Incidence estimates for well-spaced tubes

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S-T: Let $\mathcal{L}$ a set of L lines in $\mathbb{R}^2$,

$$S_r := \{x \in \mathbb{R}^2 : x \text{ lies in } \underset{\substack{\downarrow \\ \geq r,\ < 2r}}{\sim r} \text{ lines in } \mathcal{L}\}.$$

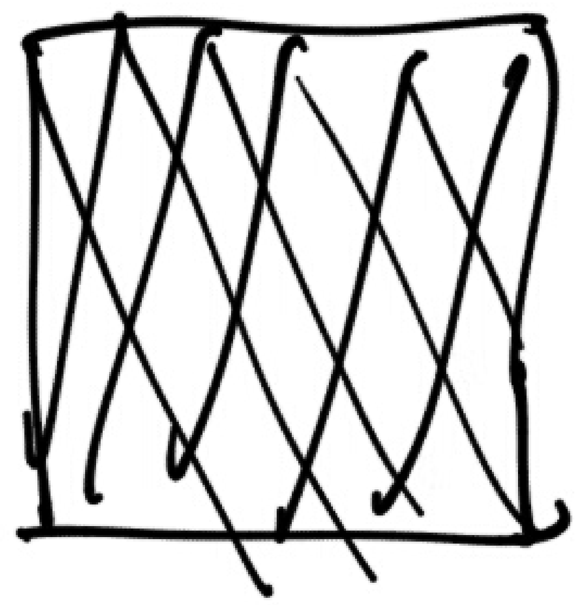
Then: $|S_r| \lesssim \frac{L^2}{r^3} + \frac{L}{r}, \quad \forall r \geq 2.$

i.e. $|S_r| \lesssim \begin{cases} \frac{L^2}{r^3}, & 2 \leq r \lesssim L^{1/2} \\ L/r, & r \gtrsim L^{1/2}. \end{cases}$

Trivial: $|S_r| \cdot r^2 \lesssim L^2$ $\left(\text{LHS} \sim \sum_{x \in S_r} \#\{\text{pairs of lines meeting at } x\} \leq \binom{L}{2} \sim L^2 \right).$

$\rightsquigarrow$ $\mathbb{T}$ a set of distinct δ-tubes in $[0,1]^2$.

($T_1 \neq T_2$ if $|T_1 \cap T_2| < \frac{1}{2}|T_1|$). $\forall\, T_1 \neq T_2 \in \mathbb{T}$, if $T_1 \cap T_2 \neq \emptyset$, then angle$(T_1, T_2) \gtrsim \delta$.



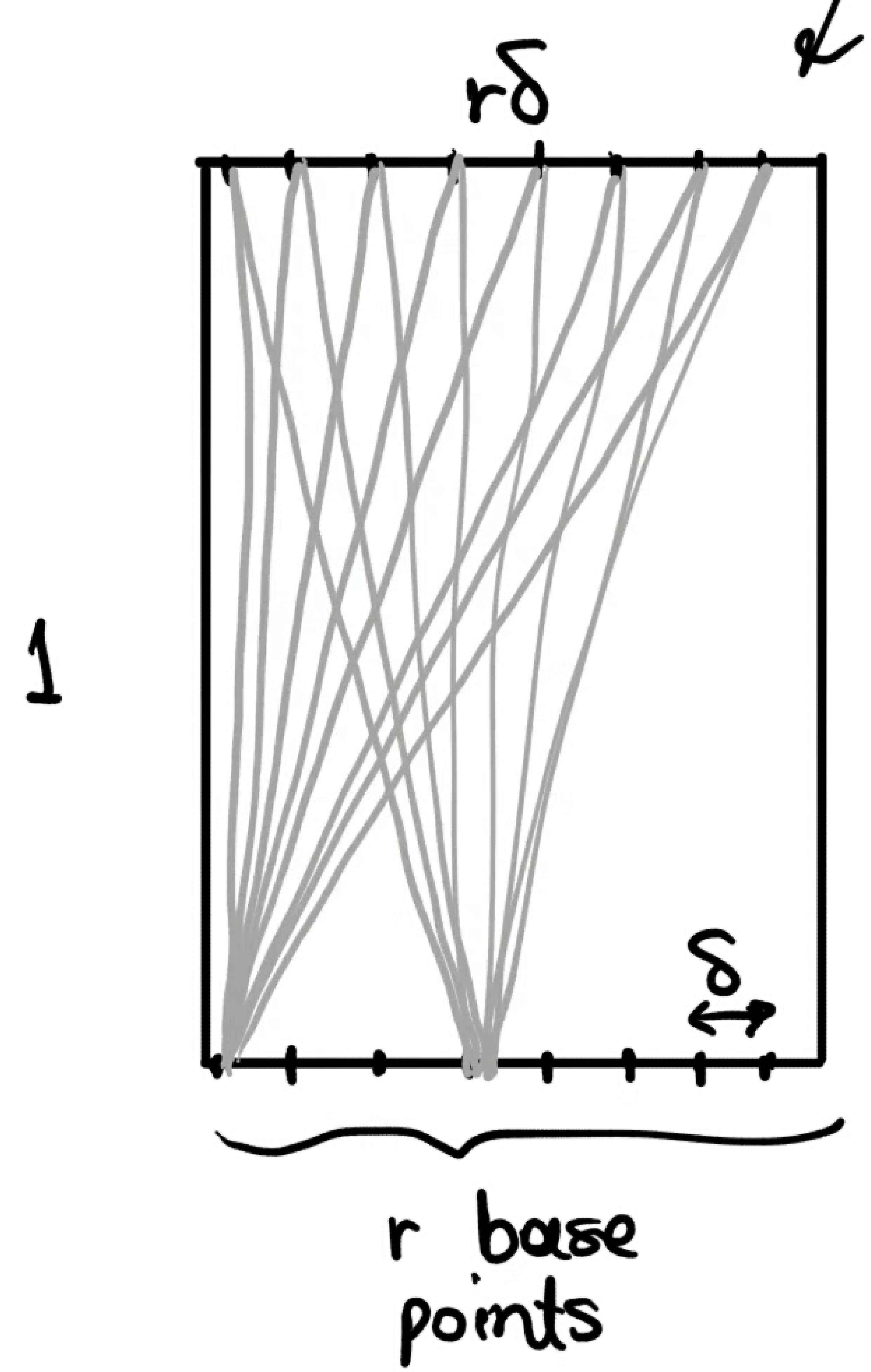
$$|\mathbb{T}| \lesssim \#(\delta\text{-sep directions}) \cdot \#(\text{tubes in each direction}) \sim \frac{1}{\delta} \cdot \frac{1}{\delta} \sim \frac{1}{\delta^2}.$$

$$P_{\underset{\gtrsim r}{r}}(\mathbb{T}) := \{ q\ \delta\text{-cubes} : q \text{ lies in } \underset{\gtrsim r}{\sim r} \text{ tubes in } \mathbb{T} \}.$$

