Discrete Geometry

The (Refined) Polynomial Method

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Outline

Generalising Joints

- Recall that, given a (multi)set of lines, $\mathcal{L} \subset \mathbb{F}^d$, a joint is a point of intersection of d lines in $\mathcal{L}$ whose directions span $\mathbb{F}^d$.
  - $\rightarrow$ Let us consider multiple multisets (families) of algebraic varieties.
- What follows is a reading of Tidor–Yu–Zhao 2020, *Joints of Varieties*, `arXiv:2008.01610`.
- Today I will give the intro + overview of the argument

Multijoint

Definition (Multijoint)

- Let $d \in \mathbb{N}$, for each $1 \leq j \leq d$, let $k_j, m_j \in \mathbb{N}$ and let $n = k_1 m_1 + \ldots + k_d m_d$. For each $1 \leq j \leq d$, let Γ_j be a set of k_j-dimensional varieties.
- For each $1 \leq j \leq d$, let $\gamma_j^1, \ldots, \gamma_j^{m_j} \in \Gamma_j$, and suppose

$$\cap_{j=1}^{d} \cap_{m=1}^{m_j} \gamma_j^m = p \in \mathbb{F}^n.$$

- If the tangent spaces $\{T_p \gamma_j^m\}_{j,m}$ span $\mathbb{F}^n$ then p is a (m_j, k_j)- multijoint.

Definition (Multiplicity)

The number of tuples (γ_j^k) which form a multijoint at p is the multiplicity of p, denoted $M(p)$.

Joints Problem

Theorem (Joints)

Let $\mathcal{L} \subseteq \mathbb{R}^d$ be a collection of lines. Then there are at most $O(|\mathcal{L}|^{\frac{d}{d-1}})$ joints.

Theorem (Tidor–Yu–Zhao (2020))

Let $\Gamma_j \subset \mathbb{F}^n$ be families of k_j-varieties for each $1 \leq j \leq d$ and let $s = m_1 + \ldots + m_d$. Then

$$\sum_{p \in \mathbb{F}^n} M(p)^{\frac{1}{s-1}} \leq C(m_j, k_j) \prod_{j=1}^{d} \left(\deg \Gamma_j\right)^{\frac{m_j}{s-1}},$$

where $\deg \Gamma_j = \sum_{\gamma_j \in \Gamma_j} \deg \gamma_j$.

Remark:

- Dimensions $k_1, \ldots, k_d$ do not affect exponents.
- Constant is independent of the field.

$(m_j, k_j)_{j=1}^d$-multijoints

Examples:

- Joints problem from before. One family of lines, choose m lines.
  - $(m, 1)_{j=1}^1$.
- Simple multijoints. As above, but with d families, choose one line from each.
  - $(1, 1)_{j=1}^d$.
- Joints formed by m k-varieties in $\mathbb{F}^{mk}$.
  - $(m, k)_{j=1}^1$.
- Joints formed by a 2-plane from one family and 2 lines from another in $\mathbb{F}^4$.
  - $((1, 2), (2, 1))$.

Remark:

- The degree of the varieties doesn't affect the (m_j, k_j) notation, nor the constant.

Plan

To prove:

Theorem

Let $\Gamma_j \subset \mathbb{F}^n$ be families of k_j-planes for each $1 \leq j \leq d$. Then

$$\sum_{p \in \mathbb{F}^n} M(p)^{\frac{1}{s-1}} \leq C(m_j, k_j) \prod_{j=1}^{d} |\Gamma_j|^{\frac{m_j}{s-1}},$$

where $s = m_1 + \ldots = m_d$.

The content remains the same. Retaining rigour while accounting for high degree varieties steps into algebraic geometry – good to know it can be done, but not important for our purposes.

Method for Joints (and Finite Field Kakeya)

- Use Parameter Counting to get a non-zero polynomial of low degree which vanishes over a specified set.
- Use geometry of the set to deduce additional properties of the polynomial.
- Use Vanishing Lemma (Bézout) to deduce that such a polynomial must either
  - be identically zero, or
  - have appropriately few zeros.

What Fails?

