Prop: Let $\mathbb{P}$ be a set of δ-cubes, $\mathbb{T}$ a family of of distinct δ-tubes in $[0,1]^n$. Suppose $\mathbb{P} \subseteq \mathbb{P}_r(\mathbb{T})$.

Then, $\forall \varepsilon > 0$, for $S = \left(\frac{1}{\delta}\right)^{\varepsilon/10n}$, either:

Thin case: $$|\mathbb{P}| \lessapprox_n S^n \cdot \frac{1}{\delta^{n-1}} \cdot \frac{|\mathbb{T}|}{r^2}$$, or

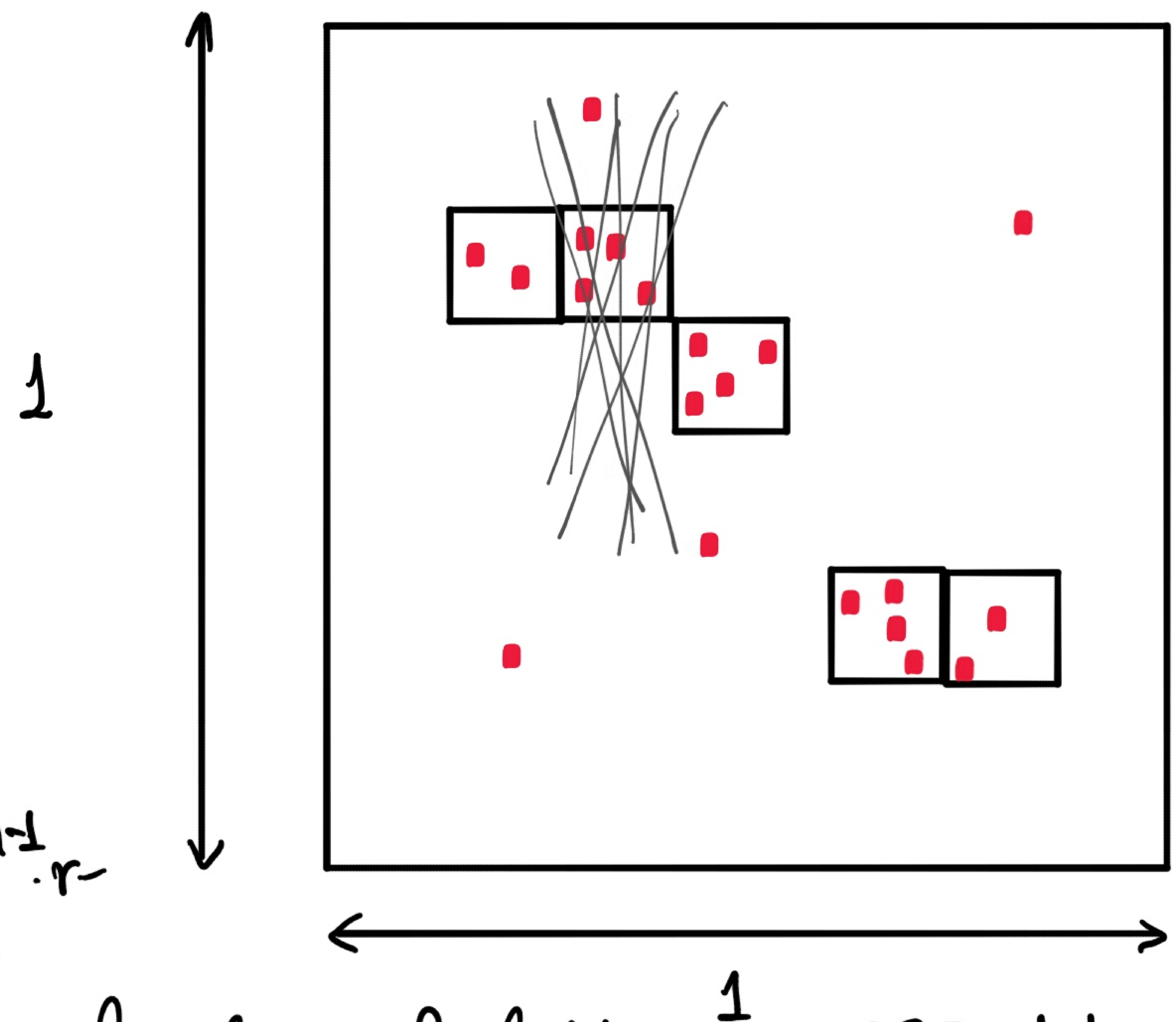
Thick case: There exist fin. overlapping $2S\delta$-cubes Q_j s.t.

(1) $\bigcup Q_j$ contain $\gtrapprox_n |\mathbb{P}|$ δ-cubes of $\mathbb{P}$.

(2) Each Q_j intersects $\gtrapprox_n S^{n-1} r$ tubes of $\mathbb{T}$.

$\geq C(\varepsilon, n) \left(\frac{1}{\delta}\right)^{10n\varepsilon^3}$

Thick case:



$\sim r$ δ-tubes through each

: δ-cube in $P \subseteq P_r(\mathbb{T})$.

: 25δ-cube

$\gtrapprox S^{n-1} r$ δ-tubes through each.

$\square$ is $\gtrapprox S^{n-1} \cdot r$-rich for $\tilde{\mathbb{T}}$, the family of fattened 25δ-tubes, as long as distinct.