<u>Question</u>: Are there conditions under which

$$|P_r(\mathbb{T})| \lesssim \frac{|\mathbb{T}|^2}{r^3} \, ?$$

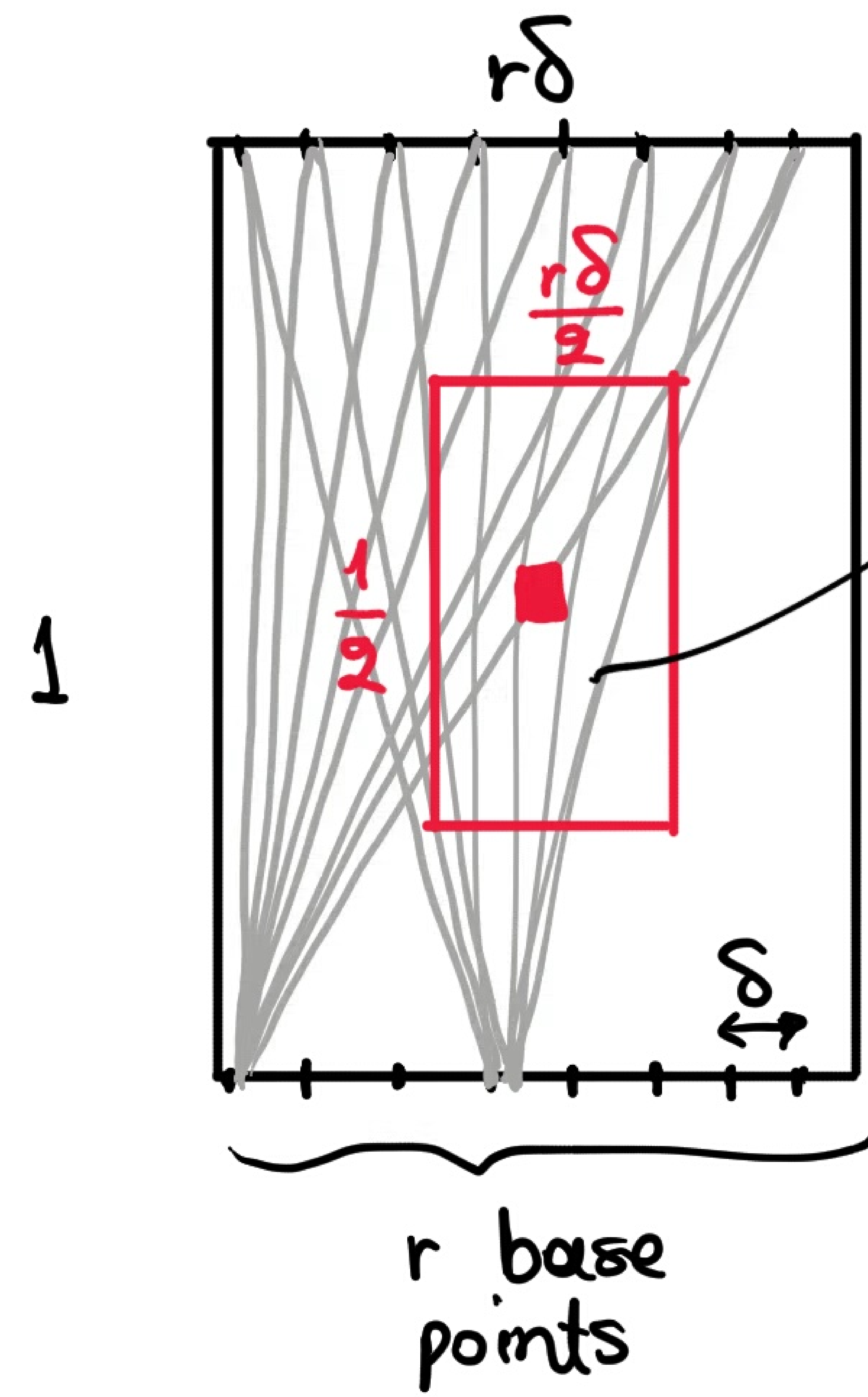
Failure under no conditions:



all distinct δ-tubes in here.

$|\mathbb{T}| \sim r^2.$

Failure under no conditions:



$|\mathbb{T}| \sim r^2$

all δ-cubes in red rectangle are r-rich.

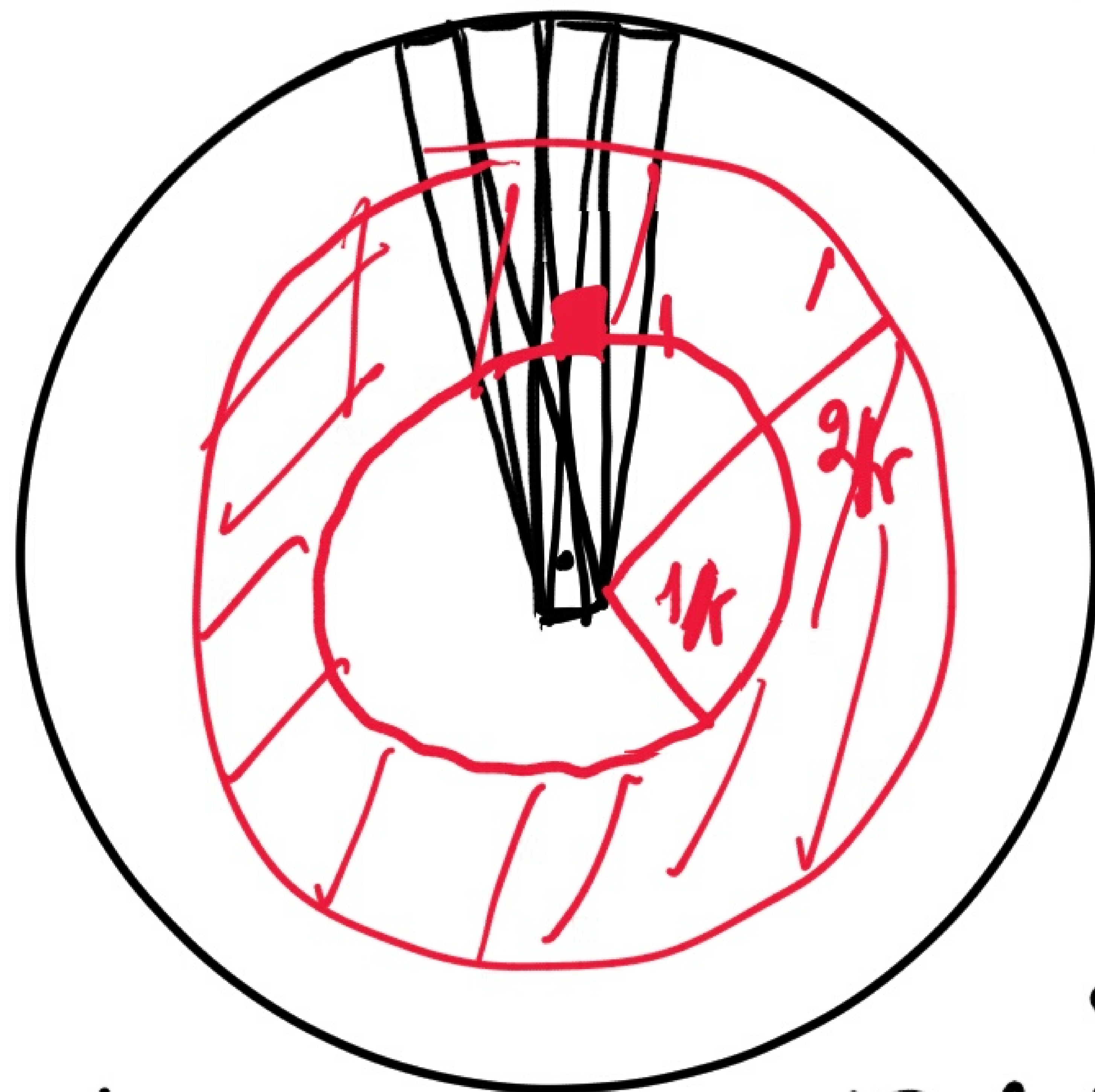
$\Rightarrow |P_r(\mathbb{T})| \gtrsim \frac{r\delta}{\delta^2} \sim \frac{r}{\delta}$,

while: $\frac{|\mathbb{T}|^2}{r^3} \sim \frac{r^4}{r^3} \sim r$

$<< |P_r(\mathbb{T})|$.

Angular separation: S-T fails $\longrightarrow$

Bush: $\underline{\text{all}}$ δ-sep. δ-tubes through 0



A δ-cube at dist $\sim x$ from 0 has $\sim \frac{1}{x}$ tubes through it.

