Kakeya maximal function conjecture

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For a set $E \subseteq \mathbb{R}^n$, define by E_δ the δ-neighbourhood of E, that is

$$E_\delta = \{x \in R^n : dist(x, E) \leq \delta\}.$$

Definition (Minkowski dimension)

Let E be a compact subset of $\mathbb{R}^n$. The Minkowski dimension of E is defined as:

$$dim_M(E) = n + \lim_{\delta \to 0} \frac{\log |E_\delta|}{\log \delta^{-1}}$$

Definition (Hausdorff dimension)

Define the Hausdorff d-measure $\mathscr{H}^d$ of set $E \subseteq \mathbb{R}^n$ by

$$\mathscr{H}^d(E) = \lim_{\epsilon \to 0} \mathscr{H}^d_\epsilon(E)$$

where $\mathscr{H}^d_\epsilon = \inf \left\{ \sum_j r(B_j)^d : r(B_j) \leq \epsilon, E \subseteq \bigcup_j B_j \right\}$.
Then we say that E has Hausdorff dimension d if $\mathscr{H}^{d+\epsilon}(E) = 0$ and $\mathscr{H}^{d-\epsilon}(E) = \infty$ for any $\epsilon > 0$.

Definition.

A Kakeya set is a subset of $\mathbb{R}^n$ which has a unit line segment in every direction.

Kakeya set conjecture.

All Kakeya sets in $\mathbb{R}^n$ have Hausdorff and Minkowski dimension n.

Kakeya maximal operator conjecture.

Let $\{T_\omega\}_{\omega\in\Omega}$ be any collection of tubes of equal sizes and $\frac{1}{\delta}$ eccentricity, whose orientation are δ-separated. Then for each $\epsilon > 0$

$$\left\|\sum_{\omega\in\Omega} \mathbb{1}_{T_\omega}\right\|_{\frac{n}{n-1}} \lesssim \delta^{-\epsilon}\left(\sum_{\omega\in\Omega} |T_\omega|\right)^{\frac{n-1}{n}} \tag{1}$$

Notice that when T_ω are disjoint, the left-hand side equals to the right-hand side in (1) except for the $\delta^{-\epsilon}$ term.

Let us look at the quantity α, which measures the overlap of the tubes:

$$|\bigcup_{\omega\in\Omega} T_\omega| = \alpha \sum_{\omega\in\Omega} |T_\omega| \qquad \text{for } \alpha \in (0,1]$$

We have

$$\begin{aligned}\sum_{\omega\in\Omega} |T_\omega| &= \left\| \sum_{\omega\in\Omega} \mathbb{1}_{T_\omega} \right\|_1 \\ &\leq \left\| \sum_{\omega\in\Omega} \mathbb{1}_{T_\omega} \right\|_{\frac{n}{n-1}} \left| \bigcup_{\omega\in\Omega} T_\omega \right|^{1/n} \\ &\lesssim \delta^{-\epsilon} \left(\sum_{\omega\in\Omega} |T_\omega| \right)^{\frac{n-1}{n}} \left(\alpha \sum_{\omega\in\Omega} |T_\omega| \right)^{1/n}\end{aligned}$$

i.e. $\delta^{\epsilon} \lesssim \alpha$.

More generally, we will use the notation $A \lessapprox B$ if $A \leq C_\epsilon \delta^{-\epsilon} B$ for any $\epsilon > 0$ and $A \approx B$ if $A \lessapprox B$ and $B \lessapprox A$. So that we now have

$$\alpha \approx 1. \tag{2}$$

Theorem

The Kakeya maximal operator conjecture implies the Kakeya set conjecture.

Proof (1/5).

Minkowski: Let K be a Kakeya set in $\mathbb{R}^n$. Then K_δ contains a $\delta \times 1$ tube in every direction and there will certainly be $C_n \delta^{1-n}$ many tubes in K_δ that are δ-separated in direction. Thus, by (2) we get that

$$|K_\delta| \geq \left| \bigcup_{\omega \in \Omega} T_\omega \right| \geq C_\epsilon^{-1} \delta^\epsilon \sum_{\omega \in \Omega} |T_\omega| \geq C_\epsilon^{-1} \delta^\epsilon C_n \delta^{1-n} |T_\omega| \gtrsim \delta^\epsilon$$

and consequently

$$dim_M(K) = n + \lim_{\delta \to 0} \frac{\log |E_\delta|}{\log \delta^{-1}} \geq n - \epsilon$$

for each $\epsilon > 0$. So that $dim_M(K) = n$.