Proposition becomes: $\rightsquigarrow$ $D = \frac{1}{\delta}$, dilate by D

Let P be a set of unit cubes and $\mathbb{T}$ a set of distinct tubes of radius 1 and length D in $[0,D]^n$.

Let $P \subseteq P_r(\mathbb{T})$. Then, $\forall \varepsilon > 0$, for $S = D^{\varepsilon/10n}$, either

Thin case: $|P| \lesssim_n S^n \, D^{n-1} \cdot \frac{|\mathbb{T}|}{r^2}$, or

Thick case: $\exists$ fin. overlapping $2S$-cubes Q_j, s.t.

(1) $\cup Q_j$ contains $\gtrsim_n |P|$ of the 1 cubes in P,

(2) Each Q_j intersects $\gtrsim_n S^{n-1} r$ tubes of $\mathbb{T}$.

$\gtrsim C(\varepsilon,n) \cdot D^{-10n\varepsilon^3}$

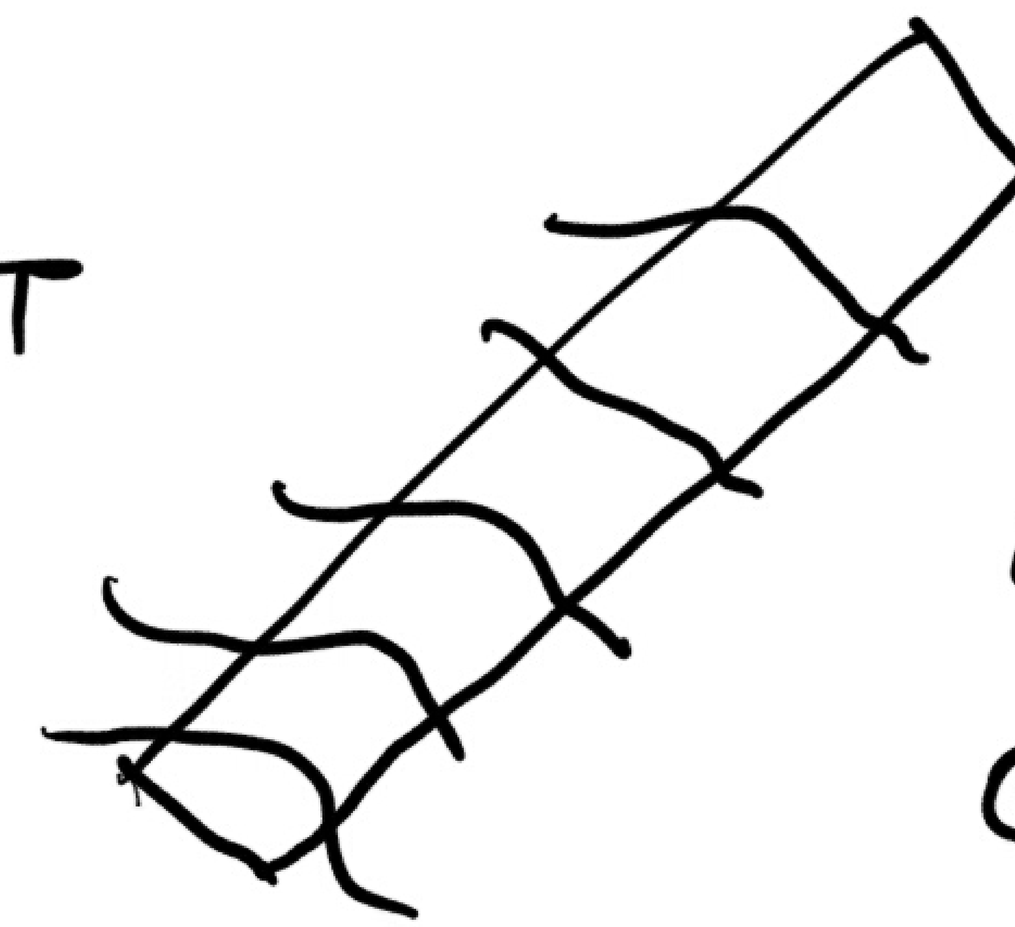
Proof: May assume (pigeonholing) that $\forall q \in P$,
$N_\delta(q)$ is intersected by the same number of δ-tubes:
Let $P_\lambda := \{ q \in P : N_\delta(q)$ intersected by $\sim \lambda$ δ-tubes$\}$.

$$\Rightarrow \exists \lambda : |P| \approx |P_\lambda| \quad \left(\Rightarrow I(P,\mathbb{T}) \sim |P| \cdot r \approx |P_\lambda| \cdot r \sim I(P_\lambda, \mathbb{T}) \right)$$

[⚠ We want: in the thick case, $\lambda \gtrsim \underline{\delta^{n-1} \cdot r}$].

q 

ψ_q : bump, 1 on q,
0 outside 2q.

T 

ψ_T : bump,
1 on T,
0 outside 2T.