S_0, $\forall r$:

All δ-cubes at dist $\sim \frac{1}{r}$ for 0 are in $|P_r(\mathbb{T})|$

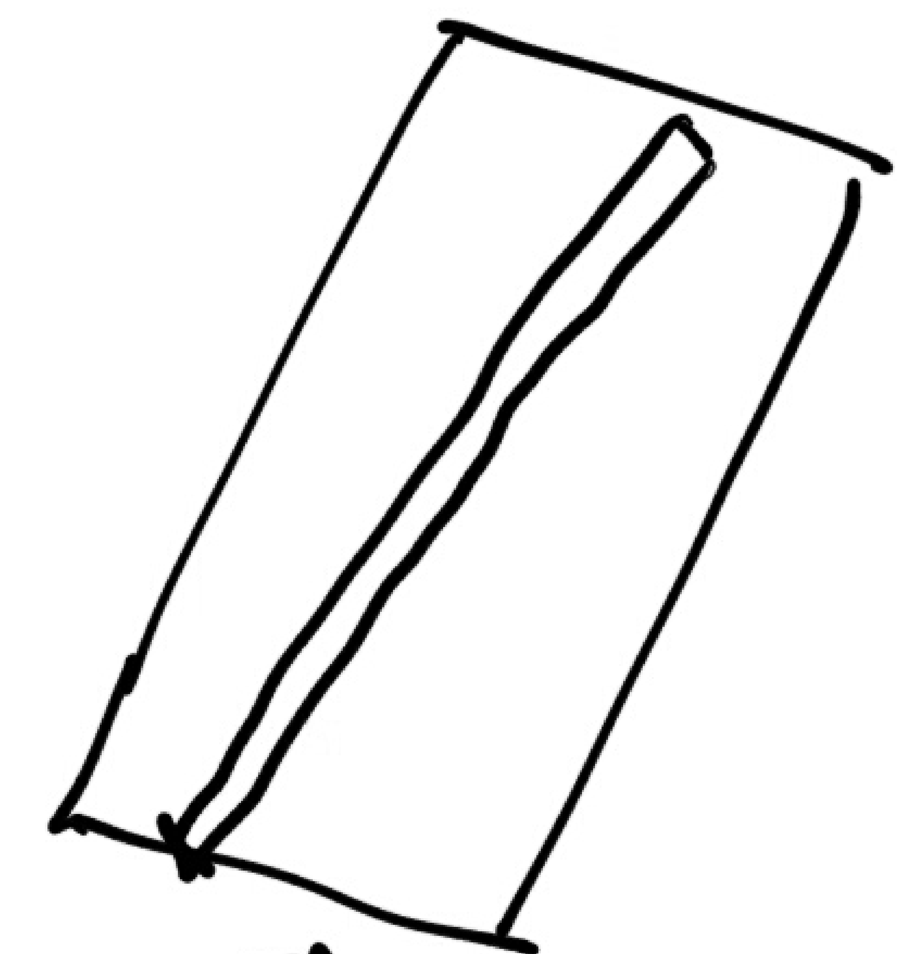
$$\Rightarrow |P_r(\mathbb{T})| \sim \frac{(1/r)^2}{\delta^2} \sim \frac{1}{r^2\delta^2} \sim \frac{|\mathbb{T}|^2}{r^2} >> \frac{|\mathbb{T}|^2}{r^3}.$$

We will impose a good spacing condition on the tubes in $\mathbb{T}$

δ-tubes

will depend on $|\mathbb{T}|$ $\left(\lesssim \frac{1}{\delta^2}\right)$

The fewer the tubes, the more well-spaced we require them to be.



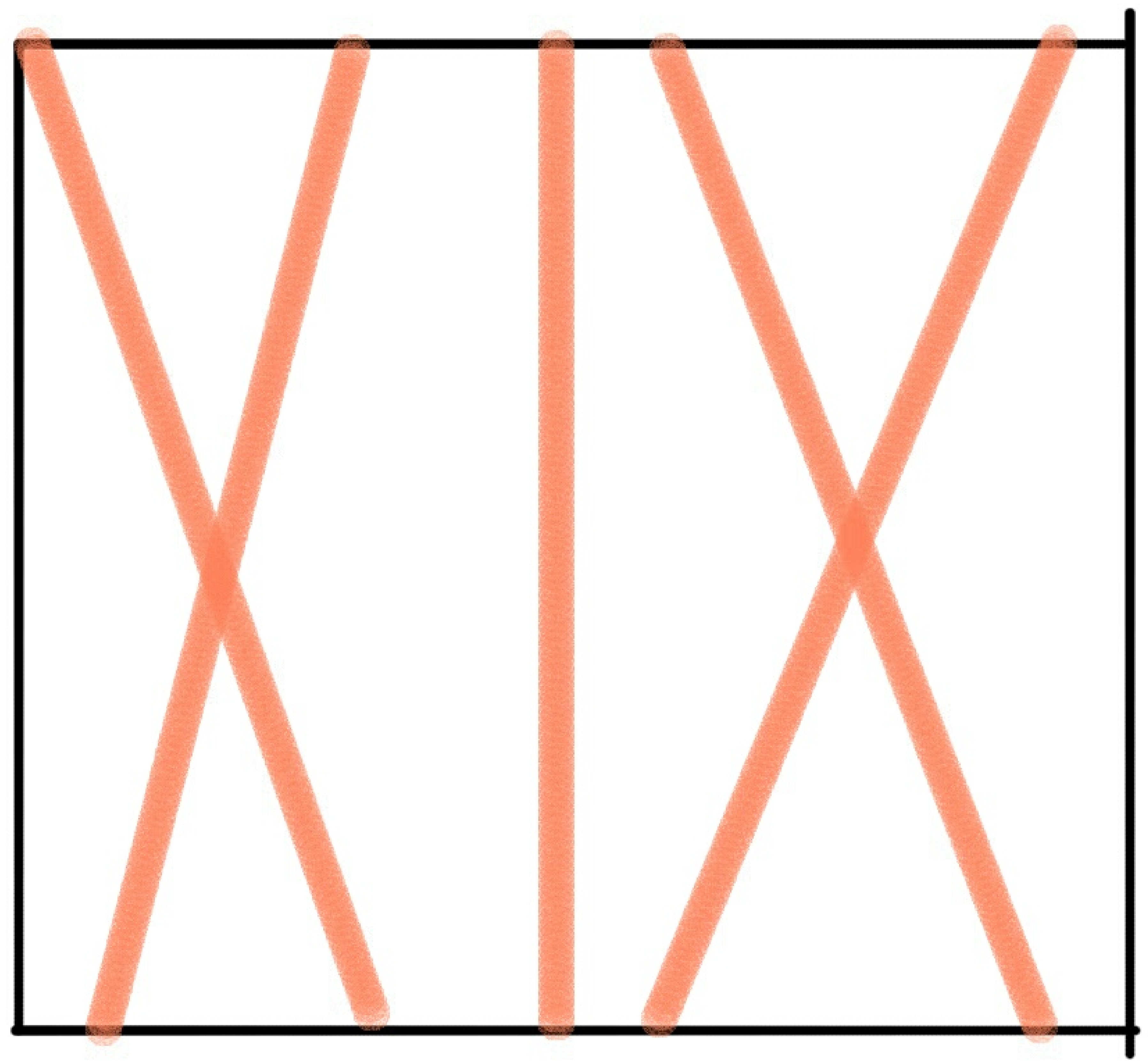
W^{-1}

$|\mathbb{T}| \sim W^2$, for some $1 \le W \le 1/\delta$.

