Harmonic Analysis Working Group

Bourgain's circular maximal theorem

Lecture 4

In the previous lecture we identified circle tangencies as the main enemy towards proving the circular maximal theorem. We also gave a characterisation of collections $\mathcal{C}$ of circles which admit many tangent pairs :- the clam shell configurations.

In order to deal with the tangent case, we require a more detailed understanding of the geometry of interacting pairs of δ-annuli C_1^δ, C_2^δ.

The essential support property.

We begin by providing a rigorous formulation of the assertion, made earlier, that each $\tilde{\chi}_{C^\delta}$ is 'essentially supported on C^δ.

Lemma 6 (Essential support) Let $0 < \eta < 1$ and $C = C(x,r)$ be a unit scale circle. Define

$$C^{\delta,*} := \{ y \in \mathbb{R}^2 : ||x-y| - r| < \delta^{1-\eta} \}$$

so that $C^{\delta,*} = C^{\delta^{1-\eta}}$ is a slight enlargement of C^δ.

If $y \in \mathbb{R}^2 \setminus C^{\delta,*}$, then

$$|\tilde{\chi}_{C^\delta}(y)| \lesssim_{N,\eta} \delta^N (1+|y|)^{-10} \quad \text{for all } N \in \mathbb{N}.$$

Proof :- Recall from the definition that

$$\tilde{\chi}_{C^\delta}(y) := \delta\, \sigma_r * \varphi_j (x-y)$$

$$= \delta \cdot \delta^{-2} \int_{S^1} \varphi(\delta^{-1}(x-y-r\omega))\, d\sigma(\omega).$$

Since $\varphi \in \mathcal{S}(\mathbb{R}^2)$, it follows that

$$|\tilde{\chi}_{C^\delta}(y)| \lesssim_N \delta^{-1} \int_{S^1} (1+\delta^{-1}|x-y-r\omega|)^{-N}\, d\sigma(\omega).$$

Now suppose $y \in \mathbb{R}^2 \setminus C^{\delta,*}$, so that

$$|x - y - r\omega| \geq \big||x-y| - |r\omega|\big| = \big||x-y| - r\big| \geq \delta^{1-\eta}$$

for all $\omega \in S^1$. Consequently,

$$|\tilde{\chi}_{C^\delta}(y)| \lesssim_{N,\eta} \delta^N \quad \text{for all } N \in \mathbb{N}_0.$$

Finally, if $|y| \geq 10$, then we may also bound

$$|x - y - r\omega| \geq \tfrac{1}{2}|y|,$$

using the hypothesis C is a unit scale circle.
Consequently,

$$|\tilde{\chi}_{C^\delta}(y)| \lesssim_{N,\eta} \delta^N (1+|y|)^{-10} \quad \text{for all } N \in \mathbb{N},$$

as required. $\square$

Corollary 7: Let C, C_1, C_2 be unit scale circles and $0 < \eta < 1$. For any measurable set $E \subseteq \mathbb{R}^2$ we have

a) $\|\tilde{\chi}_{C^\delta}\|_{L^1(E)} \lesssim_\eta |C^{\delta,*} \cap E| + \delta^{100}$

b) $|\langle \tilde{\chi}_{C_1^\delta}, \tilde{\chi}_{C_2^\delta}\rangle_{L^2(E)}| \lesssim_\eta |C_1^{\delta,*} \cap C_2^{\delta,*} \cap E| + \delta^{100}$

where $C^{\delta,*}, C_1^{\delta,*}, C_2^{\delta,*}$ are as defined as in Lemma 6.

Proof :-

a) Write $\|\tilde{\chi}_{C^\delta}\|_{L^1(E)} = \|\tilde{\chi}_{C^\delta}\|_{L^1(C^{\delta,*}\cap E)} + \|\tilde{\chi}_{C^\delta}\|_{L^1(E \setminus C^{\delta,*})}$.

