\delta$ tubes T_Q, s.t. T_Q lies in $\sim M$ δ-tubes in $\mathbb{T} \}$.

$P_Q := \{ q \in \underbrace{P_r(\mathbb{T})}_{=P} : q \in Q \}$

$\forall$ E dyadic, let

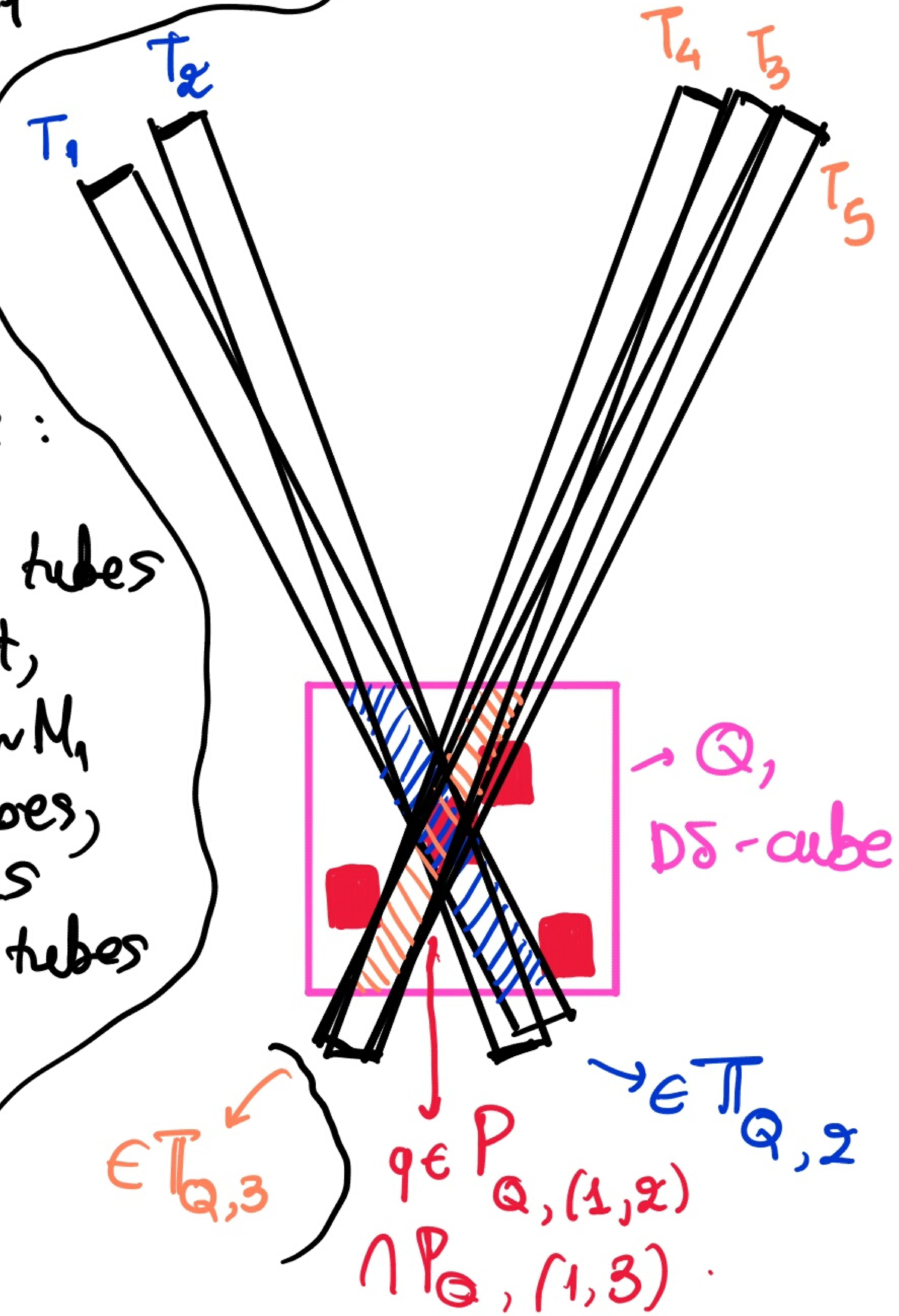
$P_{Q,(E,M)} := \{ q \in P_Q : q$ lies in $\sim E$ (short) tubes in $\mathbb{T}_{Q,M} \}$.