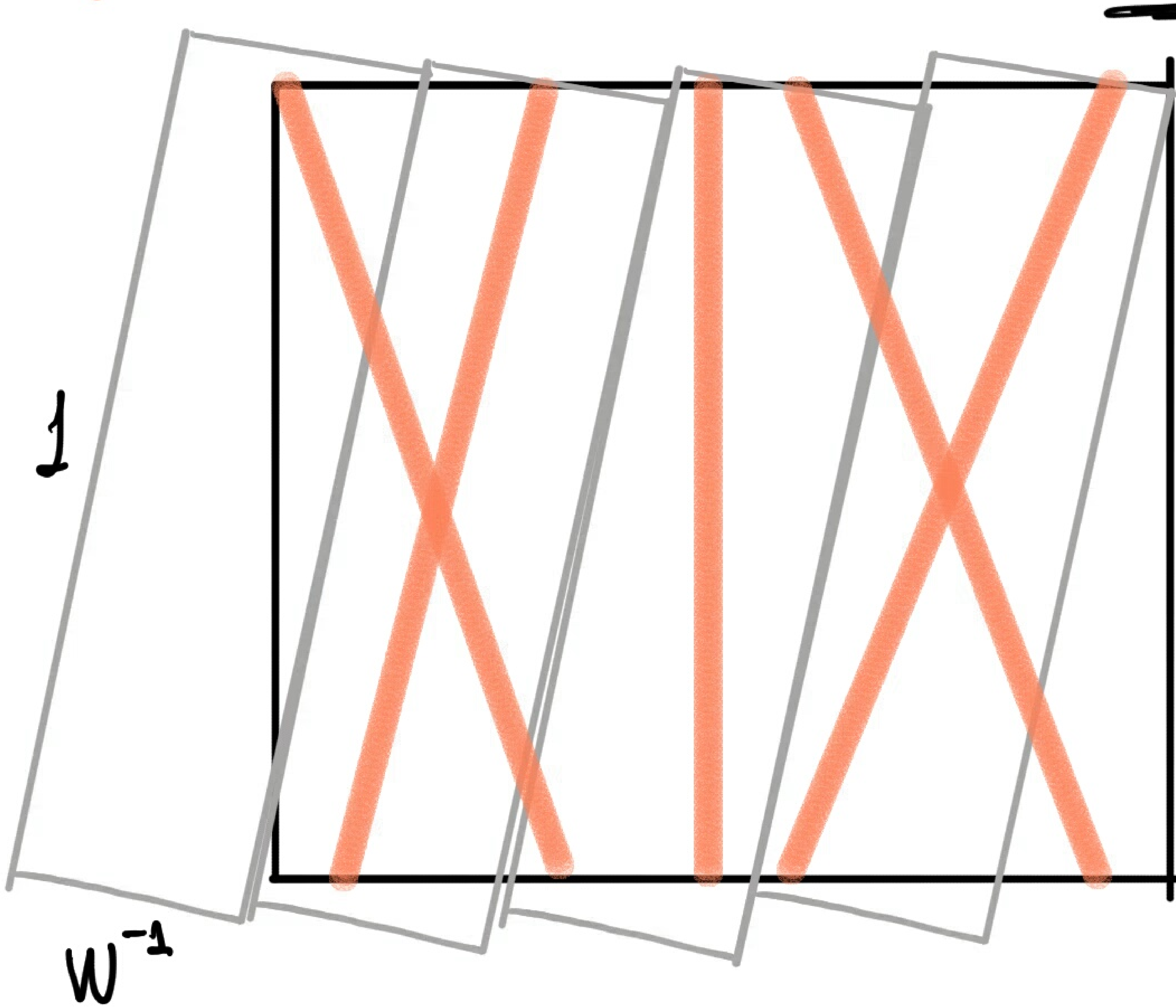
Look at $\underbrace{\text{all distinct } W^{-1}\text{-tubes}}_{\sim W^2 \text{ in total}}$ in $[0,1]^2$

The tubes in $\mathbb{T}$ are well-spaced if each W^{-1}-tube contains ≤ 1 tube in $\mathbb{T}$.