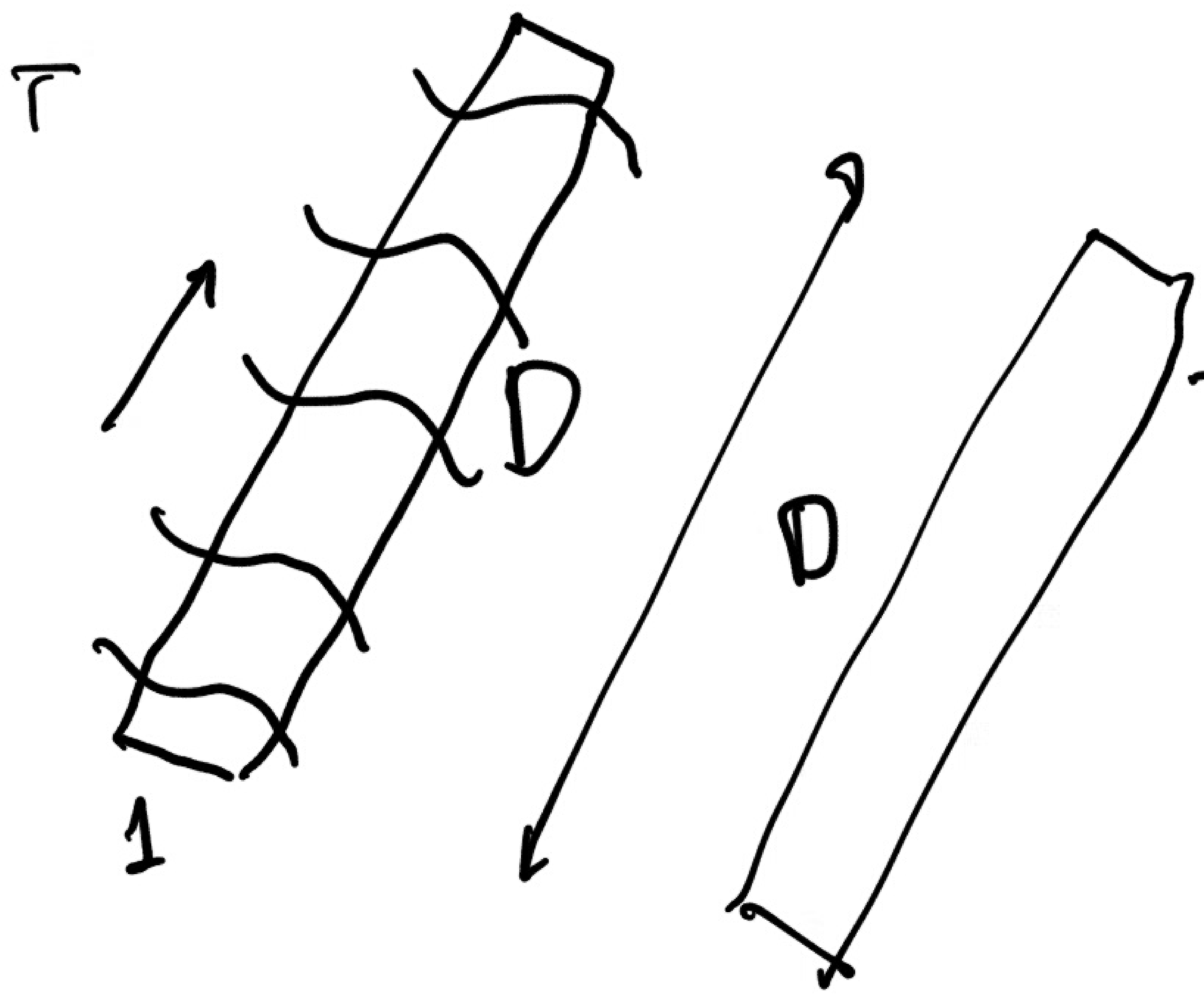
(We say that $q \cap T \neq \phi$ if $|q \cap T| \gtrsim \frac{1}{2}|q|$.)

$\rightarrow$ if $q \cap T \neq \phi$, then $\int \psi_q \cdot \psi_T \gtrsim 1$.

$$\rightarrow \underset{\underset{|P|\cdot r}{\rotatebox{90}{$\lesssim$}}}{I(P_\lambda, \Pi)} \lesssim \sum_{q \in P_\lambda} \sum_{T \in \Pi} \int \psi_q \cdot \psi_T \sim \int \underbrace{\left(\sum_{q \in P_\lambda} \psi_q\right)}_{=f} \underbrace{\left(\sum_{T \in \Pi} \psi_T\right)}_{=g}$$

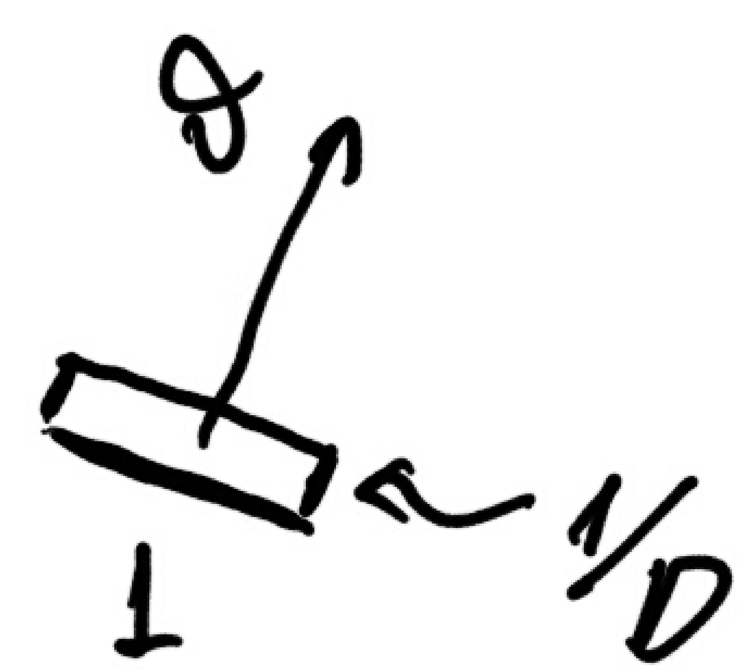
$$= \int \bar{\hat{f}} \cdot \hat{g} \; .$$

