Harmonic Analysis Working Group :-

Bourgain's circular maximal theorem

Lecture 5

We now have all the ingredients in place to prove the circular maximal theorem.

Let $\mathcal{C}$ be a collection of unit-scale circles with δ-separated centres. We will first partition

$$\mathcal{C} = \mathcal{C}_{trans} \cup \mathcal{C}_{tang}.$$

The set $\mathcal{C}_{trans}$ will satisfy the hypotheses of the transversal case in Lecture 3 and therefore

$$\Big\| \sum_{C \in \mathcal{C}_{trans}} \tilde{\chi}_{C^\delta} \Big\|_{L^2(\mathbb{R}^2)} \lesssim \delta^{\varepsilon} [\# \mathcal{C}]^{1/2} \qquad (1)$$

for some $\varepsilon > 0$. On the other hand, we will show $\mathcal{C}_{tang}$ satisfies

$$\Big\| \sum_{C \in \mathcal{C}_{tang}} \tilde{\chi}_{C^\delta} \Big\|_{L^1(\mathbb{R}^2)} \lesssim \delta^{1+\varepsilon} \# \mathcal{C} \qquad (2).$$

Combining (1) and (2), we obtain Proposition 1 from lecture 3, via the simple interpolation argument outlined earlier.

Defining the partition.

Henceforth, we fix ε to be a small, positive number, chosen so as to satisfy the forthcoming requirements of the proof. Taking $\varepsilon = \frac{1}{100}$ certainly suffices. We also let $0 < \eta << \varepsilon$ be a second small, positive number; here we may explicitly choose $\eta := \varepsilon/10$.

We construct $\mathcal{C}_{trans}$ via a greedy algorithm, successively removing circles from $\mathcal{C}$ which form too many tangent pairs.

- Let $\mathcal{C}^{(0)} := \mathcal{C}$
- Suppose $\mathcal{C}^{(0)}, \dots, \mathcal{C}^{(J)}$ have been constructed for some $J \in \mathbb{N}_0$.

If for all $C \in \mathcal{C}^{(J)}$ the set

$$\mathcal{C}_C^{(J)} := \{ \bar{C} \in \mathcal{C}^{(J)} : \Delta(C, \bar{C}) \leq \delta^{1-\eta} \}$$

satisfies $\# \mathcal{C}_C^{(J)} \lesssim \delta^{2/3 + 4\varepsilon} \delta^{-2}$, then the algorithm terminates and we define $\mathcal{C}_{trans} := \mathcal{C}^{(J)}$.

Otherwise, there exists some $C_J \in \mathcal{C}^{(J)}$ such that

$$\# \mathcal{C}_{C_J}^{(J)} > \delta^{2/3 + 4\varepsilon} \delta^{-2}$$

and we define $\mathcal{C}^{(J+1)} := \mathcal{C}^{(J)} \setminus \mathcal{C}_{C_J}^{(J)}$.

Clearly this algorithm terminates after finitely many steps. Let $J \in \mathbb{N}_0$ be the terminal index, so that $\mathcal{C}_{trans} := \mathcal{C}^{(J)}$. By construction, we have

$$\max_{C \in \mathcal{C}_{trans}} \# \{ \bar{C} \in \mathcal{C}_{trans} : \Delta(C, \bar{C}) \leq \delta^{1-\eta} \} \leq \delta^{2/3 + 4\varepsilon} \delta^{-2}$$

and so $\mathcal{C}_{trans}$ satisfies the hypothesis of Lemma 2. In particular, (1) is automatically guaranteed.

It remains to prove the L^1 bound (2) for the collection $\mathcal{C}_{tang} := \mathcal{C} \setminus \mathcal{C}_{trans}$.