Orange: π

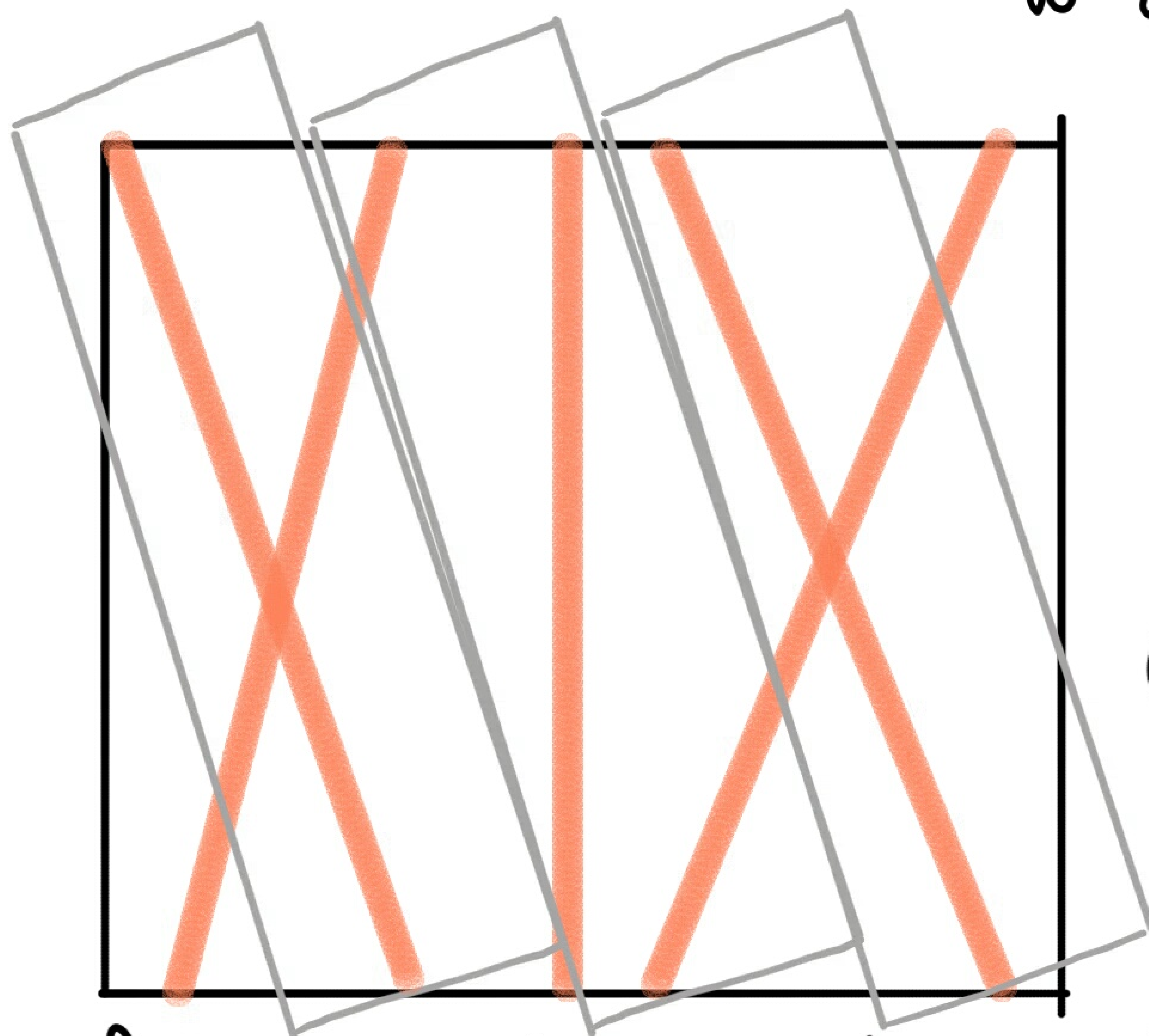


Orange: $|\mathbb{T}| \sim W^2$



W directions, W W^{-1}-tubes in each direction, each contains ≤ 1 δ-tube

Orange: $\mathbb{T}$



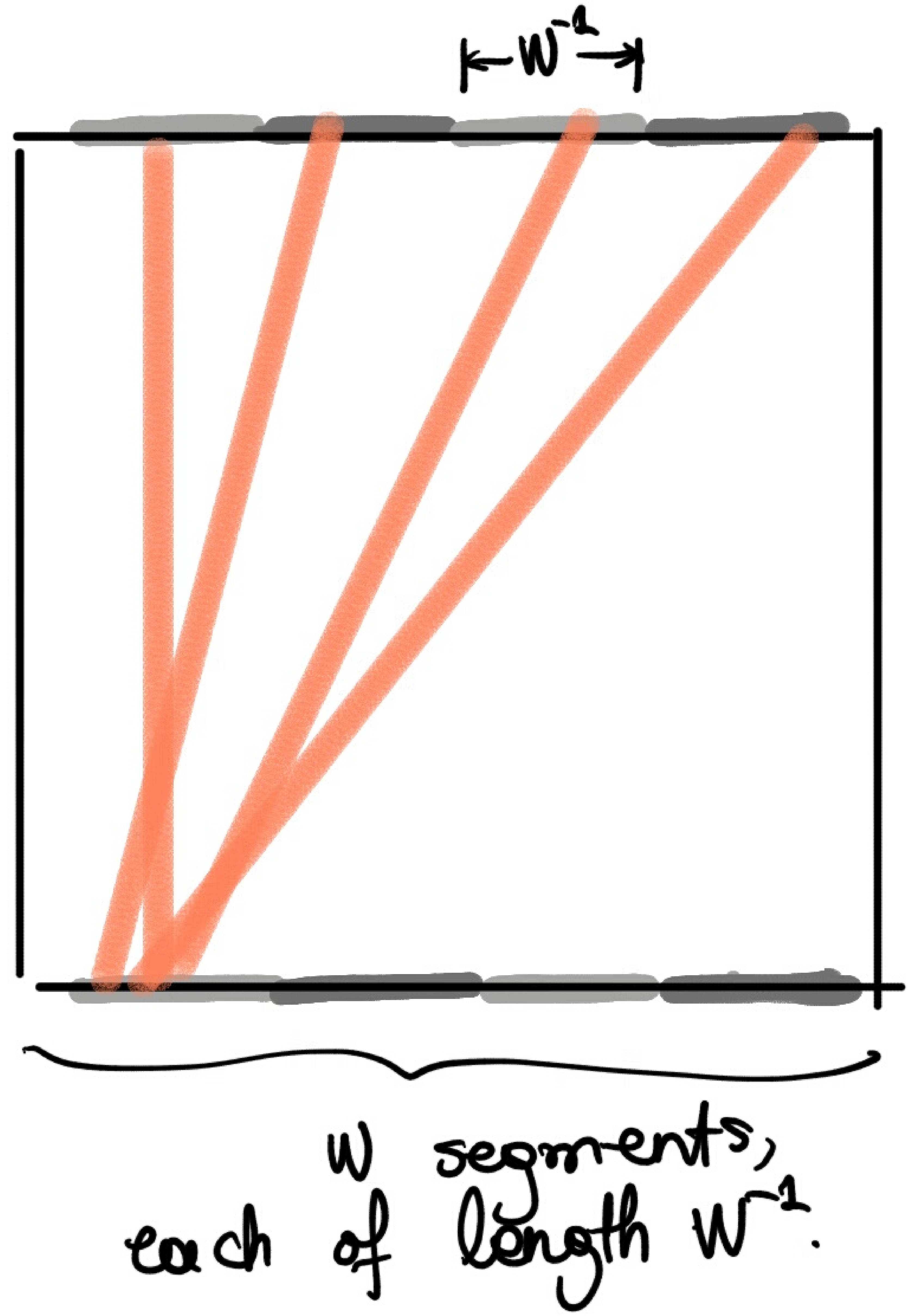
W directions, W W^{-1}-tubes in each direction, each contains ≤ 1 δ-tube

If $\mathbb{T}$ is well-spaced, then:

① If $T_1, T_2 \in \mathbb{T}$ s.t. $T_1 \cap T_2 \neq \emptyset$, then $\text{angle}(T_1, T_2) \gtrsim W^{-1}$.

② If we fatten each δ-tube so that it becomes a W^{-1}-tube, then the new family $\tilde{\mathbb{T}}$ of W^{-1}-tubes is a maximal family of distinct W^{-1}-tubes.

Good spacing condition: S–T fails for low r



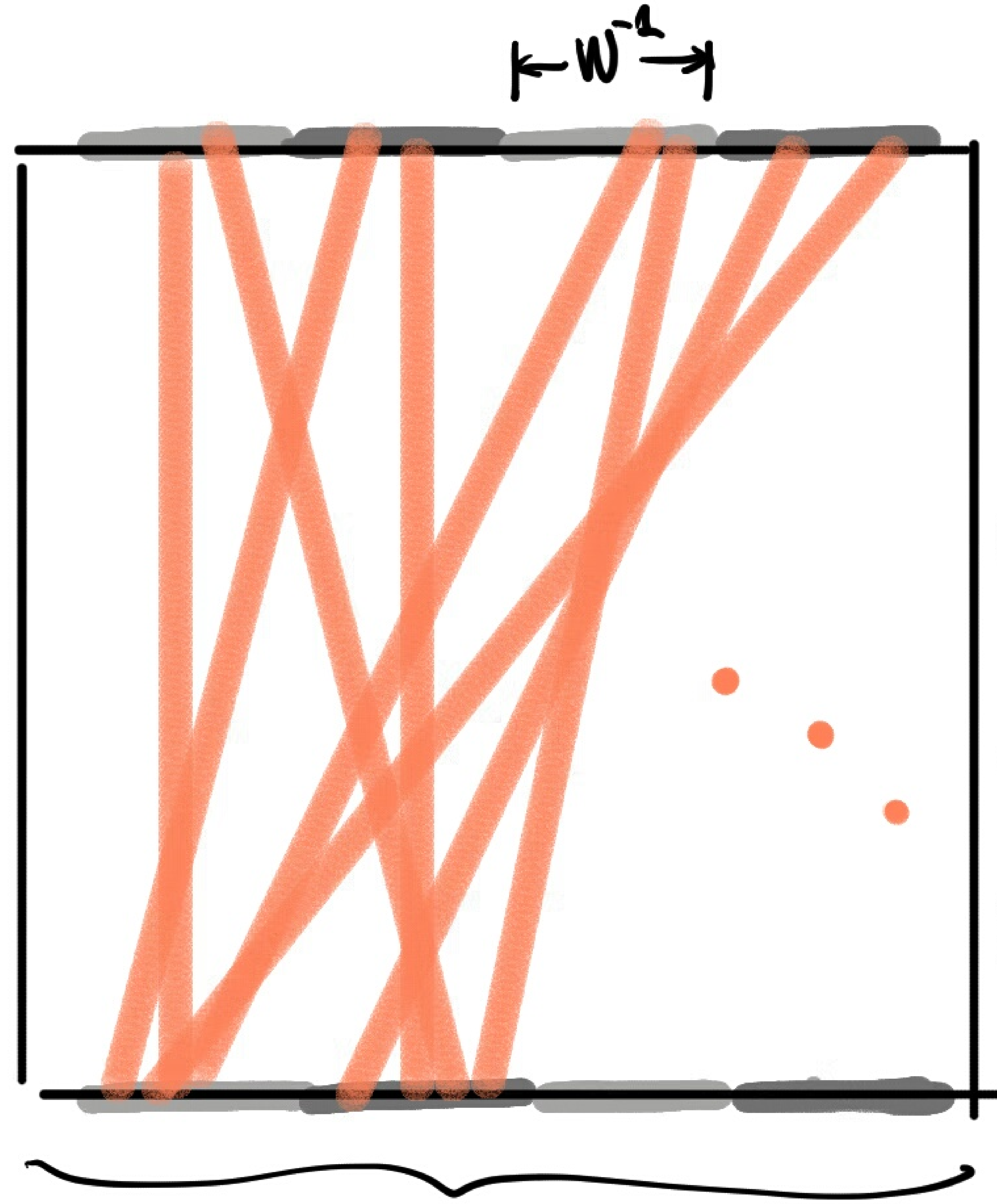
Good spacing condition: S–T fails for low r

$|P_r(\mathbb{T})| \cdot r^3 \sim \delta |\mathbb{T}|^3$, and is $\lesssim |\mathbb{T}|^2 \iff W \lesssim \left(\frac{1}{\delta}\right)^{1/2}$, false.

for lower r:

$|P_{\gtrsim r}(\mathbb{T})| \cdot r^3 \lesssim |\mathbb{T}|^2$

also fails sometimes, according to how big W is relative to $\left(\frac{1}{\delta}\right)^{1/2}$



W segments, each of length W^{-1}.