Proof (2/5).

Hausdorff: We want to show that $\mathscr{H}^{n-\epsilon_0}(K) = \infty$ for each $\epsilon_0 > 0$. More specifically we want to show that if we cover K by a collection of balls B of radius at most 2^{-j}, then

$$\sum_B r(B)^{n-\epsilon_0} > c_j, \tag{3}$$

where $c_j \to \infty$ as $j \to \infty$.
Let $\epsilon_0 > 0$, $j \gg 0$ and take a covering of E as above. Suppose for contradiction that

$$\sum_B r(B)^{n-\epsilon_0} \lesssim c_j.$$

WLOG we can assume that each ball has radius 2^{-i} with $i \geq j$ and that each ball intersects a bounded number of balls of the same size.

Proof (3/5).

For each $\omega \in \mathbb{S}^{n-1}$, let l_ω be the corresponding unit line segment in K and $\mathscr{H}^1_{|l_\omega}$ be the associated Lebesgue measure. Averaging over all these we obtain a measure $\mu := \int_{\mathbb{S}^{n-1}} \mathscr{H}^1_{|l_\omega} d\omega$ on K with total mass 1. In particular,

$$\sum_B \mu(B) \geq \mu(\bigcup_B B) = 1,$$

so there is $k \geq j$ such that

$$\sum_{B:r(B)=2^{-k}} \mu(B) \geq k^{-2}. \tag{4}$$

Fix such k and let $\delta = 2^{-k}$. Now throw away all balls B from our collection other than the ones with radius δ. Next, we have

$$\mathscr{H}^1(l_\omega \cap B) \lesssim \delta^{1-n} \int_B \mathbb{1}_{T_\omega}(x) dx$$

where T_ω is the $\delta \times 1$ tube centred around the line l_ω. Thus

$$\mu(B) \lesssim \delta^{1-n} \int_B \left(\int_{\mathbb{S}^{n-1}} \mathbb{1}_{T_\omega} d\omega \right) dx \tag{5}$$

Proof (4/5).

Using (4) together with $1 \approx k^{-2}$ and the fact that the balls are essentially disjoint (bounded number of intersections),

$$1 \lessapprox \mu(\bigcup_B B) \lessapprox \delta^{1-n} \int_{\bigcup_B B} \left(\int_{\mathbb{S}^{n-1}} \mathbb{1}_{T_\omega} \, d\omega \right) dx.$$

We then can choose a δ-separated subset of directions $\Omega \subseteq \mathbb{S}^{n-1}$ with $|\Omega| \sim \delta^{1-n}$ (e.g. maximally δ-separated tubes) such that

$$1 \lessapprox \int_{\bigcup_B B} \left(\sum_{\omega \in \Omega} \mathbb{1}_{T_\omega} \right) dx.$$

By Hölder's inequality

$$\int_{\bigcup_B B} \left(\sum_{\omega \in \Omega} \mathbb{1}_{T_\omega} \right) dx \leq \left\| \sum_{\omega \in \Omega} \mathbb{1}_{T_\omega} \right\|_{\frac{n}{n-1}} \left| \bigcup_B B \right|^{1/n}.$$

Proof (5/5).

The maximal function estimate then yields

$$1 \lessapprox \left(\sum_{\omega \in \Omega} |T_\omega| \right)^{\frac{n-1}{n}} \left| \bigcup_B B \right|^{1/n}.$$

This gives us

$$\left| \bigcup_B B \right| \approx 1.$$

As Bs have bounded overlap, the estimate above forces the number of balls to be $\approx \delta^{-n}$. But then

$$\sum_{B : r(B) = \delta} r(B)^{n-\epsilon_0} \approx \delta^{-n} \delta^{n-\epsilon_0} = \delta^{-\epsilon_0} \gg 1.$$

□

Kakeya maximal operator

Kakeya maximal function.

