Harmonic Analysis Working Group :-

Bourgain's Circular Maximal Theorem.

Lecture 1.

Setup :-

Let $d \geq 2$ and σ denote the surface area measure on S^{d-1}, normalised to have total mass 1. Thus, given, say, $f \in C(\mathbb{R}^d)$, we can interpret

$$Af(x,r) := \int_{S^{d-1}} f(x - ry)\, d\sigma(y) \quad , \quad (x,r) \in \mathbb{R}^d \times (0,\infty)$$

as the average value of f over the sphere centred at x of radius r. We use the notation

$$C(x,r) := \{ y \in \mathbb{R}^d : |x - y| = r \}$$

to denote these spheres. Note that we may write

$$Af(x,r) = f * \sigma_r(x)$$

where σ_r denotes the dilated measure, defined via the action

$$\langle \sigma_r , f \rangle := \langle \sigma, f(r\,\cdot) \rangle .$$

Since the operators $f \longmapsto Af(\cdot, r)$ are averaging operators, they behave nicely and it is very easy to show

$$\| Af(\cdot, r) \|_{L^p(\mathbb{R}^d)} \leq \| f \|_{L^p(\mathbb{R}^d)}$$

for all $1 \leq p \leq \infty$. We are interested in the L^p-mapping theory of the corresponding spherical maximal operator, defined by

$$Mf(x) := \sup_{r>0} |Af(x,r)| .$$

Thus, $Mf(x)$ corresponds to "the largest" average of f taken over all spheres centred at x.

The spherical maximal operator is of interest for a variety of reasons. We will not delve into the applications here, except to list some important instances in which this operator plays a fundamental rôle :-

- The study of the Hardy-Littlewood maximal operator M_{HL} and, in particular, dimension-free bounds for M_{HL}.
- Differentiation theory - analogues of the Lebesgue differentiation theorem for spheres.
- Geometric packing problems for circles and spheres.
- A model problem for the Kakeya conjecture.
- Relationship with the wave equation via the Kirchhoff formula - pointwise a.e. convergence questions for the wave equation, local smoothing.

It is easy to see that we cannot expect Mf to behave as well as the fixed-radius averages $Af(\cdot, r)$. For instance, let $0 < \delta << 1$ be a small parameter and take

$$f(x) := \chi_{B(0,\delta)}(x).$$

It follows that $M\chi_{B(0,\delta)}(x) \gtrsim \delta^{d-1}$ for all x belonging to the annulus

$$\{x \in \mathbb{R}^d : 1 \leq |x| \leq 2\}.$$

Thus, $\|M\chi_{B(0,\delta)}\|_{L^p(\mathbb{R}^d)} \gtrsim \delta^{d-1}$

whilst $\quad \| \chi_{B(0,\delta)} \|_{L^p(\mathbb{R}^d)} \sim \delta^{d/p}.$

If we suppose M is strong-type (p,p), then it follows

$$\delta^{d-1} \lesssim \delta^{d/p} \quad \text{for all } 0 < \delta \ll 1,$$

but this can only hold if $d-1 \geq d/p$ or, equivalently, $p \geq \frac{d}{d-1}$.

By a slight variant of this example, we can also show M fails to be strong-type (p,p) for $p = \frac{d}{d-1}$. Thus, the best we can hope for is that M is bounded on the open range $p > d/d-1$.

Theorem (Stein 1976) For $d \geq 3$, we have

$$\| Mf \|_{L^p(\mathbb{R}^d)} \lesssim_p \| f \|_{L^p(\mathbb{R}^d)}$$

for all $\frac{d}{d-1} < p \leq \infty$.

Thus, Stein's theorem together with the preceding observations give a complete characterisation of the L^p-boundedness of M. Note the restriction $d \geq 3$ – it turns out that the $d=2$ case is significantly harder to analyse! The L^p-boundedness of the circular maximal function was resolved by Bourgain some 10 years later.