From the construction, there exist circles $C_0, \dots, C_{J-1} \in \mathcal{C}$ such that

$$\mathcal{C}_{tang} = \bigcup_{j=0}^{J-1} \mathcal{C}_{C_j}^{(j)},$$

where the union is disjoint. Furthermore, each set

$$\mathcal{C}_{C_j}^{(j)} = \mathcal{C}_{C_j} \setminus \bigcup_{i=1}^{j-1} \mathcal{C}_{C_i}$$

satisfies

$$\# \mathcal{C}_{C_j}^{(j)} \geqslant \delta^{2/3 + 4\varepsilon} \delta^{-2}.$$

In order to prove (2), it suffices to show

$$\Big\| \sum_{C \in \mathcal{C}_{C_j}^{(j)}} \tilde{\chi}_{C_\delta} \Big\|_{L^1(\mathbb{R}^2)} \lesssim \delta^{\varepsilon} \# \mathcal{C}_{C_j}^{(j)}, \quad 0 \leq j \leq J-1,$$

since the estimates can then be summed using the disjointedness of the sets $\mathcal{C}_{C_j}^{(1)}$.

In particular, matters are reduced to proving the following lemma.

Lemma 11 (Tangent case $\Rightarrow L^1$ improvement) Suppose $\mathcal{A}$ is a collection of unit-scale circles with δ-separated centres, satisfying the following:-

i) There exists some $C_0 \in \mathcal{A}$ such that

$$\Delta(C, C_0) \leq \delta^{1-\eta} \quad \text{for all} \quad C \in \mathcal{A}, \qquad (3)$$

ii) $\# \mathcal{A} \geq \delta^{2/3 + 4\varepsilon} \delta^{-2}$. (

Then

$$\Big\| \sum_{C \in \mathcal{A}} \tilde{\chi}_{C^\delta} \Big\|_{L^1(\mathbb{R}^2)} \lesssim \delta^{\varepsilon} \# \mathcal{A}. \qquad ($$

Here $0 < \eta < \varepsilon \ll 1$ are as above (so we may take $\varepsilon = 1/100$ and $\eta = \varepsilon/10$).

Proof :- We break the argument into steps.

Step 1 : Dyadic pigeonholing.

We dyadically pigeonhole the distance $\mathrm{dist}(C, C_0)$ between $C \in \mathcal{A}$ and the fixed circle C_0. In particular, let

$$\mathcal{A}_0 := \{ C \in \mathcal{A} : \mathrm{dist}(C, C_0) < \delta^{1/2} \}$$

and for $1 \leq \ell \leq \lceil \log_2 \delta^{-1} \rceil$ let

$$\mathcal{A}_\ell := \{ C \in \mathcal{A} : 2^{\ell - 1} \delta^{1/2} \leq \mathrm{dist}(C, C_0) < 2^{\ell} \delta^{1/2} \}.$$

Thus, $\mathcal{A} = \bigcup_{\ell = 0}^{\lceil \log_2 \delta^{-1} \rceil} \mathcal{A}_\ell$ and it suffices to show

$$\Big\| \sum_{C \in \mathcal{A}_\ell} \tilde{\chi}_{C^\delta} \Big\|_{L^1(\mathbb{R}^2)} \lesssim \delta^{2\varepsilon} \# \mathcal{A},$$

$0 \leq \ell \leq \lceil \log_2 \delta^{-1} \rceil$. By additional pigeonholing, we can also assume either $r \geq r_0$ for ~~all~~ $C(x, r) \in \mathcal{A}_\ell$ or

Step 2: Tightly clustered case.