Parameter Counting:

- In $\mathbb{F}^d$ we find a polynomial $f \neq 0$ s.t. $\deg f \lesssim |J|^{\frac{1}{d}}$.
  - Some constraints imposed by J are redundant.
  - E.g. if $f \in \mathbb{F}_1[x_1, \ldots, x_d]$ is zero at two points then it is zero on the line they define.
  - We can improve on $|J|$.

Vanishing Lemma:

- A polynomial in more than one variable can have infinitely many zeros.
  - E.g. If many points lie in a subvariety of low degree.
- Alternatively: We don't have a suitable form of Bézout's Theorem.

Key Idea 1

Definition (Vanishing Condition)

A vanishing condition is a polynomial whose monomials are derivative evaluation maps. Alternatively, they are finite linear combinations of directional derivative evaluations. [a]

[a] Note we use derivative evaluations, not just derivative maps.

- E.g. $f(1,2) = 0$, $\partial_x(f(2,3)) = 0$, $(\partial_{xxy} - \partial_{yyx})f(0,1) = 0$.
- Carefully choose vanishing conditions so that a vanishing lemma holds.
- Using few enough to that a Bézout substitute is automatic.

Key Idea 2

Consider space $\mathbb{F}_3[t]$, and vanishing conditions of 0-th order. We can find a non-zero $f \in \mathbb{F}_3[t]$ that satisfies up to 3 distinct 0-th order vanishing conditions, but not more.

Question:

- What can we say about the 3 distinct conditions that we cannot say about 4 or more?

Answer:

- They are linearly independent.

Conclusion:

- Interpret space of vanishing conditions as the vector space that is dual to $\mathbb{F}_3[t]$, or more generally $\mathbb{F}_\lambda[x_1, \ldots, x_n]$.
- Now the number of vanishing conditions we need to consider is at most $\dim \mathbb{F}_\lambda[x_1, \ldots, x_n]$ – this will play the rôle of Bézout.

Idea

- To cycle through all the points and orders of vanishing in J, and accumulate vanishing conditions at each $p \in J$.
- But, we need to order the sets of pairs (p, r) is a sensible way.
  - Introduce handicaps and priority orders.
- Start by endowing the set of points $\mathcal{P}$ with some *preassigned order* so we can compare $(>, <, =)$ any two $p, p' \in \mathcal{P}$.

Handicaps and Priority Order

We will assign vanishing conditions by cycling through a finite set of points $\mathcal{P} \subset \mathbb{F}^n$. Let $\alpha : \mathcal{P} \to \mathbb{Z}$. The function α is referred to as a *handicap*.

We assign a total ordering, called the *priority order*, to the set $\mathcal{P} \times \mathbb{N}$. First of all, give $\mathcal{P}$ a preassigned order (any will do), and we say $(p, r) \prec (p', r')$ if

- $r - \alpha_p < r' - \alpha_{p'}$,[1] or
- $r - \alpha_p = r' - \alpha_{p'}$ and p comes before p' in the preassigned order.

Remark:

- The priority order does not change under $\alpha \mapsto \alpha + c$ for any constant $c \in \mathbb{Z}$.

Adopt the convention $0 \in \mathbb{N}$ - we will need 0-th order derivatives later.

[1] Large handicap allows r-th occurance of p to appear early.

Example

Consider a set of points $\{a < b < c < d < e\}$ with handicaps $0, 1, 3, -1, 0$, respectively. This can be represented by

$$c|c|bc|abce|acbde|abcde \cdots .$$

That is $(c, 1)$ represented by the 1-st occurance of c, etc. I.e.

$$(c, 1) \prec (c, 2) \prec (b, 1) \prec (c, 3) \prec (a, 1) \prec (b, 2) \prec (c, 4) \prec (e, 1) \prec \cdots$$

Pause for Breath/Questions

Choosing Vanishing Conditions

- Choose $\lambda \in \mathbb{N}$ which we fix. Let γ be a k-plane, and for every $r \in \mathbb{N}$, let $\mathbb{D}^r_{p,\gamma}$ be the set of derivative maps with directions parallel to $T_p\gamma$ of order at most r.[2]
- Let $\mathbb{B}^r_{p,\gamma}(\lambda)$ denote the subspace of linear operators on $\mathbb{F}_\lambda[x_1, \ldots, x_k]$ of the form $f \mapsto Df(p)$ for $D \in \mathbb{D}^r_{p,\gamma}$.