For any $\delta \in (0,1)$ and $\omega \in \mathbb{S}^{n-1}$ and $a \in \mathbb{R}^n$, let

$$T_\omega^\delta(a) = \{x \in \mathbb{R}^n : |(x-a)\cdot\omega| \le 1/2, |(x-a) - \omega[(x-a)\cdot\omega]| \le \delta\}.$$

For a function $f \in L^1_{loc}(\mathbb{R}^n)$, we define the *Kakeya maximal function* of width δ by setting $\mathcal{K}_\delta f : \mathbb{S}^{n-1} \to [0,\infty]$

$$\mathcal{K}_\delta f(\omega) = \sup_{a \in \mathbb{R}^n} \frac{1}{|T_\omega^\delta(a)|} \int_{T_\omega^\delta(a)} |f(y)| dy.$$

Basic facts:

- $\|\mathcal{K}_\delta f\|_{L^\infty(\mathbb{S}^{n-1})} \le \|f\|_{L^\infty(\mathbb{R}^n)}$
- $\|\mathcal{K}_\delta f\|_{L^\infty(\mathbb{S}^{n-1})} \lesssim \delta^{1-n} \|f\|_{L^1(\mathbb{R}^n)}$
- For $1 \le q \le \infty$, and $p < \infty$, no other bound of the form

  $$\|\mathcal{K}_\delta f\|_{L^q(\mathbb{S}^{n-1})} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$$

  holds with constant independent of δ.
- For any $\epsilon > 0$ and $1 \le p < \infty$

  $$\|\mathcal{K}_\delta f\|_{L^p(\mathbb{S}^{n-1})} \lesssim_{\epsilon,p} \delta^{-\epsilon} \|f\|_{L^p(\mathbb{R}^n)}$$

  fails for $p < n$.

Kakeya maximal operator conjecture 2.

For all $\epsilon > 0$, $n \in \mathbb{N}$ and $\delta \in (0,1)$ and $f \in L^n(\mathbb{R}^n)$ we have

$$\|\mathcal{K}_\delta f\|_{L^n(\mathbb{S}^{n-1})} \lesssim_{\epsilon,n} \delta^{-\epsilon} \|f\|_{L^n(\mathbb{R}^n)}.$$

Proposition.

Let $\epsilon > 0$ and $\delta \in (0,1)$. Then

$$\|\mathcal{K}_\delta f\|_{L^n(\mathbb{S}^{n-1})} \lesssim_{\epsilon,n} \delta^{-\epsilon} \|f\|_{L^n(\mathbb{R}^n)}$$

if and only if

$$\left\| \sum_{\omega \in \Omega} \mathbb{1}_{T_\omega} \right\|_{\frac{n}{n-1}} \lesssim \delta^{-\epsilon} \left(\delta^{n-1} \# \Omega \right)^{\frac{n-1}{n}}$$

for all collections of δ-separated δ tubes $\{T_\omega\}_{\omega \in \Omega}$.

Proof.

($\Rightarrow$) Let $n' = \frac{n}{n-1}$ and take $g \in L^{n'}(\mathbb{R}^n)$ with norm one. Then

$$\begin{aligned}\int \sum_{\omega\in\Omega} \mathbb{1}_{T_\omega} g = \sum_{\omega\in\Omega} \int_{T_\omega} g \lesssim \sum_{\omega\in\Omega} \delta^{n-1}(\mathcal{K}_\delta g)(\omega) \\ \lesssim \int_{\bigcup_{\omega\in\Omega} B(\omega,\delta)} (\mathcal{K}_\delta g)(e) d\sigma(e) \\ \leq \|\mathcal{K}_\delta g\|_{L^n(\mathbb{S}^{n-1})} \left| \bigcup_{\omega\in\Omega} B(\omega,\delta) \right|^{1/n'} \\ \lesssim \delta^{-\epsilon} M(\#\Omega\delta^{n-1})^{\frac{n-1}{n}}.\end{aligned}$$

Now take supremum over g with one norm to get the result. □

References

[1] T. Tao: *Restriction theorems and applications*, available at https://www.math.ucla.edu/~tao/254b.1.99s/ .

[2] O. Saari: *Advanced topics in analysis: Geometric Fourier Analysis*, available at https://www.math.uni-bonn.de/people/saari/teaching/luentomoniste_fourier.pdf