For $\ell = 0$, we trivially have

$$\Big\| \sum_{C \in \mathcal{A}_0} \tilde{\chi}_{C^\delta} \Big\|_{L^1(\mathbb{R}^2)} \lesssim \sum_{C \in \mathcal{A}_0} \| \tilde{\chi}_{C^\delta} \|_{L^1(\mathbb{R}^2)} \lesssim \delta \cdot \# \mathcal{A}_0 .$$

However, since the circles in $\mathcal{A}_0$ have δ-separated centres, it follows that

$$\# \mathcal{A}_0 \lesssim \left(\delta^{1/2} / \delta \right)^2 = \delta^{-1}.$$

On the other hand, by hypothesis ii) of the lemma,

$$\delta^{-1} = \delta^{1/3 - 4\varepsilon} \, \delta^{2/3 + 4\varepsilon} \, \delta^{-2} \leq \delta^{1/3 - 4\varepsilon} \# \mathcal{A} .$$

Combining these observations,

$$\Big\| \sum_{C \in \mathcal{A}_0} \tilde{\chi}_{C^\delta} \Big\|_{L^1(\mathbb{R}^2)} \lesssim \delta^{1 + 1/3 - 4\varepsilon} \# \mathcal{A} \leq \delta^{1 + 2\varepsilon} \# \mathcal{A} .$$

for $0 < \varepsilon \ll 1$ sufficiently small.

Step 3: Spatial partition.

Henceforth, let $1 \leq \ell \leq \lceil \log_2 \delta^{-1} \rceil$ and define $\rho := 2^\ell \delta^{1/2}$. We also let $\kappa := \delta^{5\varepsilon}$ and

$$\sigma =: \sigma(\rho, \kappa) := A_1 \kappa \rho$$

as in Lemma 5.

Let $\psi_{C_0^\sigma} \in C_c^\infty(\mathbb{R}^2)$ denote a bump function adapted to the annulus C_0^σ. In particular,

- $\psi_{C_0^\sigma}(x) = 1$ if $x \in C_0^\sigma$
- $\psi_{C_0^\sigma}(x) = 0$ if $x \notin C_0^{2\sigma}$
- $|\nabla \psi_{C_0^\sigma}(x)| \lesssim \sigma^{-1}$ for all $x \in \mathbb{R}^2$
- $0 \leq \psi_{C_0^\sigma}(x) \leq 1$ for all $x \in \mathbb{R}^2$.

We decompose

$$\Big\|\sum_{C\in\mathcal{A}_\ell}\tilde{\chi}_{C^\delta}\Big\|_{L^1(\mathbb{R}^2)} \leq \Big\|\sum_{C\in\mathcal{A}_\ell}\tilde{\chi}_{C^\delta}\cdot\psi_{C_0^\sigma}\Big\|_{L^1(\mathbb{R}^2)} + \Big\|\sum_{C\in\mathcal{A}_\ell}\tilde{\chi}_{C^\delta}(1-\psi_{C_0^\sigma})\Big\|_{L^1(\mathbb{R}^2)}. \qquad (4)$$

Step 4: The local contribution.

To estimate the first term on the right-hand side of (4), we note that

$$\Big\|\sum_{C\in\mathcal{A}_\ell}\tilde{\chi}_{C^\delta}\cdot\psi_{C_0^\sigma}\Big\|_{L^1(\mathbb{R}^2)} \leq \sum_{C\in\mathcal{A}_\ell}\|\tilde{\chi}_{C^\delta}\|_{L^1(C_0^{2\sigma})}$$
$$\lesssim \sum_{C\in\mathcal{A}_\ell}|C^{\delta,*}\cap C_0^{2\sigma}| + \delta^{100}, \qquad (5)$$

by the essential support property of the $\tilde{\chi}_{C^\delta}$ and, in particular, Corollary 7. The intersections satisfy

$$|C^{\delta,*}\cap C_0^{2\sigma}| \lesssim \delta^{1-\eta}\Big(\frac{\sigma}{\mathrm{dist}(C,C_0)}\Big)^{1/2}$$
$$\lesssim \delta^{1-\eta}\kappa^{1/2}. \qquad (6)$$

Combining (5) and (6), we have

$$\Big\|\sum_{C\in\mathcal{A}_\ell}\tilde{\chi}_{C^\delta}\cdot\psi_{C_0^\sigma}\Big\|_{L^1(\mathbb{R}^2)} \lesssim \delta^{1-\eta}\delta^{5\varepsilon/2}\,\#\mathcal{A}_\ell$$
$$\lesssim \delta^{1+2\varepsilon}\,\#\mathcal{A},$$

as required.