Since $\|\tilde{\chi}_{C^\delta}\|_\infty \lesssim 1$, for the first term on the right-hand side we have

$$\|\tilde{\chi}_{C^\delta}\|_{L^1(C^{\delta,*}\cap E)} \lesssim |C^{\delta,*} \cap E|.$$

On the other hand, for all $y \in E \setminus C^{\delta,*}$ it follows that $|\tilde{\chi}_{C^\delta}(y)| \lesssim_\eta \delta^{100}(1+|y|)^{-10}$ and so

$$\|\tilde{\chi}_{C^\delta}\|_{L^1(E \setminus C^{\delta,*})} \lesssim_\eta \delta^{100} \int_{\mathbb{R}^2} (1+|y|)^{-10}\, dy \lesssim \delta^{100}.$$

b) This is a minor modification of the argument used to prove part a).

Geometry of intersecting annuli.

In view of Corollary 7, we are interested in understanding intersections $C_1^\delta \cap C_2^\delta$ between two δ-annuli. One of the main goals of this subsection is to prove a 2-dimensional version of the Spherical intersection lemma introduced in Lecture 1.

We first recall some elementary facts about intersecting pairs of circles $C_1 \cap C_2$.

Suppose C_1, C_2 are circles in the plane which have non-trivial intersection. There are 3 possible scenarios:-

Fig 1: Circle intersections:-

a) $C_1 \neq C_2$ are tangent, in which case there is a unique intersection point.

b) $C_1 \neq C_2$ are non-tangent, in which case there are precisely two intersection points.

c) $C_1 = C_2$. There are infinitely many intersection points.

Our first lemma gives a quantitative description of this phenomenon.

Lemma 8 :- (Intersecting annuli :- part 1) Let C_1, C_2 be unit scale circles. Then $C_1^\delta \cap C_2^\delta$ is contained in the δ-neighbourhood of an arc on C_1 of length

$$O\left(\left(\frac{\delta + \Delta(C_1, C_2)}{\delta + \mathrm{dist}(C_1, C_2)}\right)^{1/2}\right) \qquad (1)$$

centred at

$$\omega(C_1; C_2) := x_1 - r_1 \,\mathrm{sgn}(r_1 - r_2) \frac{x_1 - x_2}{|x_1 - x_2|} .$$

Before giving the proof, we motivate the numerology in (1).

- Suppose $\Delta(C_1, C_2) \lesssim \delta$ and $\mathrm{dist}(C_1, C_2) \sim 1$. This corresponds to the case a), where C_1, C_2 are tangent and $C_1 \neq C_2$ (centres are well-separated). We see that (1) is minimised in this situation, corresponding to the fact that there is just a single point of intersection in case a).

- Continue to suppose $\Delta(C_1, C_2) \lesssim \delta$, but now consider what happens as $\mathrm{dist}(C_1, C_2)$ decreases from 1 to δ. This corresponds to transitioning from case a) to case c). We see that (1) increases from the minimal value $\delta^{1/2}$ to the maximal value 1 in this case, corresponding to the transition from a single point of intersection in case a) to infinitely many intersections in case c).

- Now suppose $\mathrm{dist}(C_1, C_2) \sim 1$ and consider increasing $\Delta(C_1, C_2)$ from δ to 1. This corresponds to transitioning from case a) to case b). We see that (1) increases here again from the minimal value $\delta^{1/2}$ to the maximal value 1. This corresponds to the transition from a single point of intersection in case a) to two points of intersection in case b), which become further and further spread out (and therefore require a longer and longer covering arc) as C_1 and C_2 become less and less tangent.