Theorem (Bourgain 1986) For $d=2$, we have

$$\| Mf \|_{L^p(\mathbb{R}^2)} \lesssim_p \| f \|_{L^p(\mathbb{R}^2)}$$

for all $2 < p \leq \infty$.

The purpose of these lectures is to present Bourgain's original proof of the circular maximal theorem, which involves a beautiful synthesis of geometric and Fourier-analytic ideas. As a first step, we will consider Stein's earlier and easier result, and we will isolate the features of the problem which cause difficulties when $d = 2$.

Stein's Spherical Maximal Theorem: L^2 bounds.

To get a feel for the problem, we will begin by proving M is bounded on $L^2(\mathbb{R}^d)$ for all $d \geq 3$ using a purely geometric argument.* This approach breaks down when $d = 2$ and, indeed, we know the circular maximal operator is unbounded on $L^2(\mathbb{R}^2)$.

The geometric argument will highlight two features of the problem:-

- The distinction between $d \geq 3$ and $d = 2$.
- The utility of Fourier analysis. In particular, we will see it is much easier to push beyond $L^2(\mathbb{R}^d)$-bounds for M when $d \geq 3$ if the geometric arguments are augmented with Fourier analytical tools.

Introducing a scale:-

Given $0 < \delta \ll 1$, let $C^\delta(x,r)$ denote the annulus formed by the δ-neighbourhood of $C(x,r)$. For $f \in C(\mathbb{R}^d)$ we can realise the spherical averages as limits

$$Af(x,r) = \lim_{\delta \to 0_+} \frac{1}{|C^\delta(x,r)|} \int_{C^\delta(x,r)} f .$$

We will often use $⨍$ to denote an "average integral" so that, for instance,

$$\frac{1}{|C^\delta(x,r)|} \int_{C^\delta(x,r)} f = ⨍_{C^\delta(x,r)} f .$$

* Strictly speaking, we will only tackle a local version of

We consider the maximal operators

$$M^\delta f(x) := \sup_{r>0} \left| \fint_{C^\delta(x,r)} f \right|$$

for $0<\delta<1$. To prove the strong-type inequality

$$\|Mf\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)}$$

it suffices (by Fatou) to show the family of strong-type inequalities

$$\|M^\delta f\|_{L^p(\mathbb{R}^d)} \leq C \|f\|_{L^p(\mathbb{R}^d)} \qquad (1)$$

hold uniformly in δ.

Local maximal operator

We will simplify the setup somewhat by considering only the local maximal operators

$$Mf(x) := \sup_{1\leq r\leq 2} |Af(x,r)|$$

$$M^\delta f(x) := \sup_{1\leq r\leq 2} \left| \fint_{C^\delta(x,r)} f \right|. \qquad (2)$$

The geometric arguments tend not to work well with the full range of radii. However, the Fourier analytic techniques developed later allow one to prove global maximal estimates as a consequence of local estimates.

Discretisation, linearisation and duality.

Here we describe a number of standard reductions which reduce the proof of maximal estimates such as (1) to counting incidences between the annuli $C^\delta(x,r)$.

Let $f \in C_c(\mathbb{R}^d)$. We first observe a 'local constant' property of M^δ. We will not use this property directly in our proof, but it will help motivate what follows.

Lemma (Locally constant property):- Let $f \geq 0$.
If $x, x' \in \mathbb{R}^d$ with $|x - x'| \leq \delta$, then

$$M^\delta f(x) \lesssim M^\delta f(x').$$

In particular, $M^\delta f(x)$ and $M^\delta f(x')$ are comparable, so we can think of $M^\delta f$ as being close to constant on scale δ cubes.