$r = E_1 M_1 + E_2 M_2$

$E_i M_i \leq r$

$\Uparrow$

a red cube: say has $\sim E_1$ short tubes through it, each in $\sim M_1$ long tubes, also has $\sim E_2$ short tubes through it, each in $\sim M_2$ long tubes.



$$|P| \cdot r \sim \sum_Q |P_Q| \cdot r \sim \sum_Q \sum_{(E,M)} |P_{Q,(E,M)}| \cdot EM$$

$$\sim \sum_{(E,M)} \sum_Q |P_{Q,(E,M)}| \cdot EM$$

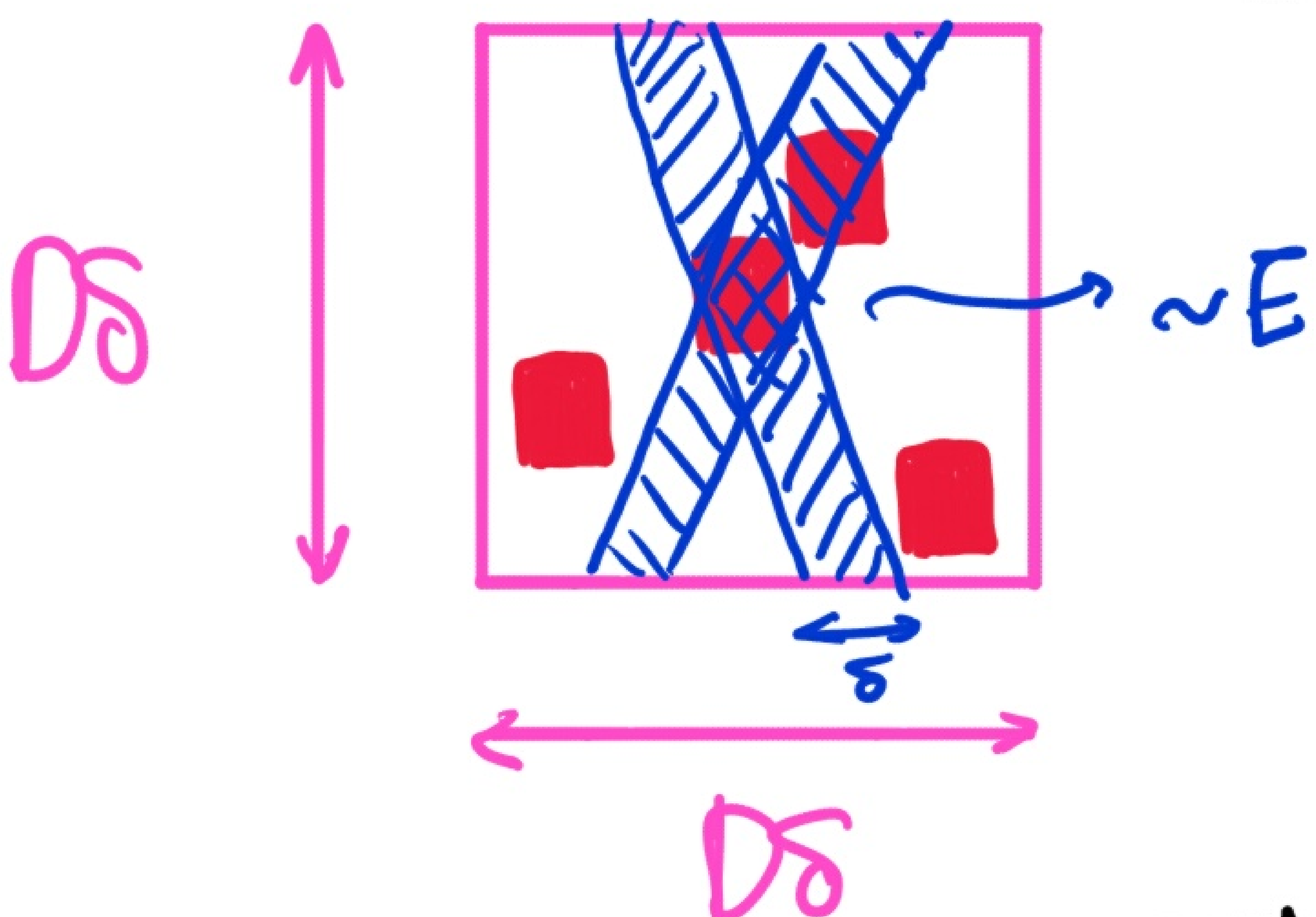
Pick a dominant (E,M): $|P| \cdot r \approx \sum_Q \underbrace{|P_{Q,(E,M)}|}_{\overset{!!}{P_Q}} \cdot \underbrace{EM}_{\leq r} \leq \underbrace{\left(\sum_Q |P_Q|\right)}_{\leq |P|} \cdot r$

we may assume that each of our δ-cubes lies in $\sim E$ short tubes, and each short tube lives in $\sim M$ long tubes

$\rightarrow$ $|P| \lessapprox \sum_Q |P_Q|$, $EM \approx r$. Also: $E \lessapprox D^{n-1}$.

$$|P| \lessapprox \sum_Q \underbrace{|P_Q|}_{\delta\text{-cubes},\ \in P_E(\underbrace{\mathbb{T}_{Q,M}}_{=:\ \mathbb{T}_Q})}$$

the short tubes are not necessarily well-spaced. $\Rightarrow$ we will use the unconditional incidence thm to bound $|P_Q|$ in each Q



thin case in Q — thick case in Q

$$\Rightarrow \quad |P| \lessapprox \underbrace{\sum_{\substack{\text{thin case}\\ \text{in } Q}} |P_Q|}_{=:\ A} + \underbrace{\sum_{\substack{\text{thick case}\\ \text{in } Q}} |P_Q|}_{=:\ B}$$