Step 5: Non-local contribution: trichotomy.

We now consider the second term on the right-hand side of (4).

First note that for each $C\in\mathcal{A}_\ell$, we have

$$C^{\delta,*} \subseteq C_0^{10\ell}$$

and so, by the essential support property and Cauchy-Schwarz,

$$\Big\|\sum_{C\in\mathcal{C}_\ell}\tilde{\chi}_{C^\delta}(1-\psi_{C_0^\sigma})\Big\|_{L^1(\mathbb{R}^2)} \lesssim \rho^{1/2}\Big\|\sum_{C\in\mathcal{C}_\ell}\tilde{\chi}_{C^\delta}(1-\psi_{C_0^\sigma})\Big\|_{L^2(\mathbb{R}^2)} + \delta^{100}. \qquad (7)$$

As usual, we express the L^2 norm interms of the inner product

$$\Big\|\sum_{C\in\mathcal{C}_\ell}\tilde{\chi}_{C^\delta}(1-\psi_{C_0^\sigma})\Big\|^2_{L^2(\mathbb{R}^2)} = \sum_{C_1,C_2\in\mathcal{C}_\ell}\langle\tilde{\chi}_{C_1^\delta}(1-\psi_{C_0^\sigma}),\tilde{\chi}_{C_2^\delta}(1-\psi_{C_0^\sigma})\rangle$$

$$= \sum_{C_1,C_2\in\mathcal{C}_\ell}\langle\tilde{\chi}_{C_1^\delta},\tilde{\chi}_{C_2^\delta}\rangle_{L^2(u)} \qquad (8)$$

where $u := |(1-\psi_{C_0^\sigma})|^2$ and

$$\langle f,g\rangle_{L^2(u)} := \int_{\mathbb{R}^2} f(x)\,\overline{g(x)}\,u(x)\,dx.$$

We perform a trichotomy of the indexing set $\mathcal{C}_\ell\times\mathcal{C}_\ell$. In particular, for each $C_1\in\mathcal{C}_\ell$ we write $\mathcal{C}_\ell$ as a disjoint union

$$\mathcal{C}_\ell = \mathcal{C}_{diag}(C_1)\cup\mathcal{C}_{tang}(C_1)\cup\mathcal{C}_{trans}(C_1)$$

where, for $\beta := \delta^{1-\eta}$ the parameter in (3),

$$\mathcal{C}_{diag}(C_1) := \{C_2\in\mathcal{C}_\ell : \operatorname{dist}(C_1,C_2)\lesssim K^{-1}\beta\}$$

$$\mathcal{C}_{tang}(C_1) := \{C_2\in\mathcal{C}_\ell : \operatorname{dist}(C_1,C_2)\gtrsim K^{-1}\beta \text{ and } \Delta(C_1,C_2)\leq K\operatorname{dist}(C_1,C_2)\}$$

$$\mathcal{C}_{trans}(C_1) := \{C_2\in\mathcal{C}_\ell : \operatorname{dist}(C_1,C_2)\geq K^{-1}\beta \text{ and } \Delta(C_1,C_2) > K\operatorname{dist}(C_1,C_2)\}.$$

This induces a corresponding decomposition of the right-hand sum in (8). We treat each term individually.

Step 6: Diagonal term.