Ideas? $\quad |P|\cdot r \lessapprox \int \overline{\hat{f}}\cdot \hat{g}\,, \qquad \hat{g} = \sum_{T\in\mathbb{T}} \hat{\psi}_T\,.$



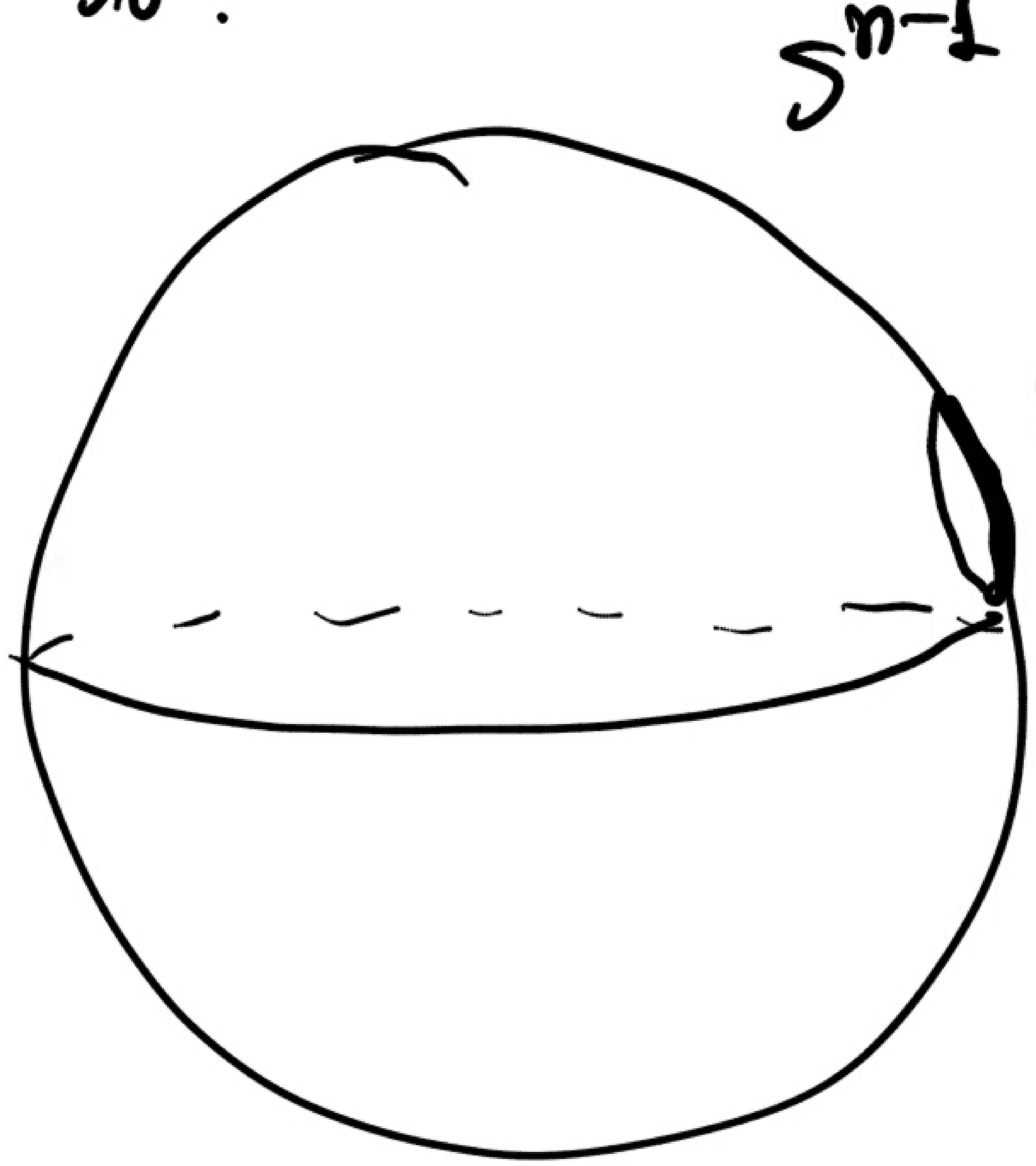
$\hat{\psi}_T$ ess. supp. in the dual object to T through 0:
$\hat{\psi}_T$ also ess. supp there.

$\Rightarrow \sum_{T \parallel \theta} \hat{\psi}_T$ ess. supp there.