Proof:- Choose $r > 0$ such that

$$M^\delta f(x) \leq 2 \cdot \left| \fint_{C^\delta(x,r)} f \right|$$

Since $|x - x'| \leq \delta$, it follows $C^\delta(x,r) \subseteq C^{2\delta}(x', r)$.
Furthermore, we can write

$$C^{2\delta}(x', r) \subseteq C^\delta(x', r_1) \cup C^\delta(x', r_2)$$

where $r_1 = r - \delta$ and $r_2 = r + \delta$. Thus, since $f \geq 0$

$$M^\delta f(x) \leq 2 \cdot \left| \fint_{C^\delta(x,r)} f \right| = 2 \fint_{C^\delta(x,r)} |f|$$

$$\leq 2 \left[\fint_{C^\delta(x', r_1)} |f| + \fint_{C^\delta(x', r_2)} |f| \right] \leq 4 \, M^\delta f(x').$$

$\square$

Remark:- There are some technical issues regarding whether r_1, r_2 lie in the admissible range $[1, 2]$, but we won't get tied up about this.

In view of the above lemma, it is natural to think of M^δ as a discrete object.
In particular, we will think of

$$M^\delta f(x) \sim \left| \fint_{C^\delta(x_Q, r_Q)} f \right|$$

for all x belonging to some δ-cube Q centred at x_Q.

In view of the definition (2), the operator M^δ is local and therefore, in order to prove (1) it suffices to show

$$\|M^\delta f\|_{L^p(Q_0)} \lesssim \|f\|_{L^p(\mathbb{R}^d)}$$

whenever $Q_0 \subseteq \mathbb{R}^d$ is a cube of side-length 1. By translation-invariance, we may assume Q_0 is centred at 0.

We let $\mathcal{Q}_\delta$ denote a decomposition of Q_0 into subcubes Q_δ of side-length δ. For each Q_δ we can find some $x_{Q_\delta} \in Q_\delta$ and $r_{Q_\delta} \in [1,2]$ such that

$$M^\delta f(x) \lesssim 2 M^\delta f(x_{Q_\delta}) \leq 4 \Big| \fint_{C^\delta(x_{Q_\delta}, r_{Q_\delta})} f \Big|$$

for all $x \in Q_\delta$. Here we just take suprema - we don't need to use the local constant property. However, the local constant property does help to reassure us that this step makes sense (ie is not too lossy).

Writing $\|M^\delta f\|_{L^p(Q_0)} = \Big(\sum_{Q_\delta \in \mathcal{Q}_\delta} \|M^\delta f\|^p_{L^p(Q_\delta)} \Big)^{1/p}$

and applying the above, we see

$$\|M^\delta f\|_{L^p(Q_0)} \lesssim \Big(\sum_{C \in \mathcal{C}} \delta^d \Big| \fint_{C^\delta} f \Big|^p \Big)^{1/p} \qquad (3)$$

where $\mathcal{C}$ is a collection of circles such that if $C_1, C_2 \in \mathcal{C}$ with $C_1 \neq C_2$, then the centres of C_1 and C_2 lie in disjoint δ-cubes. By pigeonholing, we can pass to a subcollection of $\mathcal{C}$ for which the centres are δ-separated.

Taking the RHS of (3) we use ℓ^p-duality to write

$$\Big(\sum_{C \in \mathcal{C}} \delta^d \Big| \fint_{C^\delta} f \Big|^p \Big)^{1/p} = \delta^{d/p - 1} \Big(\sum_{C \in \mathcal{C}} \Big| \int_{C^\delta} f \Big|^p \Big)^{1/p}$$

$$= \delta^{d/p-1} \sum_{c\in\mathcal{C}} a_c \int_{C^\delta} f$$

for some sequence $(a_c)_{c\in\mathcal{C}}$ satisfying

$$\Big(\sum_{c\in\mathcal{C}} |a_c|^{p'}\Big)^{1/p'} \leq 1.$$

We now write

$$\Big|\sum_{c\in\mathcal{C}} a_c \int_{C^\delta} f\Big| = \Big|\int_{\mathbb{R}^d} \sum_{c\in\mathcal{C}} a_c \chi_{C^\delta} \cdot f\Big|$$

$$\leq \Big\|\sum_{c\in\mathcal{C}} a_c \chi_{C^\delta}\Big\|_{L^{p'}(\mathbb{R}^d)} \|f\|_{L^p(\mathbb{R}^d)},$$

by Hölder. Thus, combining all the above steps we see that

$$\|M^\delta f\|_{L^p(Q_0)} \lesssim \delta^{d/p-1} \Big\|\sum_{c\in\mathcal{C}} a_c \chi_{C^\delta}\Big\|_{L^{p'}(\mathbb{R}^d)} \cdot \|f\|_{L^p(\mathbb{R}^d)}.$$