For convenience, $\mathbb{D} \leftrightarrow$ derivatives, and $\mathbb{B} \leftrightarrow$ basis for linear functionals.

[2]Note that $T_p\gamma$ is independent of p in this special case of planes.

Choosing Vanishing Conditions

For each γ, for each $(p, r) \in (\mathcal{P} \cap \gamma) \times \mathbb{N}$, we choose a set

$$\mathcal{B}^r_{p,\gamma}(\alpha, \lambda) \subset \mathbb{B}^r_{p,\gamma}(\lambda).$$

We fixed α, λ and now γ so we may suppress them. The above now reads $\mathcal{B}^r_p \subset \mathbb{B}^r_p$.

Suppose we are at step (p, r), and we have chosen $\mathcal{B}^{r'}_{p'}$ for each $(p', r') \prec (p, r)$ so that

- The sets $\mathcal{B}^{r'}_{p'}$ are disjoint, and
- $\cup_{(p',r')\prec(p,r)}\mathcal{B}^{r'}_{p'}$ is a basis for $\text{span}\left(\cup_{(p',r')\prec(p,r)}\mathbb{B}^{r'}_{p'}\right)$.

Now choose $\mathcal{B}^r_p$, disjoint from every previous $\mathcal{B}^{r'}_{p'}$, and

$$\text{span}\left(\cup_{(p',r')\preceq(p,r)}\mathcal{B}^{r'}_{p'}\right) = \text{span}\left(\cup_{(p',r')\prec(p,r)}\mathbb{B}^{r'}_{p'}\right).$$

Fixed $p, \alpha, \lambda, \gamma$.

Choosing Vanishing Conditions

Remark:

- Although we have freedom to choose *which* elements we choose to include in $\mathcal{B}_p^r$, we do not have freedom on *how many* we add.
- Eventually, $\mathcal{B}_p^r$ will be empty as r exceeds λ.

By the form of the sets $\mathbb{B}_p^r$, we can choose corresponding sets of derivatives[3] $\mathcal{D}_{p,\gamma}^r(\alpha,\lambda) = \mathcal{D}_p^r$ which realise the basis elements in each $\mathcal{B}_p^r$.

Let

$$\mathcal{B}_{p,\gamma}(\alpha,\lambda) := \cup_{r\in\mathbb{N}}\mathcal{B}_p^r, \quad \text{and} \quad \mathcal{D}_{p,\gamma}(\alpha,\lambda) := \cup_{r\in\mathbb{N}}\mathcal{D}_p^r.$$

By construction

$$\sum_{p\in\gamma\cap\mathcal{P}} |\mathcal{B}_{p,\gamma}(\alpha,\lambda)| = \dim \mathbb{F}_\lambda[x_1,\dots,x_k] = \binom{\lambda+k}{k}.$$

[3] Think of usual derivative over $\mathbb{R}$. For arbitrary $\mathbb{F}$, this is generalised by the Hasse derivative.

Fixed $p, \alpha, \lambda, \gamma$.

What Was The Point of the Construction

Given our fixed handicap α, for every $1 \leq j \leq d$, $\gamma_j \in \Gamma_j$ and $p \in \gamma_j \cap J$, we now have numbers

$$g_j(p, \gamma, \alpha, \lambda) = g_j(p, \gamma) := \frac{1}{\binom{\lambda+k_j}{k_j}} |\mathcal{B}_{p,\gamma}(\alpha, \lambda)|.$$

We will show that for any $G : J \to \mathbb{R}_{\geq 0}$, there is a special handicap α so that, with sufficiently large λ,

$$\frac{1}{G(p)^d} \prod_{(\gamma_j^m)_{j,m} \in \mathcal{M}(p)} \left(\prod_{j,m} g_j(p, \gamma_j^m) \right)^{\frac{1}{M(p)}} = O_{m_j k_j}(1)$$

for every joint p, where $\mathcal{M}(p)$ is the set of all possible tuples $(\gamma_j^m)_{j,m}$ which form a multijoint at p, and has cardinality $M(p)$.