$1\times 1\times\cdots\times 1\times \frac{1}{D} =: \theta^*$

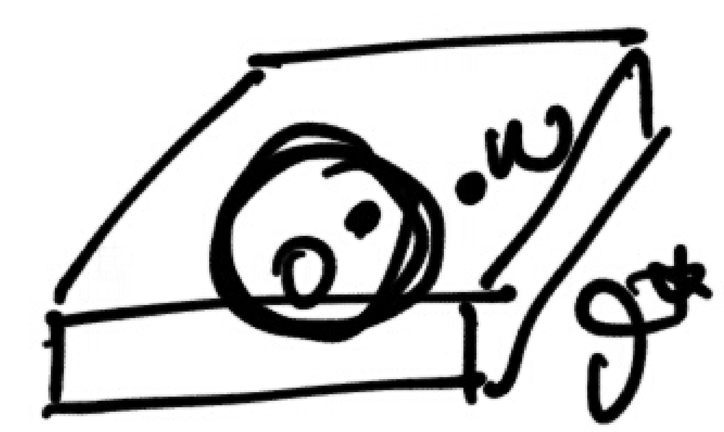
So:



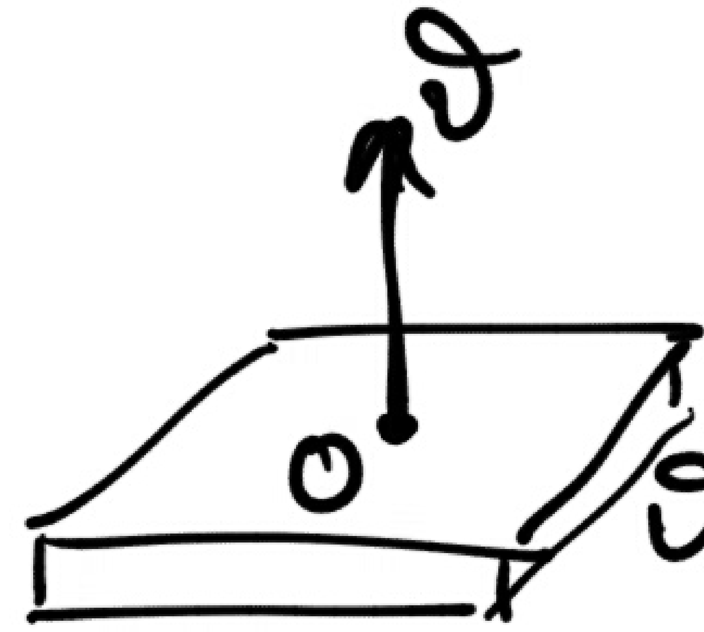
$\mathbb{T}_\theta := \{T \in \mathbb{T} : T \parallel \theta\}$

θ, $1/D$-cap. $\quad g_\theta := \sum_{T \in \mathbb{T}_\theta} \psi_T$

very few θ^* contain ω



$\hat{g}_\theta$ ess. supp. in θ^*:



θ^*, $1 \times \dots \times 1 \times 1/D$ slab.

C-S?

$$|P| \cdot r \lessapprox \underline{\int \bar{\hat{f}} \cdot \hat{g}} \sim \int \bar{\hat{f}} \cdot \Big(\sum_\theta \hat{g}_\theta\Big) = \int \bar{\hat{f}}(\omega) \cdot \underline{\overbrace{\Big(\sum_\theta \hat{g}_\theta(\omega)\Big)}} \, d\omega.$$

very few θ will contribute to this sum, if ω long.

So: $|P|\cdot r \approx \int \bar{\hat{f}}\cdot\hat{g} = \int \bar{\hat{f}}\cdot\hat{g}\cdot\eta + \int \bar{\hat{f}}\,\hat{g}\cdot(1-\eta)$

low frequencies $\leq \rho$

$\Downarrow$

thick case.

high freq. $\geq \rho$

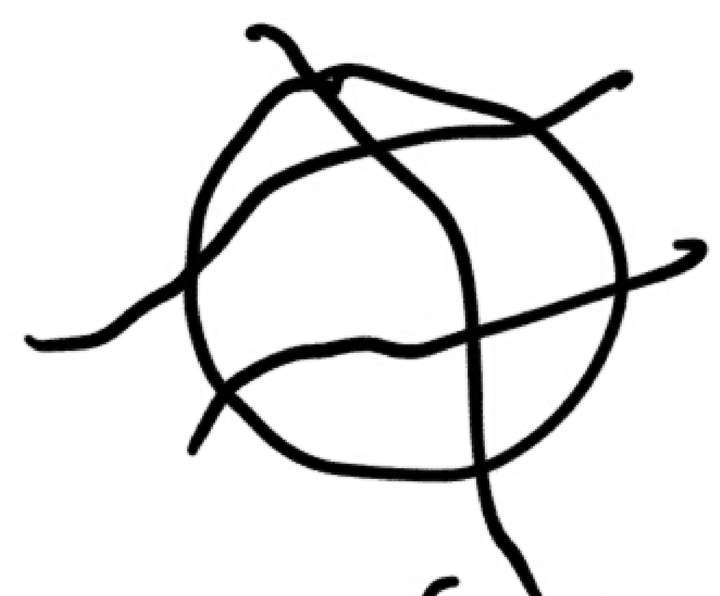
$\Rightarrow$ good incidence est. (thin case)