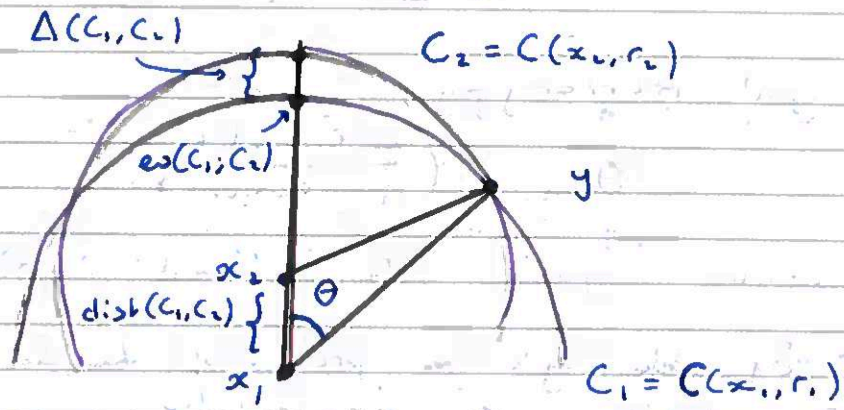


Fig 2: Intersecting circles/annuli.

Proof :- Without loss of generality, assume $r_1 \geq r_2$ and $C_1^\delta \cap C_2^\delta \neq \emptyset$. Then, there exists some $y \in \mathbb{R}^2$ such that

$$||x_1 - y| - r_1| < \delta \qquad \text{and} \qquad ||x_2 - y| - r_2| < \delta . \qquad (2)$$

Consequently,

$$\begin{aligned} |r_1 - r_2| &\leq \big||x_1 - y| - |x_2 - y|\big| + 2\delta \\ &\leq |x_1 - x_2| + 2\delta \end{aligned}$$

and so

$$\begin{aligned} \Delta(C_1, C_2) = \big||x_1 - x_2| - |r_1 - r_2|\big| &\leq |x_1 - x_2| + |r_1 - r_2| \\ &\leq 2(|x_1 - x_2| + \delta) . \end{aligned}$$

Thus, if $\mathrm{dist}(C_1, C_2) \leq \delta$, then we also have $\Delta(C_1, C_2) \lesssim \delta$, in which case the desired upper bound is trivial. Hence, we may assume $\mathrm{dist}(C_1, C_2) > \delta$.

We now apply the quantitative triangle inequality from Lecture 3 to vertices x_1, x_2, y and angle $\theta := \measuredangle(x_2 - x_1, y - x_1)$, giving

$$|x_2 - y| + |x_1 - x_2| - |x_1 - y| \gtrsim |x_1 - x_2|\,\theta^2 ; \qquad (3)$$

see Figure 2.

Combining (2) and (3),

$$\theta^2 |x_1 - x_2| \lesssim r_2 + |x_1 - x_2| - r_1 + 2\delta$$
$$\lesssim \Delta(C_1, C_2) + \delta$$

and rearranging,

$$\theta \lesssim \left(\frac{\Delta(C_1, C_2) + \delta}{\operatorname{dist}(C_1, C_2) + \delta} \right)^{1/2},$$

which implies the desired result. $\square$

By adapting the proof of Lemma 8 slightly, we can prove the following:-

Corollary 9: Let C_1, C_2 be unit scale circles. If

$$\sigma \geq 2(\Delta(C_1, C_2) + \delta), \qquad (4)$$

then $C_1^\delta \cap C_2^\sigma$ contains the δ-neighbourhood of an arc of C_1 of length

$$\gtrsim \left(\frac{\sigma}{\operatorname{dist}(C_1, C_2) + \delta} \right)^{1/2}$$

centred at $\omega(C_1; C_2)$.

Proof:- We first replace C_2 with an auxiliary circle $\tilde{C}_2$ concentric to C_2 but with radius $\tilde{r}_2$ satisfying

$$|r_2 - \tilde{r}_2| = \Delta(C_1, C_2)$$

so that $\Delta(C_1, \tilde{C}_2) = 0$. From the hypothesis (4) on σ, it follows that

$$\tilde{C}_2^{\sigma/2} \subseteq C_2^\sigma$$

and so it suffices to show $\tilde{C}_2^{\sigma/2} \cap C_1^\delta$ contains an arc of C_1 of length

$$\gtrsim \left(\frac{\sigma}{\operatorname{dist}(C_1, \tilde{C}_2) + \delta} \right)^{1/2}$$

centred at $\omega(C_1; \tilde{C}_2) = \omega(C_1, C_2)$. However, this follows from a simple variant of the argument used to prove Lemma 8, measuring the angle between the intersection points of C_1 and either

$C(x_2, \tilde{r}_2 + \sigma/2)$ or $C(x_2, \tilde{r}_2 - \sigma/2)$. $\square$

Using these observations, we can prove the key Lemma 5, which was stated without proof in the previous lecture.