Fixed α, λ, relaxed p, γ.

What is special about this?

Let $s = m_1 + \ldots + m_d$. Then this is equivalent to:

$$\left(\prod_{(\gamma_j^m)_{j,m} \in \mathcal{M}(p)} \left(\prod_{j,m} g_j(p, \gamma_j^m) \right)^{\frac{1}{M(p)}} \right)^{\frac{1}{s}} \sim_d G(p).$$

Note that this may be satisfied by additionally insisting that $g_j(p, \gamma_j^m) = 0$ whenever $p \notin \gamma_j^m$.

Recall, by construction, that a Bézout-type equality holds:

$$\sum_{p \in J \cap \gamma_j^m} g_j(p, \gamma_j^m) = \frac{1}{\binom{\lambda + k_j}{k_j}} \sum_{p \in J \cap \gamma_j^m} |\mathcal{B}_{p, \gamma_j^m}(\alpha, \lambda)| = 1$$

for all $\gamma_j^m \in \Gamma_j$ for all $1 \leq j \leq d$.

This may right a bell if you attended Brascamp–Lieb inequalities.

We Will Show

Want to show,

$$\sum_{p \in \mathbb{F}^n} M(p)^{\frac{1}{s-1}} \lesssim \prod_{j=1}^{d} |\Gamma_j|^{\frac{m_j}{s-1}},$$

or, equivalently,

$$\left\| M^{\frac{1}{s}} \right\|_{\ell^{\frac{s}{s-1}}(J)} \lesssim \prod_{j=1}^{d} |\Gamma_j|^{\frac{m_j}{s}}.$$

So we will show that for G with $\|G\|_s = 1$,

$$\sum_{p \in J} M(p)^{\frac{1}{s}} G(p) \lesssim \prod_{j=1}^{d} |\Gamma_j|^{\frac{m_j}{s}}.$$

Tidor–Yu–Zhao use the maximising G, but this ready more easily in my opinion.

Symbol Pushing

$$\sum_{p\in J} M(p)^{\frac{1}{s}} G(p) \lesssim \sum_{p\in J} \left(M(p) \prod_{\mathcal{M}(p)} \left(\prod_{j,m} g_j(p,\gamma_j^m) \right)^{\frac{1}{M(p)}} \right)^{\frac{1}{s}}$$

$$\leq \sum_{p\in J} \left(\sum_{(\gamma_j^m)_{j,m}\in\mathcal{M}(p)} \prod_{j,m} g_j(p,\gamma_j^m) \right)^{\frac{1}{s}}$$

$$\leq \sum_{p\in J} \left(\sum_{(\gamma_j^m)_{j,m}\in\Gamma_1^{m_1}\times\cdots\Gamma_d^{m_d}} \prod_{j,m} g_j(p,\gamma_j^m) \right)^{\frac{1}{s}}$$

$$= \sum_{p\in J} \left(\prod_{j,m} \sum_{\gamma_j\in\Gamma_j} g_j(p,\gamma_j) \right)^{\frac{1}{s}}$$

More Symbol Pushing

$$
\begin{aligned}
\sum_{p\in J}\left(\prod_{j,m}\sum_{\gamma_j\in\Gamma_j} g_j(p,\gamma_j)\right)^{\frac{1}{s}} &\le \prod_{j,m}\left(\sum_{p\in J}\sum_{\gamma_j\in\Gamma_j} g_j(p,\gamma_j)\right)^{\frac{1}{s}} \\
&= \prod_{j}\left(\sum_{\gamma_j\in\Gamma_j}\sum_{p\in J} g_j(p,\gamma_j)\right)^{\frac{m_j}{s}} \\
&= \prod_{j}\left(\sum_{\gamma_j\in\Gamma_j} 1\right)^{\frac{m_j}{s}} \\
&= \prod_{j} |\Gamma_j|^{\frac{m_j}{s}}.
\end{aligned}
$$

Next Time

We will see the detail of the extension to the Polynomial Method:

- Formally understand how the numbers $g_j(p, \gamma, \alpha, \lambda)$ vary with respect to α.
- Establish new "bespoke" Vanishing Lemma.
- Prove existence of nice handicap.