Discrete Geometry

The Polynomial Method

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Rough Outline

(1) Classical Polynomial Method and applications,

(2) Overview of Tidor–Yu–Zhao, *Joints of Varieties*, 2020,

(3/4) Technical Work:

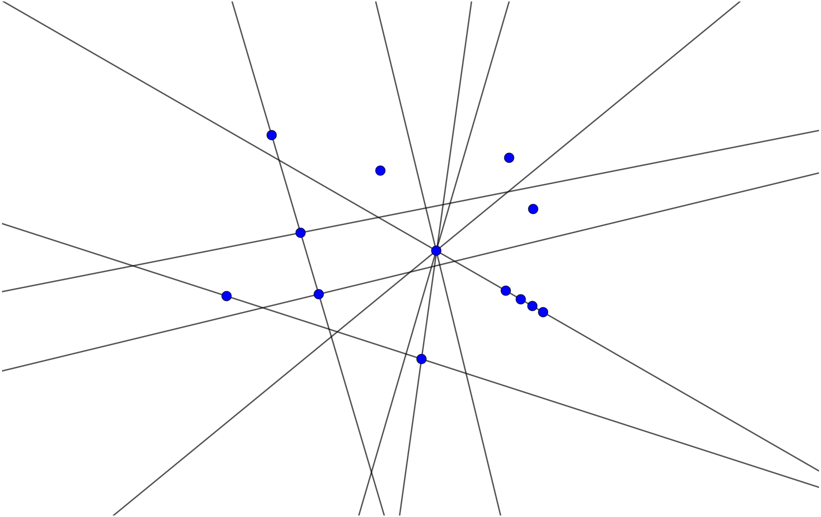
- Discrete continuity and monotonocity of construction,
- New Vanishing Lemma and conclusion.

This is essentially self-contained, requiring only elementary arguments and basic linear algebra and facts about polynomials.
There will be A LOT of parameters floating around. If you realise that you have lost track, please interrupt me and ask!

Outline

1. Introduction
   - Discrete Problems
2. Polynomial Method
   - Parameter Counting
   - Vanishing Lemma
3. Proof of Joints Theorem (Quilodrán)

Point - Line Incidence



Given a set of points and a set of lines in the plane, how many incidences can occur?

Szemerédi–Trotter

Theorem (Szemerédi–Trotter)

A collection of n points and m lines in the Euclidian plane can have at most

$$O(n^{\frac{2}{3}}m^{\frac{2}{3}} + n + m)$$

incidences.

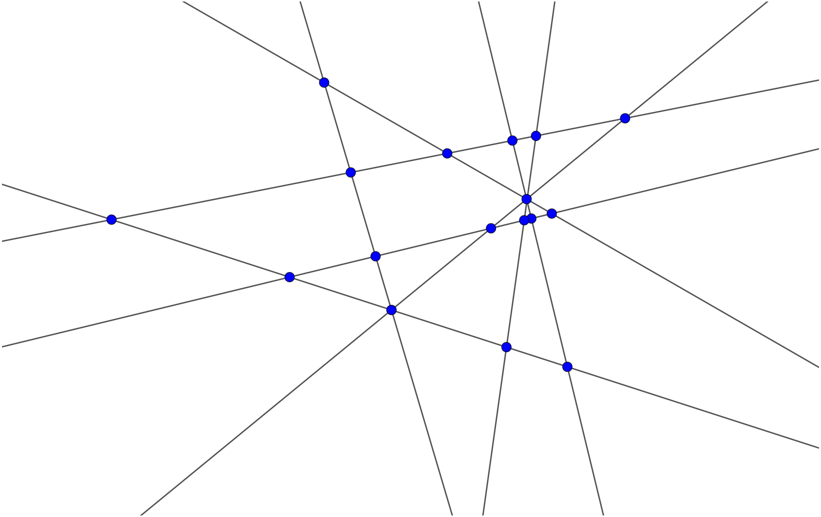
Might have seen this in the Harmonic Analysis Reading Group last term.

Line - Line Incidence: The Joints Problem

- Let $\mathbb{F}$ be a field ($\mathbb{R}$ will do), and let $\mathcal{L} \subset \mathbb{F}^d$ be a collection of lines.
- How many points of intersection $\mathcal{P}$ are there between pairs of lines in $\mathcal{L}$?

Line - Line Incidence: The Joints Problem

How many incidences?



Line - Line Incidence: The Joints Problem

- When $d = 2$, the problem is trivial and $|\mathcal{P}| \leq |\mathcal{L}|^d$.
- The same trivial estimate holds for $d > 2$.
- In fact, this is sharp without any additional hypotheses.

