If the low frequency term dominates:

$S = D^{\varepsilon/10n}$

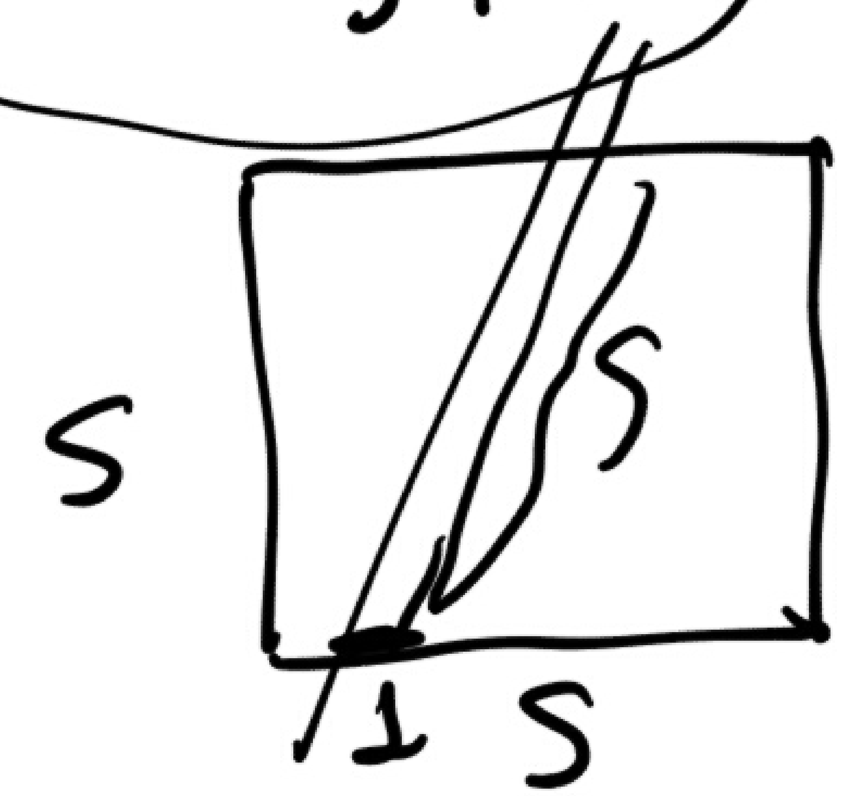
$$|P|\cdot r \lessapprox \int \bar{\hat f}\,\hat g\cdot\eta = \int (f * \eta^\vee)\cdot g = \int \left[\left(\sum_{q\in P_\lambda}\psi_q\right) * \eta^\vee\right]\cdot\left(\sum_{T\in\mathbb{T}}\psi_T\right)$$

$$= \sum_{q\in P_\lambda}\sum_{T\in\mathbb{T}}\int_{N_S(q)} \overbrace{\underbrace{(\psi_q * \eta^\vee)}_{\text{supp. in } N_S(q)}}^{\lesssim \rho^n}\cdot\,\psi_T$$

η is ess. 1 on $B(0,\rho) \rightarrow |\eta^\vee(\xi)| \lesssim |B(0,\rho)| \sim \rho^n$,

$$\lesssim \sum_{q\in P_\lambda}\rho^n\cdot\sum_{T\in\mathbb{T} \text{ that intersect } N_S(q)} |T\cap N_S(q)| = S$$

also: $\eta^\vee$ is ess. supp. in $B(0,\overset{\lesssim S}{\rho^{-1}})$, decays rapidly outside $B(0,S)$.



So: $|P|\cdot r \lessapprox \sum_{q\in P_\lambda} \underbrace{\#\{T\in\mathbb{T} : T\cap N_S(q)\neq\emptyset\}}_{\approx\lambda}\cdot \rho^n S$

$$\approx |P_\lambda|\cdot\lambda\rho^n\cdot S \lessapprox |P|\cdot\lambda\cdot D^{n\varepsilon^3}\cdot\frac{1}{S^{n-1}}$$

$\rho = D^{\varepsilon^3}/S$ $\qquad\qquad$ $S = D^{\varepsilon/10n}$

$$\to \quad \lambda \gtrapprox D^{-n\varepsilon^3}\left(S^{n-1}\cdot r\right)$$

So: $\bigcup_{q\in P_\lambda} \boxed{N_S(q)}$ contains the cubes in P_λ, i.e.

$\gtrapprox |P|$ of the cubes in P.

Take finitely overlapping $2S$-cubes covering $\bigcup_{q\in P_\lambda} N_S(q)$ ✓