$$\int \bar{\hat{f}}\,\hat{g}\cdot\eta$$

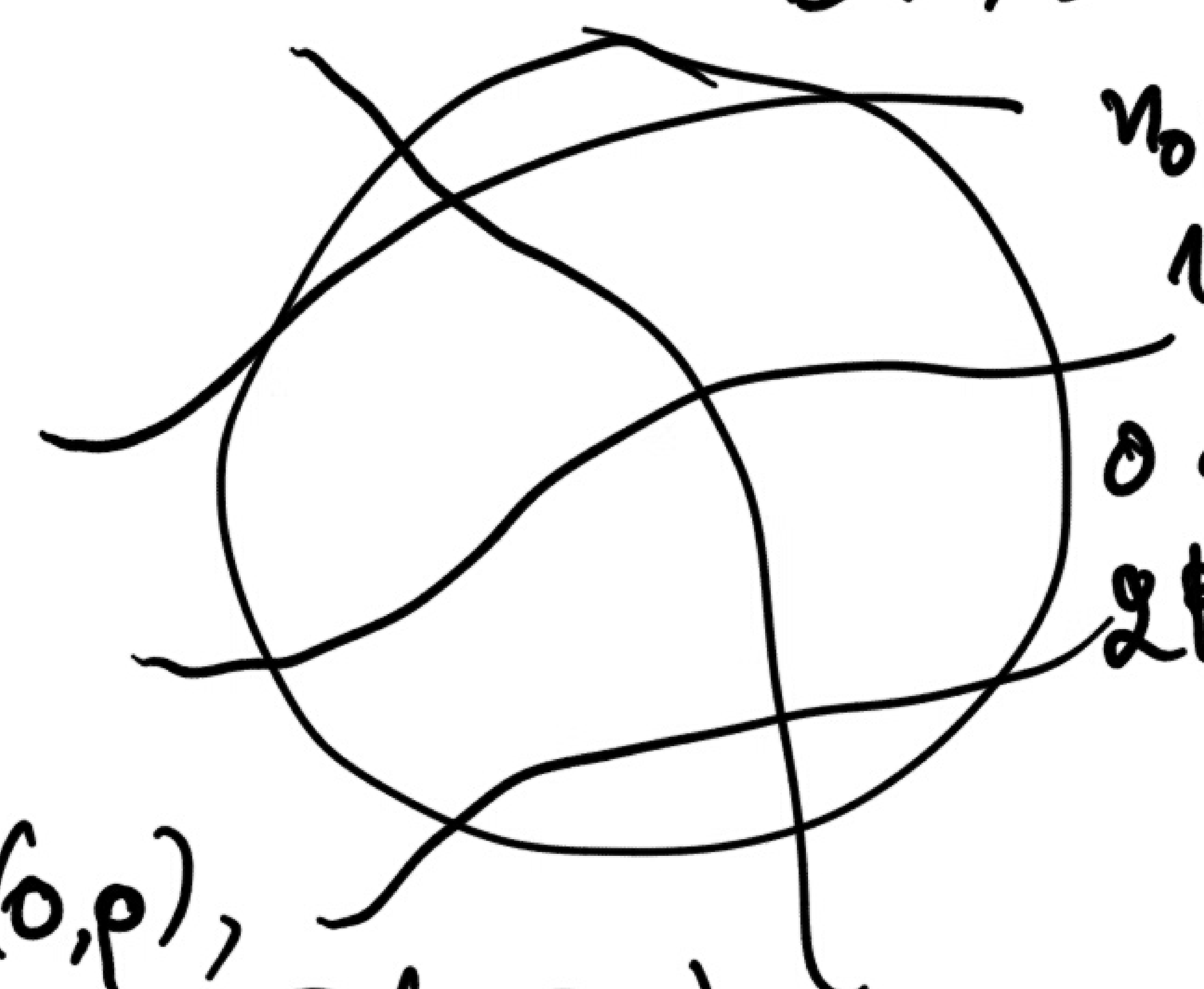
$$= \int (f * \eta^\vee)\cdot g$$

$$f = \sum_{q\in P} \varphi_q$$

$B(0,\rho)$



$B(0,1)$



η_0: bump, 1 on $B(0,1)$, 0 outside $2B(0,2)$

$\eta := \eta_0\left(\frac{\cdot}{\rho}\right)$, 1 on $B(0,\rho)$, 0 outside $B(0,2\rho)$.

$\longrightarrow$ If high freq. term dominates: $\underline{|P| \cdot r} \lesssim \underline{\int \overline{\hat{f}} \cdot \hat{g}\,(1-\eta)}$

$$\leq \underbrace{\left(\int |\hat{f}|^2 (1-\eta) \right)^{1/2}} \left(\int |\hat{g}|^2 (1-\eta) \right)^{1/2}$$