Thus, the L^2 case of Stein's theorem is reduced to the following :-

Lemma For $\underline{d \geq 3}$ and $\underline{\text{any}}$ collection $\mathcal{C}$ of spheres in $\mathbb{R}^d$ with δ-separated centres lying in $[-1/2, 1/2]^d$, we have

$$\Big\|\sum_{c\in\mathcal{C}} a_c \chi_{C^\delta}\Big\|_{L^2(\mathbb{R}^d)} \lesssim \delta^{1-d/2} \Big(\sum_{c\in\mathcal{C}} |a_c|^2\Big)^{1/2}$$

for any sequence of complex coefficients $(a_c)_{c\in\mathcal{C}}$.

Proof :- We exploit heavily the Hilbert space structure of $L^2(\mathbb{R}^d)$.

Write

$$\Big\| \sum_{C\in\mathcal{C}} a_C \chi_{C^\delta} \Big\|^2_{L^2(\mathbb{R}^d)} = \Big\langle \sum_{C_1\in\mathcal{C}} a_{C_1}\chi_{C_1^\delta}, \sum_{C_2\in\mathcal{C}} a_{C_2}\chi_{C_2^\delta} \Big\rangle$$

$$= \sum_{C_1, C_2\in\mathcal{C}} a_{C_1}\overline{a_{C_2}} \langle \chi_{C_1^\delta}, \chi_{C_2^\delta}\rangle .$$

$$= \sum_{C\in\mathcal{C}} |a_C|^2 \|\chi_{C^\delta}\|^2_{L^2(\mathbb{R}^d)}$$

$$+ \sum_{\substack{C_1, C_2\in\mathcal{C}\\ C_1\neq C_2}} a_{C_1}\overline{a_{C_2}} \langle \chi_{C_1^\delta}, \chi_{C_2^\delta}\rangle .$$

First consider the diagonal contribution. Since $|C^\delta| \sim \delta$, it follows that

$$\sum_{C\in\mathcal{C}} |a_C|^2 \|\chi_{C^\delta}\|_2^2 \sim \delta \sum_{C\in\mathcal{C}} |a_C|^2$$

which is a substantially better estimate than what we are aiming for.

The main contribution will come from the off-diagonal terms. Note that

$$\langle \chi_{C^\delta_1}, \chi_{C^\delta_2}\rangle = |C_1^\delta \cap C_2^\delta| .$$

The key ingredient in the proof is the following volume bound.

<u>Lemma (Spherical intersection bound)</u>:– For $\underline{\underline{d \geq 3}}$,

$$|C_1^\delta \cap C_2^\delta| \lesssim \frac{\delta}{1 + \delta^{-1}\mathrm{dist}(C_1, C_2)} \qquad (4)$$

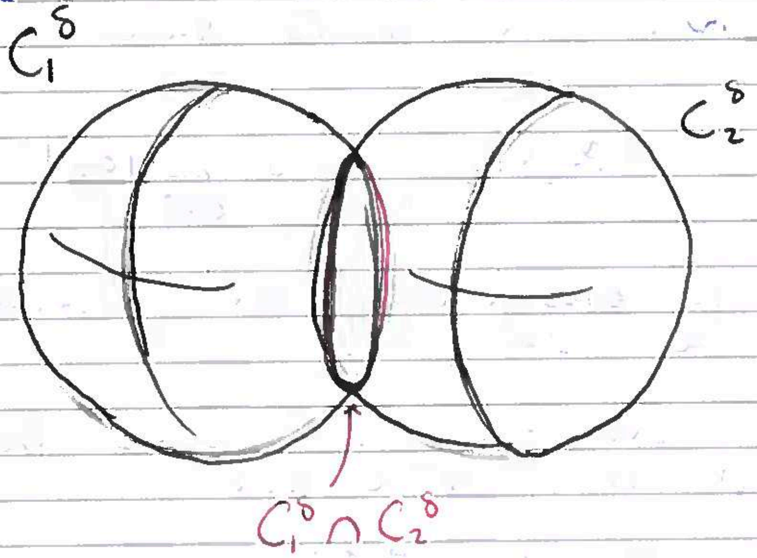
where $\mathrm{dist}(C_1, C_2) := |x_1 - x_2|$ is the distance between the centres of the spheres C_1 and C_2.