By the pointwise bound $|(1-\psi_{C_0^\sigma})|\leq 1$ and a 3-fold application of Cauchy-Schwarz,

$$\Sigma_{diag} := \sum_{C_1\in\mathcal{C}_\ell}\ \sum_{C_2\in\mathcal{C}_{diag}(C_1)} |\langle\tilde{\chi}_{C_1^\delta},\tilde{\chi}_{C_2^\delta}\rangle_{L^2(u)}|$$

$$\leq \sum_{C_1 \in \mathcal{A}_e} \| \tilde{\chi}_{C_1^\delta} \|_{L^2(\mathbb{R}^2)} \sum_{C_2 \in \mathcal{A}_{diag}(C_1)} \| \tilde{\chi}_{C_2^\delta} \|_{L^2(\mathbb{R}^2)}$$

$$\leq \max_{C \in \mathcal{A}_e} \# \mathcal{A}_{diag}(C) \sum_{C \in \mathcal{A}_e} \| \tilde{\chi}_{C^\delta} \|^2_{L^2(\mathbb{R}^2)} .$$

Since the circles in $\mathcal{A}_e$ have δ-separated centres,

$$\max_{C \in \mathcal{A}_e} \# \mathcal{A}_{diag}(C) \lesssim (K^{-1}\beta/\delta)^2 \lesssim \delta^{-11\varepsilon} .$$

Thus, we have

$$\sum_{C_1 \in \mathcal{A}_e} \sum_{C_2 \in \mathcal{A}_{diag}(C_1)} | \langle \tilde{\chi}_{C_1^\delta}, \tilde{\chi}_{C_2^\delta} \rangle_{L^2(\omega)} |$$

$$\lesssim \delta^{-11\varepsilon} \sum_{C \in \mathcal{A}_e} \| \tilde{\chi}_{C^\delta} \|^2_{L^2(\mathbb{R}^2)}$$

$$\lesssim \delta^{1-11\varepsilon} \# \mathcal{A} .$$

Recall from hypothesis ii) that $\# \mathcal{A} \geq \delta^{2/3 + 4\varepsilon} \delta^{-2}$ and so

$$\# \mathcal{A} = \frac{1}{\# \mathcal{A}} [\# \mathcal{A}]^2 \leq \delta^{4/3 - 4\varepsilon} [\# \mathcal{A}]^2 . \qquad (9)$$

Thus,

$$\sum_{diag} \lesssim \delta^{2+4\varepsilon} \delta^{1/3 - 19\varepsilon} [\# \mathcal{A}]^2$$

$$\lesssim \delta^{2+4\varepsilon} [\# \mathcal{A}]^2 \qquad (10).$$

Step 7 : Tangent term.

Note that $\operatorname{supp}(1 - \chi_{C_0^\sigma}) \subseteq \mathbb{R}^2 \setminus C_0^\sigma$ and so

$$\sum_{tang} := \sum_{C_1 \in \mathcal{A}_e} \sum_{C_2 \in \mathcal{A}_{tang}(C_1)} | \langle \tilde{\chi}_{C_1^\delta}, \tilde{\chi}_{C_2^\delta} \rangle_{L^2(\omega)} |$$

$$\lesssim \sum_{C_1 \in \mathcal{A}_e} \sum_{C_2 \in \mathcal{A}_{tang}(C_1)} | C_1^{\delta,*} \cap C_2^{\delta,*} \cap \mathbb{R}^2 \setminus C_0^\sigma | + \delta^{100} [\# \mathcal{A}]^2 \qquad (11)$$

by the essential support property of the $\tilde{\chi}_{C^\delta}$ and, in particular, Corollary 7.

Observe that for $C_1 \in \mathcal{A}_e$, $C_2 \in \mathcal{A}_{tang}(C_1)$ we have

i) $\Delta(C_0, C_1)$, $\Delta(C_0, C_2) \leq \beta$ by (3);
ii) $e/2 \leq \mathrm{dist}(C_0, C_1), \mathrm{dist}(C_0, C_2) \leq e$ by the definition of $\mathcal{A}_e$
iii) $r_1, r_2 \geq r$ or $r_1, r_2 \leq r$, again by the definition of $\mathcal{A}_e$.