$$\leq \left(\int |\hat{f}|^2 \right)^{1/2} = \left(\int |f|^2 \right)^{1/2}$$

$$\approx |P_\lambda|^{1/2}$$

$$\lesssim \underline{|P|^{1/2}}$$

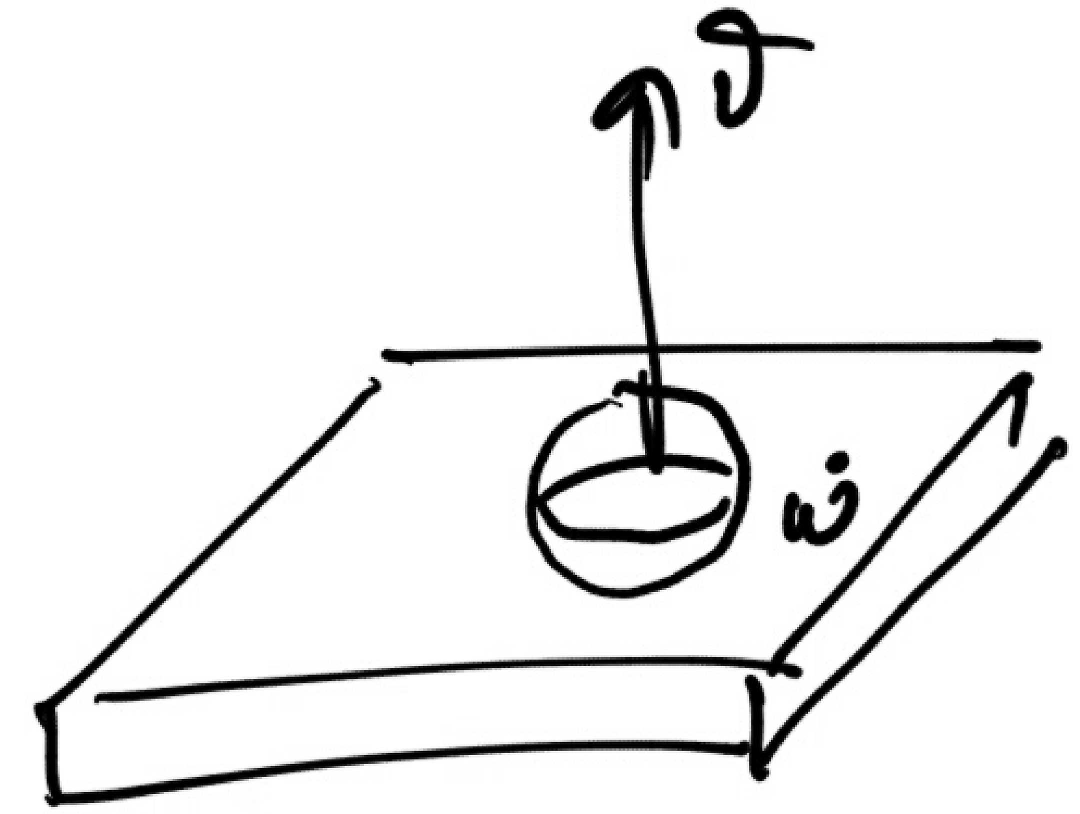
$$f = \sum_{q \in P_\lambda} \psi_q$$

$$\int |\hat{g}(\omega)|^2 \, (1-n(\omega))\, d\omega = \int \underbrace{\Big|\sum_\vartheta \hat{g}_\vartheta(\omega)\Big|^2}_{\text{How many } \vartheta\text{'s contribute to sum } \forall \text{ given } \omega, \text{ for } |\omega| \gtrsim \rho \,?} \cdot \underline{(1-n(\omega))}\, d\omega$$

$g = \sum_\vartheta g_\vartheta$

$g_\vartheta = \sum_{T \in \mathbb{T}_\vartheta} \psi_T$

$\hat{g}_\vartheta$ ess. supp on ϑ^* (negligible outside $\underline{D^{\varepsilon^3}\vartheta^*}$).