We will discuss the proof of this bound below, but for now we motivate the numerology :-

- If $\text{dist}(C_1, C_2) \lesssim \delta$, then C_1^δ and C_2^δ can essentially completely overlap (provided they have similar radii). In this case, we can't do any better than

$$|C_1^\delta \cap C_2^\delta| \leq |C_1^\delta| \sim \delta \qquad (4a)$$

- If $\text{dist}(C_1, C_2) \sim 1$, then C_1^δ and C_2^δ should intersect around a $(d-2)$-dimensional sphere



In particular, $|C_1^\delta \cap C_2^\delta|$ is bounded by the volume of the δ-neighbourhood of this $(d-2)$-dimensional sphere, which gives

$$|C_1^\delta \cap C_2^\delta| \lesssim \delta^2 \qquad (4b)$$

Note that (4) agrees with (4a) and (4b) in these special (extreme) cases and "interpolates" between them.

We will now temporarily assume (4) and complete the proof of the main lemma.

By (4), we have,

$$\Big| \sum_{\substack{c_1, c_2 \in \mathcal{C} \\ c_1 \neq c_2}} a_{c_1} \bar{a}_{c_2} \langle X_{c_1}, X_{c_2} \rangle \Big|$$

$$\lesssim \delta^2 \sum_{\substack{c_1, c_2 \in \mathcal{C} \\ c_1 \neq c_2}} | a_{c_1} \bar{a}_{c_2} | \operatorname{dist}(c_1, c_2)^{-1}$$

$$= \delta^2 \sum_{l=0}^{\log_2 \delta^{-1}} \sum_{c_1 \in \mathcal{C}} |a_{c_1}| \sum_{\substack{c_2 \in \mathcal{C} \\ \operatorname{dist}(c_1,c_2) \sim 2^l \delta}} |\bar{a}_{c_2}| \operatorname{dist}(c_1, c_2)^{-1}$$

$$= \delta^2 \sum_{l=0}^{\log_2 \delta^{-1}} \sum_{c_1 \in \mathcal{C}} |a_{c_1}| \sum_{\substack{c_2 \in \mathcal{C} \\ \operatorname{dist}(c_1,c_2) \sim 2^l \delta}} |\bar{a}_{c_2}| \, 2^{-l} \delta^{-1} \qquad (5)$$

By Cauchy-Schwarz, we can bound

$$\Big| \sum_{c_1 \in \mathcal{C}} |a_{c_1}| \sum_{\substack{c_2 \in \mathcal{C} \\ \operatorname{dist}(c_1,c_2) \sim 2^l \delta}} |\bar{a}_{c_2}| \Big| \leq \Big(\sum_{c_1 \in \mathcal{C}} |a_{c_1}|^2 \Big)^{1/2} .$$

$$\Big(\sum_{c_1 \in \mathcal{C}} \Big| \sum_{\substack{c_2 \in \mathcal{C} \\ \operatorname{dist}(c_1,c_2) \sim 2^l \delta}} \bar{a}_{c_2} \Big|^2 \Big)^{1/2}$$