We first recall the statement.

Lemma 5 : Let $\delta \leq \beta, e, \varkappa \lll 1$ and suppose $C_\ell = C(x_\ell, r_\ell)$, $\ell = 1, 2, 3$, are unit scale circles such that

i) $\Delta(C_0, C_1), \Delta(C_0, C_2) \leq \beta$
ii) $\frac{e}{2} \leq \mathrm{dist}(C_0, C_1), \mathrm{dist}(C_0, C_2) \leq e$
iii) Either $r_1, r_2 \geq r_0$ or $r_1, r_2 \leq r_0$.

Further suppose

$$\Delta(C_1, C_2) + \beta \leq \varkappa\, \mathrm{dist}(C_1, C_2) . \qquad (5)$$

Then there exists an absolute constant $A_1 \geq 1$ such that for $\sigma := \sigma(e, \varkappa) := A_1 e \varkappa$ we have

$$C_1^\delta \cap C_2^\delta \subseteq C_0^\sigma .$$

Recall, the conditions i), ii), iii) are precisely those of the "tangent case" described in lecture 3. The additional tangency condition between C_1 and C_2 in (5) is that of Lemma 3.

Proof : We will assume $r_1, r_2 \geq r_0$; the case $r_1, r_2 \leq r_0$ is treated almost identically. Without loss of generality, $r_1 \geq r_2$.

In view of i), ii), iii) and (5), we may apply Lemma 3 to conclude C_0, C_1, C_2 are in "clam shell configuration". In particular,

$$\Delta(C_1; C_0, C_2) \lesssim \varkappa^{1/2} . \qquad (6)$$

By Lemma 8, the intersection $C_1^\delta \cap C_2^\delta$ is contained in the δ-neighbourhood of an arc of C_1 of length

$$\lesssim \left(\frac{\Delta(C_1, C_2) + \delta}{\mathrm{dist}(C_1, C_2) + \delta} \right)^{1/2} \lesssim \kappa^{1/2} \qquad (7)$$

centred at $\omega(C_1 ; C_2)$. Here we have used the hypothesis (5) to estimate the length.

On the other hand, by the triangle inequality $\mathrm{dist}(C_1, C_2) \leq 2e$ and so

$$2(\Delta(C_1, C_2) + \delta) \leq 2(\Delta(C_1, C_2) + \delta) \leq 2\kappa \, \mathrm{dist}(C_1, C_2) \leq 4\kappa e.$$

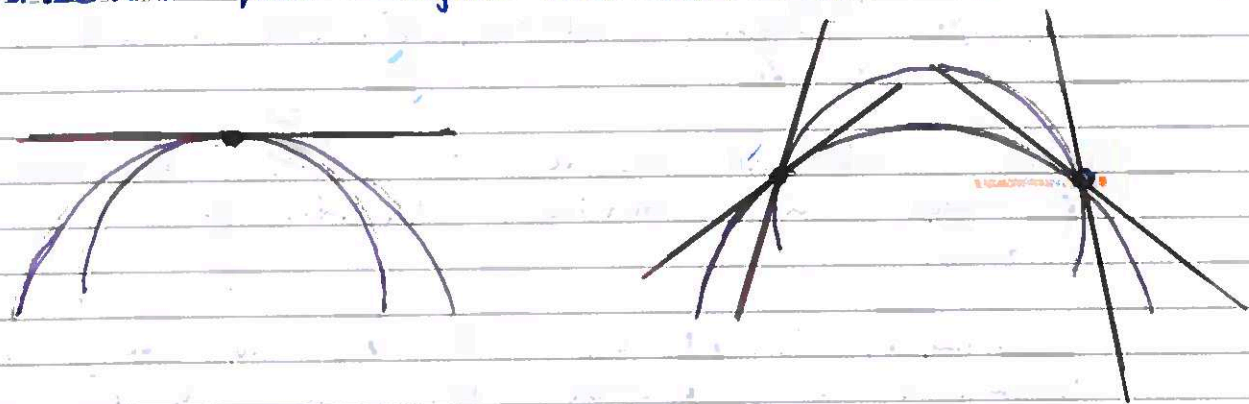
Hence, provided A_1 is sufficiently large, $\sigma \geq 2(\Delta(C_1, C_2) + \delta)$ and so Corollary 9 implies that $C_1^{\delta} \cap C_0^{\sigma}$ contains the δ-neighbourhood of an arc of C_1 of length