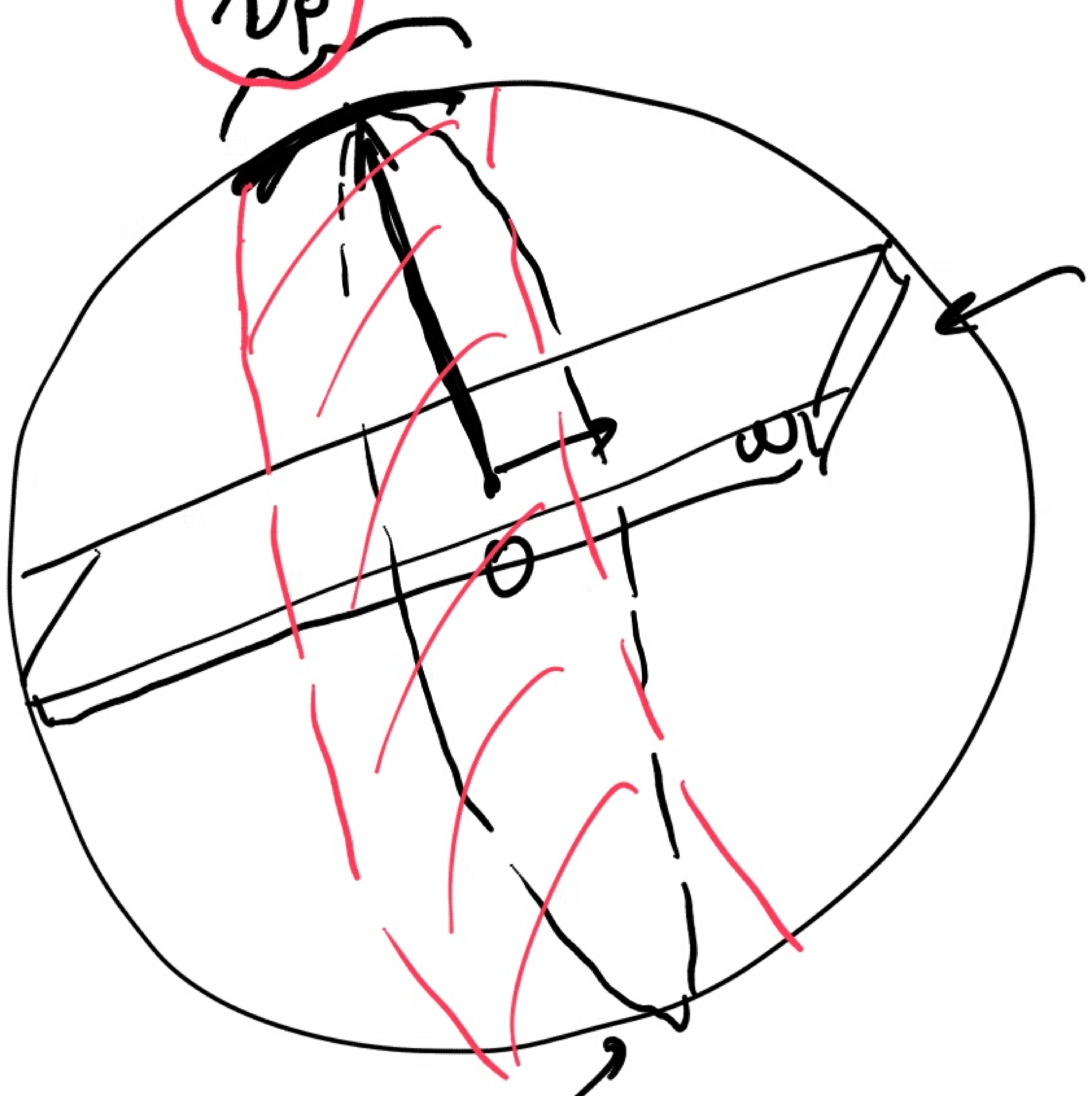


$1 \times \dots \times 1 \times 1/D$

the ϑ's that contribute to $\sum_\vartheta \hat{g}_\vartheta(\omega)$ are as many as the $D^{\varepsilon^3}\vartheta^*$ containing ω: the ϑ's are the normals to these slabs.

How many $D^{\varepsilon^3}\vartheta^*$ contain ω? $|\omega|\sim p$

D^{ε^3}/Dp



$1\times 1\cdots\times 1\times 1/D$ - slab containing ω

The normals to the ϑ^* containing ω live in the red strip of thickness $1/Dp$ on S^{n-1}

$S^{n-2}\perp\omega$

$$\Longrightarrow \frac{D^{\varepsilon^3}/Dp}{\left(\frac{1}{D}\right)^{n-1}} \sim \frac{D^{\varepsilon^3}}{p}\cdot D^{n-2}$$ different normals, also diff. $D^{\varepsilon^3}\vartheta^*$ containing ω.

They use: $\lesssim p^{-n} D^{n-2+n\varepsilon^3}$

So, $\forall \omega : |\omega| \gtrsim \rho$, $\quad \Big| \underbrace{\sum_\theta \hat{g}_\theta(\omega)}_{\hat{g}(\omega)} \Big|^2 \lesssim \#\{\theta\text{'s contributing}\} \cdot \Big(\sum_\theta |\hat{g}_\theta(\omega)|^2 \Big)$

$$\lesssim \rho^{-n} D^{n-2+n\varepsilon^3} \cdot \sum_\theta |\hat{g}_\theta(\omega)|^2 .$$

So, $\quad |P| \cdot r \lessapprox |P|^{1/2} \cdot \Big(\rho^{-n} D^{n-2+n\varepsilon^3} \cdot \sum_\theta \int |g_\theta|^2 \Big)^{1/2}$

$g_\theta = \sum_{T \in \mathbb{T}_\theta} \psi_T \qquad \sim |P|^{1/2} \Big(\rho^{-n} D^{n-2+n\varepsilon^3} \cdot \sum_\theta \sum_{T \in \mathbb{T}_\theta} \underbrace{\int |\psi_T|^2}_{|T| \sim D} \Big)^{1/2}$

$$\sim |P|^{1/2} \Big(\rho^{-n} D^{n-1+n\varepsilon^3} \cdot |\mathbb{T}| \Big)^{1/2}$$

So, $$|P|\cdot r \lessapprox |P|^{1/2}\left(\rho^{-n} D^{n-1+n\varepsilon^3} \cdot |\Pi|\right)^{1/2}$$

$$\Rightarrow |P|\, r^2 \lessapprox \rho^{-n} D^{n-1+n\varepsilon^3} |\Pi|$$

$$\rightarrow |P| \lessapprox \underbrace{\rho^{-n} D^{n\varepsilon^3}}_{= S^n} \cdot D^{n-1} \cdot \frac{|\Pi|}{r^2}$$

Define $$\rho := S^{-1} D^{\varepsilon^3} = D^{\varepsilon^3}\cdot\left(\frac{1}{S}\right). \quad (<1).$$

Trivial: $$|P|\, r^2 \lesssim |\Pi|^2 \cdot D^1.$$