Furthermore

$$\Delta(C_1, C_2) + \beta \lesssim \kappa \,\mathrm{dist}(C_1, C_2)$$

by the definition of $\mathcal{A}_{tang}(C_1)$. Thus, we may apply Lemma 5 to conclude that

$$C_1^{\delta, *} \cap C_2^{\delta, *} \subseteq C_0^{\sigma}.$$

Thus, from (11) we see

$$\Sigma_{tang} \lesssim \delta^{100} [\# \mathcal{A}]^2 \qquad (12).$$

Step 8: Transverse term.

Finally, we turn to the transverse term

$$\Sigma_{trans} := \sum_{C_1 \in \mathcal{A}_e} \sum_{C_2 \in \mathcal{A}_{trans}(C_1)} |\langle \tilde{\chi}_{C_1^\delta}, \tilde{\chi}_{C_2^\delta} \rangle_{L^2(\omega)}|.$$

The idea is to use the transversality to obtain a good bound for the $|\langle \tilde{\chi}_{C_1^\delta}, \tilde{\chi}_{C_2^\delta} \rangle_{L^2(\omega)}|$, as in the transverse case dealt with in Lemma 2. However, we cannot appeal directly to the weak-orthogonality lemma here, since the inner products now include the cut-off function ω.

Claim: For $C_1 \in \mathcal{A}_e$ and $C_2 \in \mathcal{A}_{trans}(C_1)$, we have

$$|\langle \tilde{\chi}_{C_1^\delta}, \tilde{\chi}_{C_2^\delta} \rangle_{L^2(\omega)}| \lesssim \sigma^{-1} \frac{\delta^{1-\eta}}{\Delta(C_1, C_2)} |C_1^{\delta,*} \cap C_2^{\delta,*}| + \delta^{100}.$$

(13)

Proof (of Claim). The proof is based on modifying the weak orthogonality lemma.

First note, by Corollary 7,

$$|\langle \tilde{\chi}_{C_1^\delta}, \tilde{\chi}_{C_2^\delta} \rangle_{L^2(u)}| \lesssim |C_1^{\delta,*} \cap C_2^{\delta,*}| + \delta^{100}$$

and so (13') trivially holds if

$$\sigma^{-1} \frac{\delta^{1-\eta}}{\Delta(C_1, C_2)} \gtrsim 1.$$

Thus, henceforth we assume $\sigma^{-1} \frac{\delta^{1-\eta}}{\Delta(C_1,C_2)} << 1$ so that

$$\Delta(C_1, C_2) >> \sigma^{-1}\delta^{1-\eta}.$$

We can assume $C_1^{\delta,*} \cap C_2^{\delta,*} \neq \emptyset$, or else we are done by the essential support property. Under this hypothesis, it is easy to see

$$|r_1 - r_2| \lesssim |x_1 - x_2| + \delta^{1-\eta} \quad \text{and so} \quad \Delta(C_1, C_2) \lesssim \mathrm{dist}(C_1, C_2).$$

Consequently, $\sigma >> \dfrac{\delta^{1-\eta}}{\Delta(C_1,C_2)^{1/2}\,\mathrm{dist}(C_1,C_2)^{1/2}}$

From the geometric observations of Lemma 4, this means each connected component of $C_1^{\delta,*} \cap C_2^{\delta,*}$ is contained in a cube of side-length σ.

We decompose $u = u_1 + u_2$ where each u_j is supported in a cube of side-length σ and satisfies

$$|\partial_x^\alpha u_j(x)| \lesssim \sigma^{-|\alpha|} \quad \text{for all } x \in \mathbb{R}^2, \ \alpha \in \mathbb{N}_0^2,$$

$$\langle \tilde{\chi}_{C_1^\delta}, \tilde{\chi}_{C_2^\delta} \rangle_{L^2(u)} = \sum_{j=1,2} \langle \tilde{\chi}_{C_1^\delta}, \tilde{\chi}_{C_2^\delta} \rangle_{L^2(u_j)} + \delta^{100}.$$