$$\gtrsim \left(\frac{\sigma}{\mathrm{dist}(C_1, C_0) + \delta} \right) \gtrsim A_1^{1/2} \kappa^{1/2} \qquad (8)$$

centred at $\omega(C_1 ; C_0)$.

Note that the angle between the centres $\omega(C_1 ; C_2)$ and $\omega(C_1 ; C_0)$ is $\measuredangle(C_1 ; C_0, C_2)$ which is bounded by $\kappa^{1/2}$ in (6). Comparing the lengths in (7) and (8) we see, provided A_1 is sufficiently large, the desired containment must hold. $\square$

We return to consider the two types of circle intersections from Figure 1 a) and 1 b).



a) Tangent intersection: A single intersection of multiplicity 2

b) Transverse intersection: Two intersections, each of multiplicity 1.

Figure 3: Circle intersections revisited.

A tangent intersection has multiplicity 2 and is therefore in a sense "larger" than the individual transverse intersections of multiplicity 1. Correspondingly, in the continuum setting, the measure $|C_1^\delta \cap C_2^\delta|$ will depend on the degree of tangency between C_1 and C_2, as measured by $\Delta(C_1, C_2)$.

Lemma 10 (Intersecting annuli : part 2) :- Let C_1, C_2 be unit scale circles.

a) $C_1^\delta \cap C_2^\delta$ has at most 2 connected components, each of diameter

$$O\left(\frac{\delta}{(\Delta(C_1,C_2)+\delta)^{1/2}(\mathrm{dist}(C_1,C_2)+\delta)^{1/2}}\right). \tag{9}$$

b)

$$|C_1^\delta \cap C_2^\delta| \lesssim \frac{\delta^2}{(\Delta(C_1,C_2)+\delta)^{1/2}(\mathrm{dist}(C_1,C_2)+\delta)^{1/2}}. \tag{10}$$

Note that part b) easily follows from part a).

Again, we pause to motivate the numerology.

- Suppose $\Delta(C_1, C_2) \lesssim \delta$ and $\mathrm{dist}(C_1, C_2) \sim 1$. This corresponds to the case a) in Figure 3, where there is a single intersection of multiplicity 2. Here we have

$$|C_1^\delta \cap C_2^\delta| \lesssim \delta \cdot \delta^{1/2}$$

where the (relatively large) $\delta^{1/2}$ factor encodes the multiplicity 2 intersection.

- Suppose $\Delta(C_1, C_2) \sim 1$ and $\mathrm{dist}(C_1, C_2) \sim 1$. This corresponds to the case b) in Figure 3, where the points of intersection have multiplicity 1. Here we have

$$|C_1^\delta \cap C_2^\delta| \lesssim \delta \cdot \delta$$

where the (small) δ factor encodes the multiplicity 1 intersection.

Proof: As already noted, it suffices to show c) only. Furthermore, if $\Delta(C_1, C_2) \lesssim \delta$, then the bounds (9) and (10) follow easily from Lemma 8.