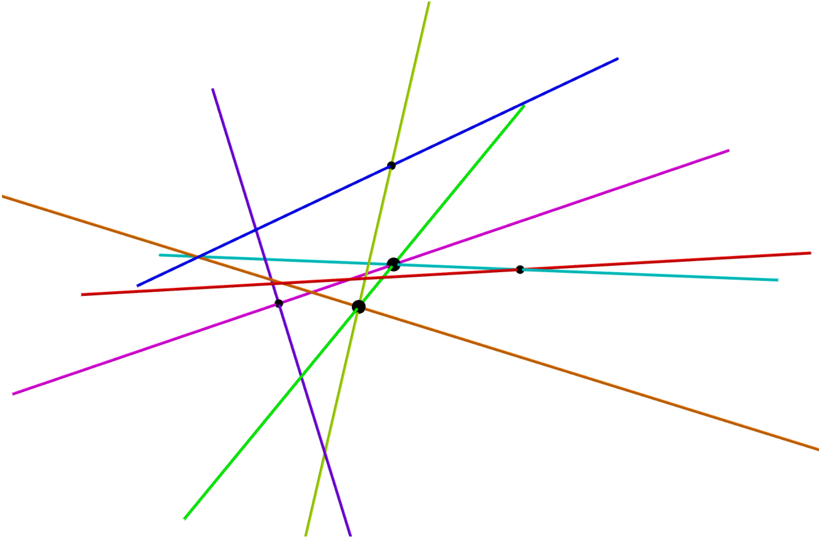
Joints

So that the problem is not trivial, we only count special incidences.

Definition (Joint)

Let $\mathcal{L} \subset \mathbb{F}^d$ be a collection of lines. A point $p \in \mathbb{F}^d$ is a *joint* if it is a point of intersection of d lines whose directions span $\mathbb{F}^d$.

- The lines are “well spaced” at joints.

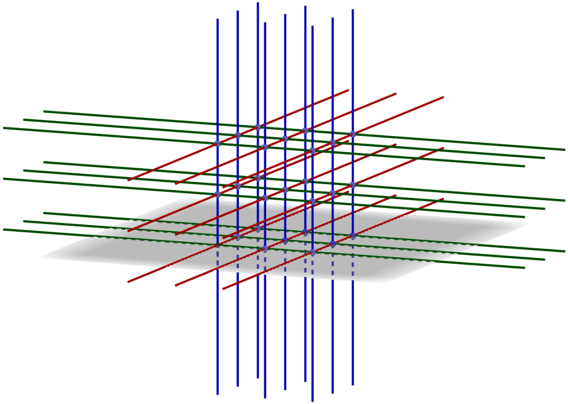


Joints

- It is no longer obvious whether or not we can construct examples where $|\mathcal{P}| \sim_d |\mathcal{L}|^d$.
- What should we expect to see on the RHS?

Loomis–Whitney Type Example

So called because we have three colours of lines, each colour having a fixed direction.



$$|\mathcal{P}| = 27 = 9^{\frac{3}{2}} = (N/3)^{\frac{3}{2}} \sim_d N^{\frac{3}{2}}.$$

Joints Theorem

Theorem (Joints)

Let $\mathcal{L} \subseteq \mathbb{R}^d$ be a collection of lines. Then there are at most $O(|\mathcal{L}|^{\frac{d}{d-1}})$ joints.

- Guth, Katz ($d = 3$) 2008,
- Elekes, Kaplan, Sharir (simpler, $d = 3$) 2009,
- Quilodrán (simplest) 2009.

Parameter Counting

Lemma

The vector space $\mathbb{F}_n[x_1, \ldots, x_d]$ of d-variate polynomials over $\mathbb{F}$ of degree at most n has dimension

$$\binom{n+d}{d}.$$

Proof

Proof.

- Count the number of monomials.
- Same as number of d-tuples of non-negative integers whose sum is at most n.

Use the "stars and bars" argument to place n stars into d bins by using d bars. Stars in a common bin all lie to the immediate left of a bar. So $| * *| * ||*$ corresponds to $x_2^2 x_3 \in \mathbb{F}[x_1, \dots, x_4]$. There are $(n + d)$ objects in total. By choosing which of these places are occupied by stars determines the locations of the bars and vice versa. Hence, the total number of monomials is $\binom{n+d}{d} = \binom{n+d}{n}$. $\square$

Consequences

Corollary (Parameter Counting)

Given at most $\binom{n+d}{d}$ distinct points $\mathcal{P}$, we can find a nonzero polynomial of degree at most n which vanishes on $\mathcal{P}$.