and

$$\sum_{c_1 \in \mathcal{C}} \Big| \sum_{\substack{c_2 \in \mathcal{C} \\ \operatorname{dist}(c_1,c_2) \sim 2^l \delta}} |\bar{a}_{c_2}| \Big|^2 \leq \sum_{c_1 \in \mathcal{C}} \#\{ c_2 \in \mathcal{C} : \operatorname{dist}(c_1, c_2) \sim 2^l \delta \} . \sum_{\substack{c_2 \in \mathcal{C} \\ \operatorname{dist}(c_1,c_2) \sim 2^l \delta}} |a_{c_2}|^2$$

$$\lesssim 2^{dl} \sum_{c_2 \in \mathcal{C}} \#\{ c_1 \in \mathcal{C} : \operatorname{dist}(c_1, c_2) \sim 2^l \delta \} |a_{c_2}|^2$$

$$\lesssim 2^{2dl} \sum_{c_2 \in \mathcal{C}} |a_{c_2}|^2$$

where we have used the fact that
$\sup_{c_1 \in \mathcal{C}} \#\{ c_2 \in \mathcal{C} : \operatorname{dist}(c_1, c_2) \sim 2^l \delta \} \lesssim 2^{ld}$
by the δ-separation hypothesis.

Combining these observations,

$$\Big|\sum_{c_1\in\mathcal{C}} |a_{c_1}| \sum_{\substack{c_2\in\mathcal{C}\\ \mathrm{dist}(c_1,c_2)\sim 2^\ell\delta}} |a_{c_2}|\Big| \lesssim 2^{d\ell} \sum_{c\in\mathcal{C}} |a_c|^2$$

and plugging this into (5) we see

$$\Big|\sum_{\substack{c_1,c_2\in\mathcal{C}\\ c_1\neq c_2}} a_{c_1}\bar{a}_{c_2}\langle \chi_{c_1^\delta}, \chi_{c_2^\delta}\rangle\Big| \lesssim \delta\cdot\sum_{\ell=0}^{\log_2\delta^{-1}} 2^{(d-1)\ell}\sum_{c\in\mathcal{C}}|a_c|^2$$

$$\lesssim \delta^{-(d-2)}\sum_{c\in\mathcal{C}}|a_c|^2$$

From all this, we conclude that

$$\Big\|\sum_{c\in\mathcal{C}} a_c\chi_{c^\delta}\Big\|_{L^2(\mathbb{R}^d)} \lesssim \delta^{1-d/2}\Big(\sum_{c\in\mathcal{C}}|a_c|^2\Big)^{1/2},$$

as required. □

Remark:- We can avoid a lot of annoying and distracting Cauchy-Schwarz gymnastics if we simply prove a bound of the form

$$\Big\|\sum_{c\in\mathcal{C}}\chi_{c^\delta}\Big\|_{L^2(\mathbb{R}^d)} \lesssim \delta^{1-d/2}[\#\mathcal{C}]^{1/2}$$

This can be interpreted as a 'restricted weak-type' (2,2) estimate an is sufficient to prove the L^p-boundedness of M in the open range $p>2$. We will return to this point later.

Proof of the Spherical Intersection Bound :-

First consider the intersection between two (discrete) spheres $C(x_1, r_1)$ and $C(x_2, r_2)$. Without loss of generality, assume $x_1 \neq x_2$.
If $y \in C(x_1, r_1) \cap C(x_2, r_2)$, then

$$|y - x_1|^2 = r_1^2 \quad ; \quad |y - x_2|^2 = r_2^2$$

and so

$$|y|^2 - 2\langle y, x_1 \rangle + |x_1|^2 = r_1^2$$
$$|y|^2 - 2\langle y, x_2 \rangle + |x_2|^2 = r_2^2$$