Now suppose $\Delta(C_1, C_2) >> \delta$ and $C_1^\delta \cap C_2^\delta \neq \emptyset$. Given any $y \in C_1^\delta \cap C_2^\delta$, we can determine its position from the lengths

$$r_1 = r_1(y) = |y - x_1| \quad , \quad r_2 = r_2(y) = |y - x_2|,$$

up to reflective symmetry. Consider the angle $\theta = \measuredangle(x_2 - x_1, y - x_1)$, which featured in the proof of Lemma 8. We think of this as a function of r_1, r_2 (our choice of coordinates).

We want to show the set of possible angles

$\{\theta(r) : y(r) \in C_1^\delta \cap C_2^\delta\}$ is small, so that $C_1^\delta \cap C_2^\delta$ lies in a union of small arcs. To do this, we bound the r-gradient of θ.

By the proof of Lemma 8, we have

$$\theta(r) \sim \left(\frac{\Delta(C_1, C_2)}{\operatorname{dist}(C_1, C_2)}\right)^{1/2} \tag{11}$$

under the hypothesis $\Delta(C_1, C_2) >> \delta$.

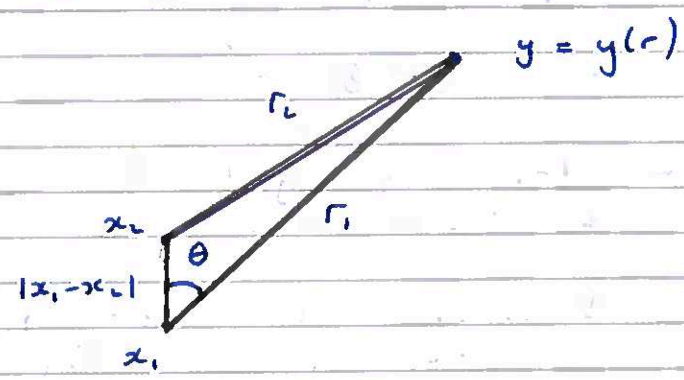


Fig 4: The Angle θ.

By the law of cosines

$$r_2^2 = r_1^2 + |x_1 - x_2|^2 - 2 r_1 |x_1 - x_2| \cos\theta$$

Implicitly differentiating with respect to r_1 and r_2 gives

$$0 = 2r_1 - 2|x_1 - x_2|\cos\theta + 2r_1|x_1 - x_2|(\partial_{r_1}\theta)\cdot\sin\theta \qquad (12a)$$
$$2r_2 = 2r_1|x_1 - x_2|(\partial_{r_2}\theta)\sin\theta \qquad (12b)$$

Rearranging (12a), we have

$$r_1|x_1 - x_2|(\partial_{r_1}\theta)\cdot\sin\theta = -r_1 + |x_1 - x_2|\cos\theta$$

so, since $1 \le r_1 \le 2$ and $|x_1 - x_2| \le 1/2$, we have

$$|x_1 - x_2|\cdot|\partial_{r_1}\theta(r)|\cdot\theta \sim 1 \iff |\partial_{r_1}\theta(r)| \sim \frac{1}{\theta\cdot|x_1 - x_2|}$$

Hence, recalling (11), we have

$$|\partial_{r_1}\theta(r)| \lesssim \frac{1}{\Delta(C_1, C_2)^{1/2}\,\mathrm{dist}(C_1, C_2)^{1/2}}$$

A similar calculation involving (12 b) also shows

$$|\partial_{r_2}\theta(r)| \lesssim \frac{1}{\Delta(C_1, C_2)^{1/2}\,\mathrm{dist}(C_1, C_2)^{1/2}}.$$

Since r_1 and r_2 can only vary in an interval of length δ, from this we conclude

$$|C_1^\delta \cap C_2^\delta| \lesssim \frac{\delta^2}{\Delta(C_1, C_2)^{1/2}\,\mathrm{dist}(C_1, C_2)^{1/2}}$$

by the change of variables formula. On the other hand, fixing r_2 and considering only the r_1 derivative gives part a).

$\square$