For $W \gtrsim \left(\frac{1}{\delta}\right)^{1/2}$, $\sim \delta^{-2}$ δ-cubes are r-rich, for $r \sim \delta|\mathbb{T}|$:

(1) follow proof Kakeya max conj in $\mathbb{R}^2$

$\to$ #(≥ 2-rich pts) $\gtrsim \delta^{-3}$

(2) #incidences $\sim |\mathbb{T}| \cdot \frac{1}{\delta} \delta^{-2} \cdot (\delta|\mathbb{T}|)$

$\to$ typical δ-cube is $\sim \delta|\mathbb{T}|$ rich.

Thm: Let $1 \le W \le \delta^{-1}$. Let $\mathbb{T}$ a set of distinct δ-tubes in $[0,1]^2$, with ≤ 1 in each W^{-1}-tube.

If $r > \max(\delta^{1-\varepsilon}|\mathbb{T}|, 1)$, then

$$|P_r(\mathbb{T})| \lesssim_\varepsilon \delta^{-\varepsilon} \frac{|\mathbb{T}|^2}{r^3}.$$

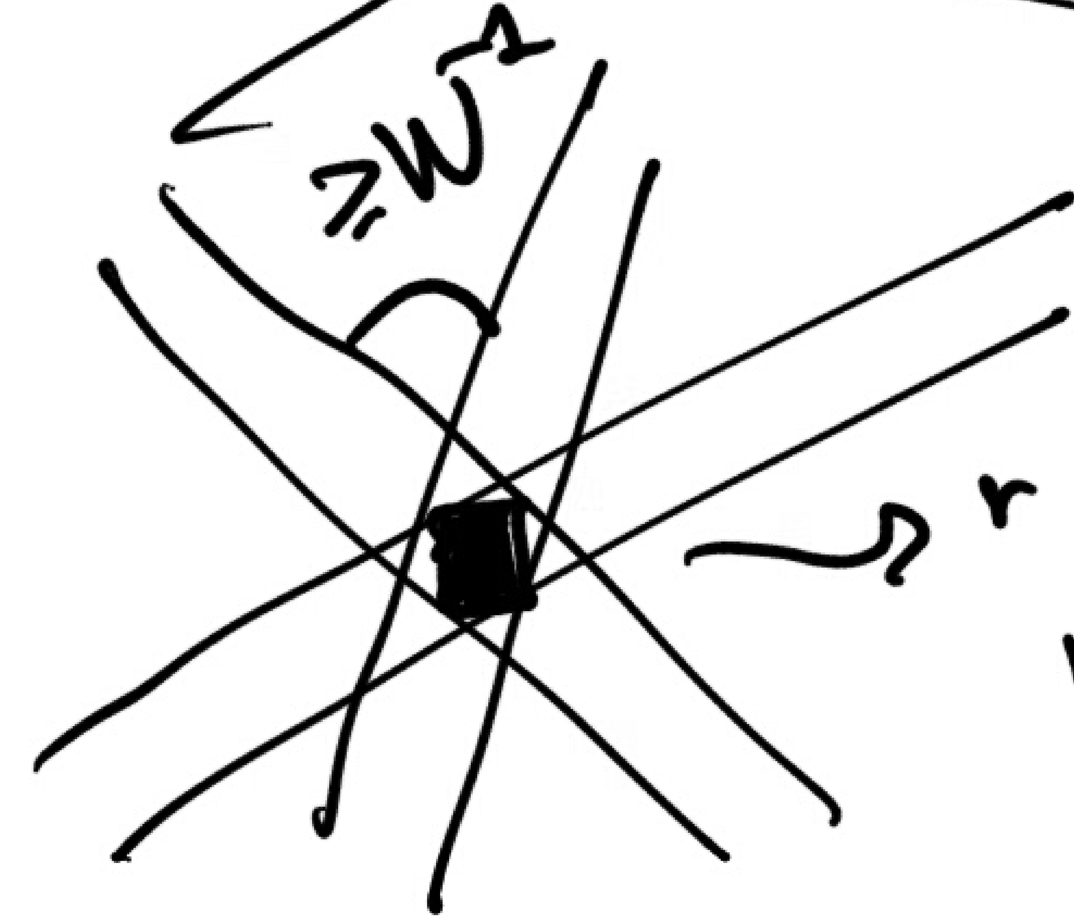
The following fails for smaller r:

$$|P_{\gtrsim r}(\mathbb{T})| \lesssim_\varepsilon \delta^{-\varepsilon} \frac{|\mathbb{T}|^2}{r^3}$$

$|S_r| \lesssim \frac{L^2}{r^3}$, for $2 \le r \lesssim L^{1/2}$

⚠ → for $r \gtrsim |\mathbb{T}|^{1/2}$, $|P_r(\mathbb{T})| = 0$

for appropriate const.



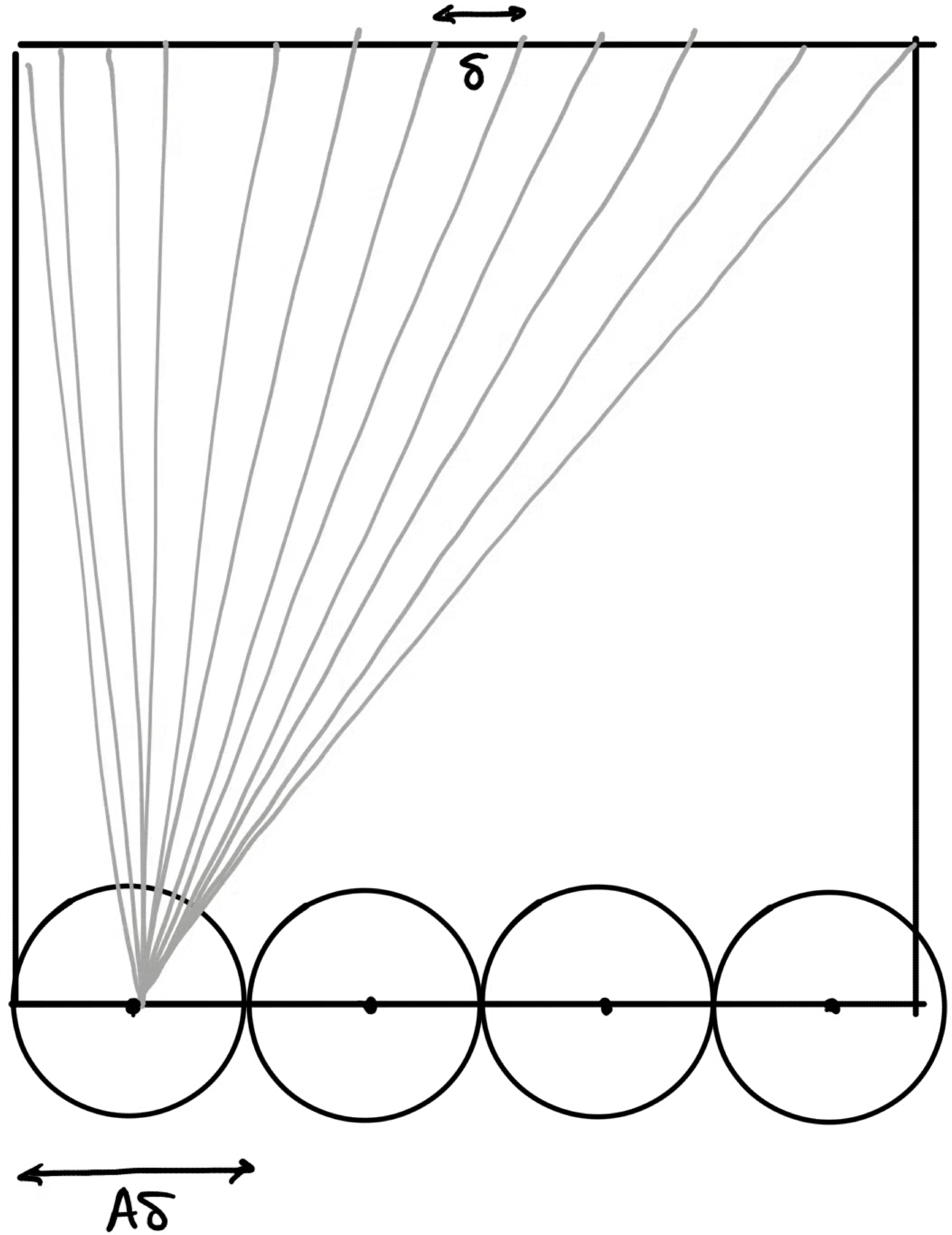
→ r rich for $r \lesssim W \sim |\mathbb{T}|^{1/2}$.