Write

$$\langle \tilde{\chi}_{C_1^\delta}, \tilde{\chi}_{C_2^\delta} \rangle_{L^2(u_j)} = \int_{\mathbb{R}^2} (\tilde{\chi}_{C_1^\delta} \cdot \overline{\tilde{\chi}_{C_2^\delta}})(x)\, u_j(x)\, dx$$

$$= \int_{\mathbb{R}^2} (\tilde{\chi}_{C_1^\delta})^\wedge * (\overline{\tilde{\chi}_{C_2^\delta}})^\wedge(\zeta) \cdot \hat{u}_j(\zeta)\, d\zeta$$

by the Fourier multiplication formula. Now

$$(\tilde{\chi}_{C_1^\delta})^\wedge * (\overline{\tilde{\chi}_{C_2^\delta}})^\wedge(\zeta) = \int_{\mathbb{R}^2} (\tilde{\chi}_{C_1^\delta})^\wedge(\zeta - \xi)\, (\overline{\tilde{\chi}_{C_2^\delta}})^\wedge(-\xi)\, d\xi$$

$$= \delta^2 \int_{\mathbb{R}^2} \hat{\sigma}(r_1(\xi - \zeta))\, \beta(\delta|\xi - \zeta|)\, e^{2\pi i x_1 \cdot (\zeta - \xi)} \cdot \overline{\hat{\sigma}(r_2(-\xi))}\, \beta(\delta|\xi|)\, e^{2\pi i x_2 \cdot \xi}\, d\xi$$

We can express

$$\hat{\sigma}(\xi) = \sum_{\pm} e^{\pm 2\pi i |\xi|} a_\pm(\xi)$$

where $a_\pm \in S^{-1/2}$ are symbols of order $-1/2$. As in the weak orthogonality lemma, only certain sign choices give non-negligible contributions and we are left to consider

$$\delta^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{2\pi i (r_1|\xi - \zeta| - r_2|\xi| - (x_1 - x_2)\cdot\xi + x_1 \cdot \zeta)} a_1(|\xi - \zeta|)\, a_2(|\xi|)\, \beta(\delta|\xi - \zeta|)\, \beta(\delta|\xi|)\, d\xi\, \hat{u}_j(\zeta)\, d\zeta$$

where $a_1, a_2 \in S^{-1/2}$.

We rescale $\xi \mapsto \delta^{-1}\xi$ and $\zeta \mapsto \sigma^{-1}\zeta$, noting that the function

$$w(\zeta) := \sigma^{-2}\hat{u}_j(\sigma^{-1}\zeta)$$

is Schwarz with bounded derivatives. We now have

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{2\pi i \delta^{-1}(r_1|\xi - \sigma^{-1}\delta\zeta| - r_2|\xi| - (x_1 - x_2)\cdot\xi + x_1 \cdot \sigma^{-1}\delta\zeta)} a_1(\delta^{-1}|\xi - \sigma^{-1}\delta\zeta|)\, a_2(\delta^{-1}|\xi|)\, \beta(|\xi - \sigma^{-1}\delta\zeta|)\, \beta(|\xi|)\, d\xi\, w(\zeta)\, d\zeta$$

If we restrict the ς-integration to $|\varsigma| \gtrsim \delta^{-\eta}$, then

$$|\omega(\varsigma)| \lesssim \delta^{100} (1 + |\varsigma|)^{-10}$$

on this set, so we obtain the desired result.

We therefore restrict the ς-integration to $|\varsigma| \leq \delta^{-\eta}$.
Note that the ξ-gradient of the phase is

$$\delta^{-1}\left[(r_1 - r_2)\frac{\xi}{|\xi|} - (x_1 - x_2) + r_1\left(\frac{\xi - \sigma^{-1}\delta\varsigma}{|\xi - \sigma^{-1}\delta\varsigma|} - \frac{\xi}{|\xi|}\right)\right].$$

Notice that we always have

$$\left|\frac{\xi - \sigma^{-1}\delta\varsigma}{|\xi - \sigma^{-1}\delta\varsigma|} - \frac{\xi}{|\xi|}\right| \lesssim \sigma^{-1}\delta|\varsigma| \leq \sigma^{-1}\delta^{1-\eta} \ll \Delta(c_1, c_2)$$

since $|\xi - \sigma^{-1}\delta\varsigma| \sim |\xi| \sim 1$ over the domain of integration.