Proof.

$\mathcal{P}$ imposes at most $\binom{n+d}{d} - 1$ conditions, but polynomials of degree at most n have $\binom{n+d}{d}$ degrees of freedom. Hence, there is a subspace of dimension at least 1 satisfying the constraints. □

$$\binom{n+d}{d} = \frac{(n+d)!}{d!n!} = \frac{(n+d)\cdots(n+1)}{d!} \sim_d n^d$$

In other words, given a set of N distinct points $\mathcal{P} \subset \mathbb{F}^d$, we can find a non-zero polynomial f in d-variables such that $f(p) = 0$ for all $p \in \mathcal{P}$, such that $\deg f \leq CN^{\frac{1}{d}}$ for some $C = C(d)$.

Vanishing Lemma

Lemma (Vanishing)

Let $f \in \mathbb{F}[t]$ be a polynomial. If $|Z(f)| > \deg f$ then $f \equiv 0$.

Proof.

Contrapositive of the Fundamental Theorem of Algebra. $\square$

Joints Theorem

Lemma

There is a constant $C = C(d)$ such that: For any $J' \subset J$, if $m \in \mathbb{N}$ is such that every $l \in \mathcal{L}$ which intersects J' is such that $|l \cap J'| \geq m$, then $|J'| \geq Cm^d$.

That is, joints configurations behave like the Loomis–Whitney style lattice.

Proof of Lemma (1/3).

WLOG, assume $m > 1$. For a contradiction, suppose otherwise. Then for any $K > 0$, we can find a collection of lines $\mathcal{L}$ and subset of joints $J' \subset J$ satisfying the hypothesis for some m so that $|J'| < \frac{1}{K}m^d$.

Joints Theorem

Proof of Lemma (2/3).

By Parameter Counting, we can find a non-zero polynomial f so that $f(p) = 0$ for all $p \in J'$, and $\deg f \leq \frac{D}{K^{\frac{1}{d}}} m$ for some $D = D(d)$. Of all possible f, choose one which has the least degree. Since $m > 1$, J' is not co-linear and so $\deg f > 1$. We are free to choose K large enough so that $\deg f < m$. Let $p \in J'$, and $l \ni p$, then $|J' \cap l| \geq m$. So $|Z(f) \cap l| \geq m$. By the Vanishing Lemma, the one-variable polynomial $f|_l$ is identically 0. Hence $(\nabla f(p)) \cdot e(l) = (\nabla \cdot e(l)) f(p) = 0$.

Proof of Lemma (3/3).

Since p is a joint, there are d linearly independent such lines l. Therefore, $\nabla f(p)$ is perpendicular to a spanning set of vectors, and so $\nabla f(p) = 0$. Since $p \in J'$ was arbitrary, every component of ∇f is zero at every $p \in J'$, but has strictly smaller degree than f and since $\deg f > 1$, there is a component of ∇f which is not identically zero. □

Proof of Joints Theorem (Quilodrán)

Proof.

Let $m = K|J|^{\frac{1}{d}}$, where K is chosen large enough, but depending only on d, so that $K^d C > 1$, where C is the constant from the Lemma. Let $\mathcal{L}' \subseteq \mathcal{L}$, and let $J' \subseteq J$ be the set of joints formed by $\mathcal{L}'$. If every line $l' \in \mathcal{L}'$ contains at least m points of J', then

$$|J'| \geq Cm^d = CK^d|J| > |J|,$$

a contradiction. Hence, for every pair $\mathcal{L}', J'$, there is a line $l' \in \mathcal{L}'$ that contains at most m elements of J'. So, by iteratively removing lines which contain fewer than m joints, we may cover J by at most $|\mathcal{L}|$ sets of at most m joints. By subadditivity of the counting measure,

$$|J| \leq m|\mathcal{L}| \leq K(d)|\mathcal{L}||J|^{\frac{1}{d}}.$$

□

Remark

Without going into detail, the cover into sets of at most m joints can be constructed using converse vanishing lemma – i.e. Fundamental Theorem of Algebra, or more generally, Bézout's Theorem.

Next Time

- How might we generalise the Joints Theorem.
  - Joint “multiplicity”
  - Lines $\rightarrow$ planes?
  - Lines $\rightarrow$ curves?
  - Lines $\rightarrow$ varieties?
  - Single collection of algebraic objects $\rightarrow$ many collections?
  - Does this geometric problem have a functional form?
- In what way do our tools fail for these generalisations?
- My remaining contribution: A digest of Tidor–Yu–Zhao (2020) which solves the problem in full generality.