So interesting r:

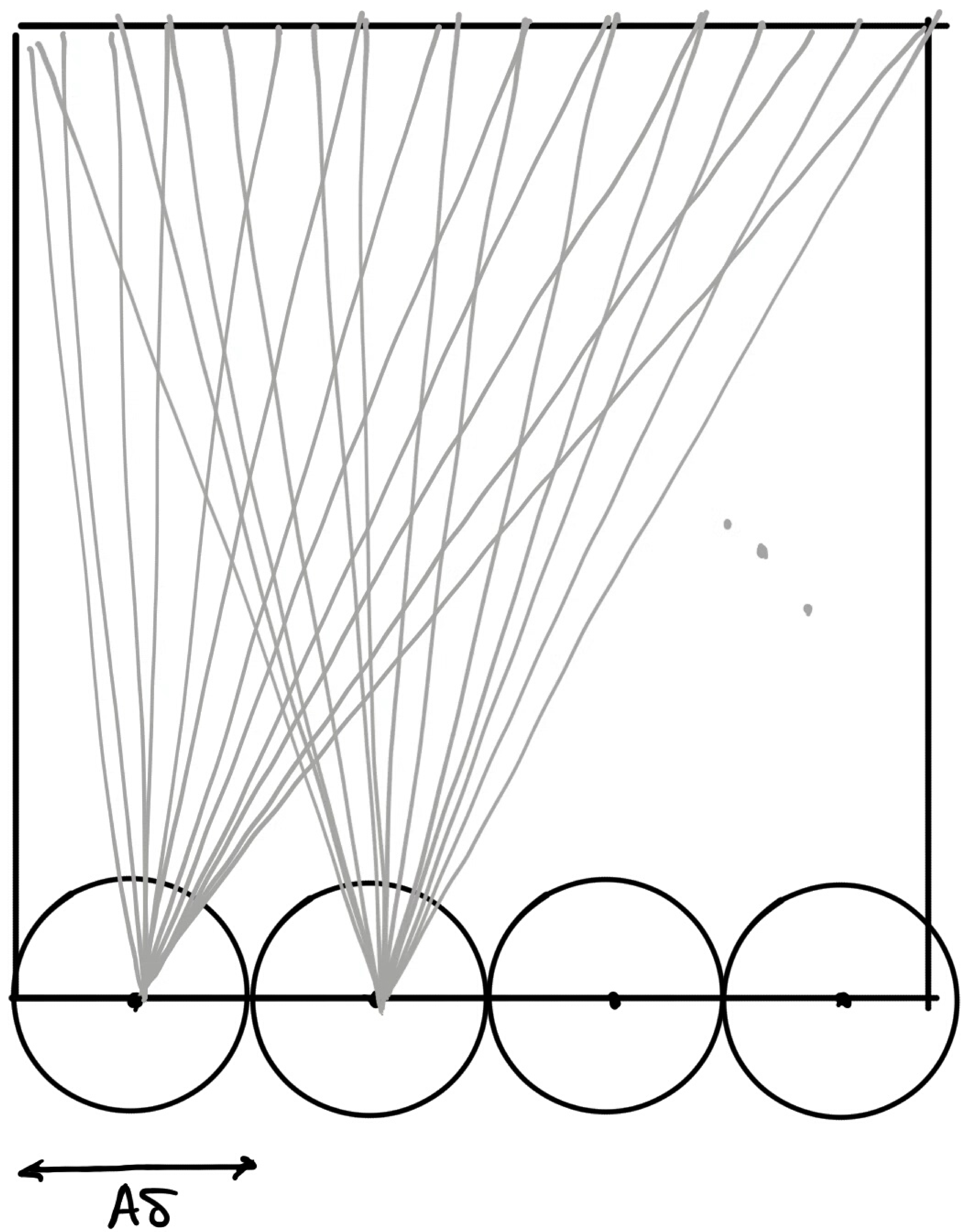
$\max(\delta^{1-\varepsilon}|\mathbb{T}|, 1) < r \lesssim |\mathbb{T}|^{1/2}$, non-empty for $W \lesssim \delta^{-1+\varepsilon}$

→ A tight situation:

all δ-tubes through centre (that reach across).



→ A tight situation:



→ A tight situation:

for $r \sim \frac{1}{A\delta}$, all δ-cubes in red region are r-rich. $\to P$

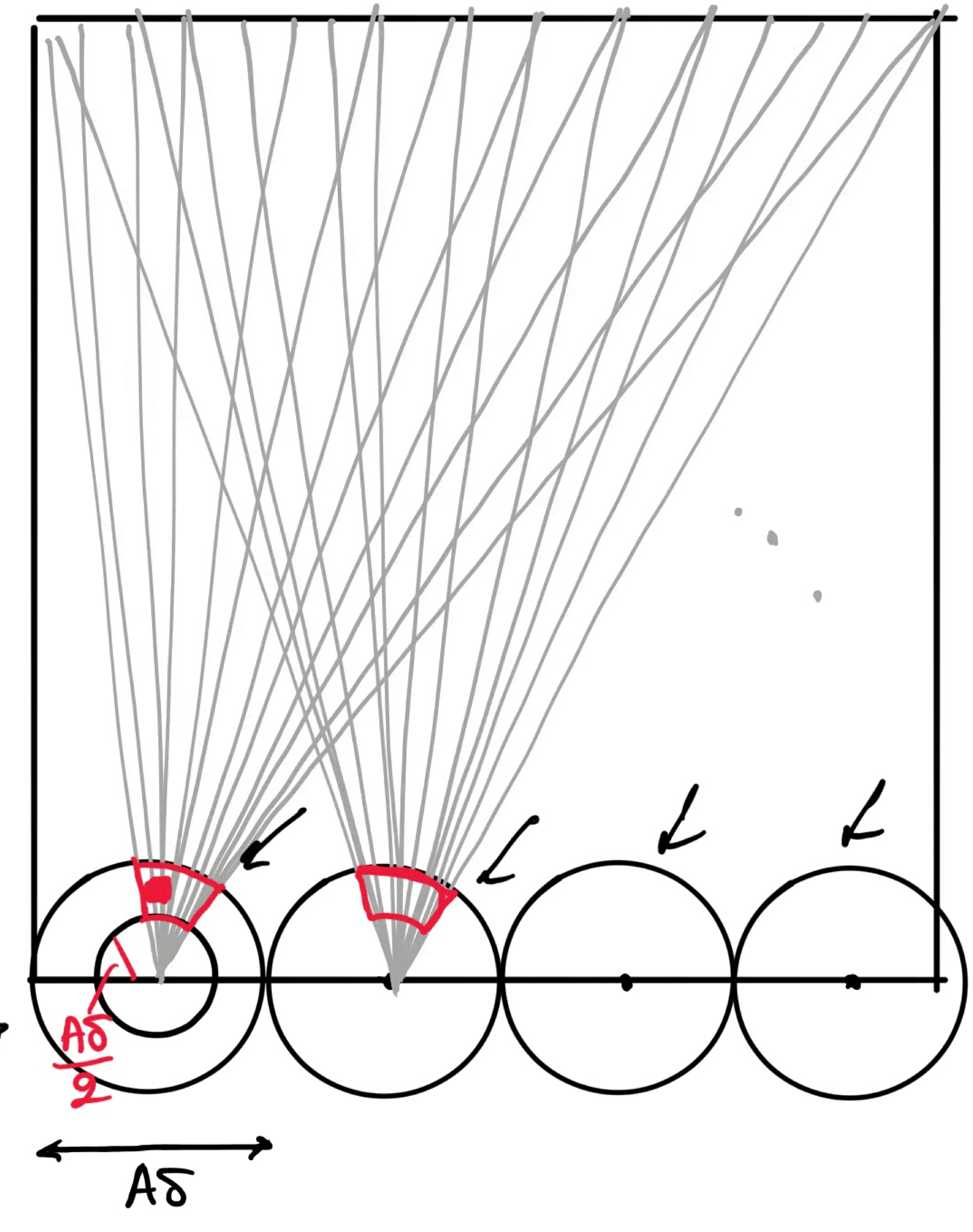
$$|P| \sim \frac{1}{A\delta} \cdot \frac{(A\delta)^2}{\delta^2} \sim \frac{1}{\delta^2} \cdot \frac{1}{r}$$