Thus, the gradient of the phase is bounded below in magnitude by

$$\delta^{-1}\left[\inf_{\omega \in S^1} |(r_1 - r_2)\omega - (x_1 - x_2)| - O(\delta^{1-\eta}\sigma^{-1})\right]$$

$$\gtrsim \delta^{-1}\Delta(c_1, c_2) \gtrsim \delta^{-1}\varkappa \cdot \mathrm{dist}(c_1, c_2) \gtrsim \delta^{-1}\beta = \delta^{-2}$$

Thus, repeated integration-by-parts implies that

$$|\langle \tilde{\chi}_{c_1^\delta}, \tilde{\chi}_{c_2^\delta} \rangle_{L^2(u)}| \lesssim \delta^{100}$$

in this case. $\square$

It is now a simple task to conclude the argument for the transverse case. By the claim and Lemma 10,

$$\Sigma_{trans} \lesssim \sum_{c_1 \in \mathcal{A}_c} \sum_{c_2 \in \mathcal{A}_{trans}(c_1)} \sigma^{-1} \frac{\delta^{1-\eta}}{\Delta(c_1, c_2)} |C_1^{\delta,\sigma} \cap C_2^{\delta,\sigma}| + \delta^{100} [\# \mathcal{A}]^2$$

$$\lesssim \delta^{3(1-\eta)} \sigma^{-1} \sum_{c_1 \in \mathcal{A}_c} \sum_{c_2 \in \mathcal{A}_{trans}(c_1)} \Delta(c_1, c_2)^{-3/2} \mathrm{dist}(c_1, c_2)^{-1/2} + \delta^{100} [\# \mathcal{A}]^2$$

From the definition of $\mathcal{A}_{trans}(C_1)$, we have

$$\Delta(C_1, C_2) \gtrsim \kappa \cdot \mathrm{dist}(C_1, C_2)$$

and so

$$\Sigma_{trans} \lesssim \delta^{1-3\eta} \kappa^{-5/2} e^{-1} \sum_{C_1, C_2 \in \mathcal{A}_L} (1 + \delta^{-1} \mathrm{dist}(C_1, C_2))^{-2} + \delta^{100} [\# \mathcal{A}]^2$$

$$\lesssim \delta^{1-3\eta-25\varepsilon/2} e^{-1} |\log_2 \delta_2^{-1}| \cdot \# \mathcal{A} + \delta^{100} [\# \mathcal{A}]^2$$

$$\lesssim \delta^{2+4\varepsilon} e^{-1} [\# \mathcal{A}]^2, \qquad (14)$$

again by (9) and choosing ε sufficiently small.

Step 9: Concluding the argument

From (7), we have

$$\Big\| \sum_{C \in \mathcal{A}_L} \tilde{\chi}_{C^\delta} (1 - \chi_{C_0^\sigma}) \Big\|_{L^1(\mathbb{R}^2)} \lesssim e^{1/2} (\Sigma_{diag} + \Sigma_{tang} + \Sigma_{trans})^{1/2}.$$

By (10), (12) and (14), we have

$$\Sigma_{diag} + \Sigma_{tang} + \Sigma_{trans} \lesssim e^{-1} \cdot \delta^{2+4\varepsilon} [\# \mathcal{A}]^2$$

and so

$$\Big\| \sum_{C \in \mathcal{A}_L} \tilde{\chi}_{C^\delta} (1 - \chi_{C_0^\sigma}) \Big\|_{L^1(\mathbb{R}^2)} \lesssim \delta^{1+2\varepsilon} \# \mathcal{A}.$$

Plugging this into (4), along with the bound from Step 4, gives the desired result. $\square$.