$$|T| \sim \frac{1}{A\delta} \cdot \frac{1}{\delta} \sim \frac{r}{\delta}.$$

$\Rightarrow$

$$\Rightarrow \; |P| \sim \frac{|T|^2}{r^3} \to \text{sharp for } P.$$

P is contained in fatter balls.



Prop: Let P be a set of δ-cubes, $\mathbb{T}$ a family of of distinct δ-tubes in $[0,1]^n$. Suppose $P \subseteq P_r(\mathbb{T})$.

Then, $\forall \varepsilon > 0$, for $S = \left(\frac{1}{\delta}\right)^{\varepsilon/10n}$, either:

Thin case: $|P| \lessapprox_n S^n \cdot \boxed{\frac{1}{\delta^{n-1}} \cdot \frac{|\mathbb{T}|}{r^2}}$ $\to$ better than S–T sometimes (ex: $|\mathbb{T}| = \max \sim \left(\frac{1}{\delta^{n-1}}\right)^2$), or

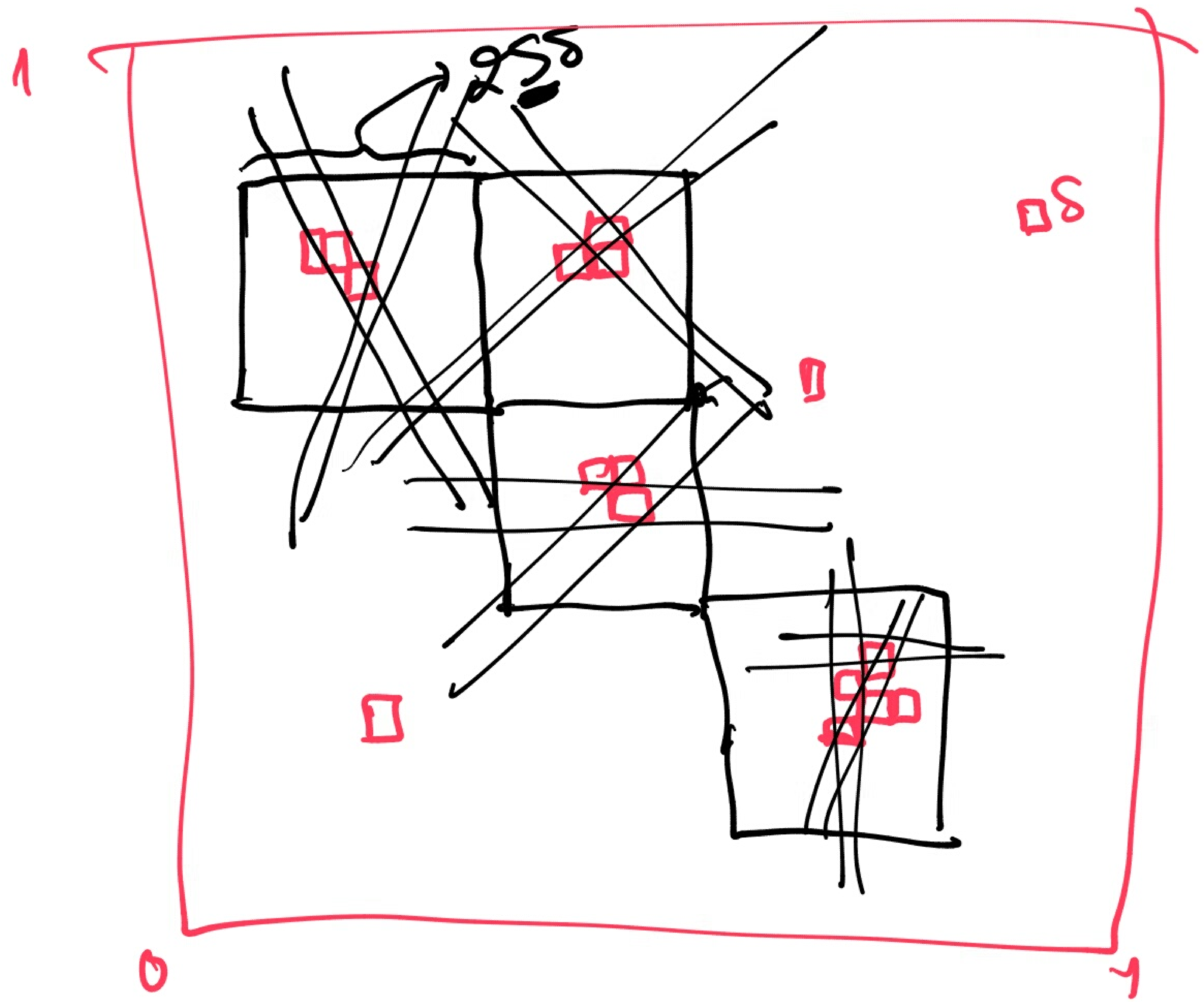
$\leq C(\varepsilon, n) \cdot \left(\frac{1}{\delta}\right)^{10n\varepsilon^3}$

Thick case: There exist fin. overlapping $2S\delta$-cubes Q_j s.t.

(1) $\bigcup Q_j$ contain $\gtrapprox_n |P|$ δ-cubes of P.

(2) Each Q_j intersects $\underline{\gtrapprox_n S^{n-1} r}$ tubes of $\mathbb{T}$.

If I fatten each ~~δ-tube~~ to become an $S\delta$-tube, then each $Q_j \in P_{\gtrsim S^{n-1} r}(\tilde{\mathbb{T}})$ $\to$ family of fat tubes (as long as distinct).


1
S
0
1

Proposition becomes:

Let P be a set of unit cubes and $\mathbb{T}$ a set of distinct tubes of radius 1 and length D in $[0,D]^n$.

Let $P \subseteq P_r(\mathbb{T})$. Then, $\forall \varepsilon > 0$, for $S = D^{\varepsilon/10n}$, either

Thin case: $|P| \lesssim_n S^n \, D^{n-1} \cdot \frac{|\mathbb{T}|}{r^2}$, or

$\hookrightarrow \leq C(\varepsilon, n) D^{10n\varepsilon^3}$

Thick case: $\exists$ fin. overlapping $2S$-cubes Q_j, s.t.

(1) $\cup Q_j$ contains $\gtrsim_n |P|$ of the 1-cubes in P,

(2) Each Q_j intersects $\gtrsim_n S^{n-1} r$ tubes of $\mathbb{T}$.