Combining these equations,

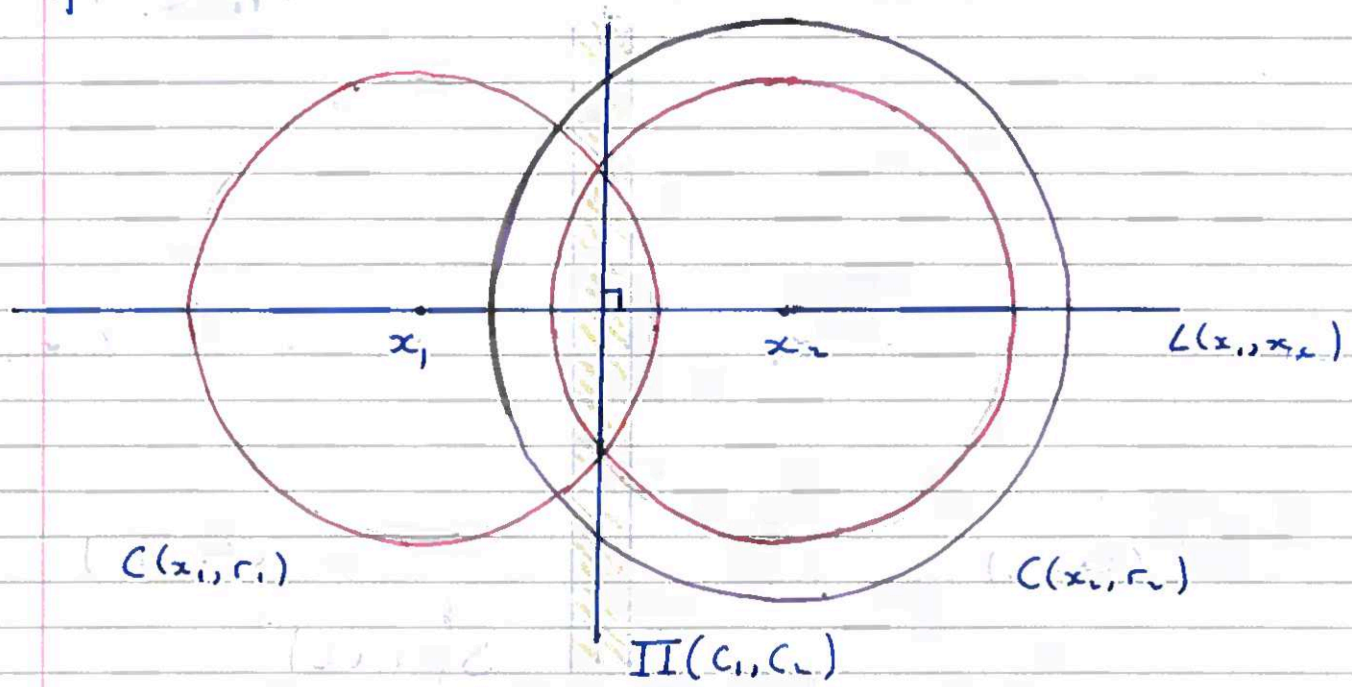
$$2\langle y, x_2 - x_1 \rangle + |x_1|^2 - |x_2|^2 = r_1^2 - r_2^2$$

which can be written as

$$\langle y - \frac{x_1 + x_2}{2}, x_2 - x_1 \rangle = \frac{r_1^2 - r_2^2}{2}, \qquad (6)$$

using the difference of squares identity
$|x_1|^2 - |x_2|^2 = \langle x_1 + x_2, x_1 - x_2 \rangle$.

From (6) we see that $C(x_1, r_1) \cap C(x_2, r_2)$ lies on an affine hyperplane normal to the line $L(x_1, x_2)$ passing through the centres of the two spheres:-



Let's call this hyperplane $\Pi(C_1, C_2)$. If $r_1 = r_2$, then $\Pi(C_1, C_2)$ passes through the midpoint $\frac{x_1+x_2}{2}$. If $r_2 > r_1$, then $\Pi(C_1, C_2)$ pass through $L(x_1, x_2)$ at a point closer to x_1 than x_2.

In what follows it will be important to normalize (6) by writing it as

$$\left\langle y - \frac{x_1+x_2}{2}, \frac{x_2-x_1}{|x_2-x_1|} \right\rangle = \frac{r_1^2-r_2^2}{2|x_2-x_1|} \qquad (6').$$

We now consider the "continuum analogue" of this computation, replacing the C_j with their fattened versions C_j^δ.

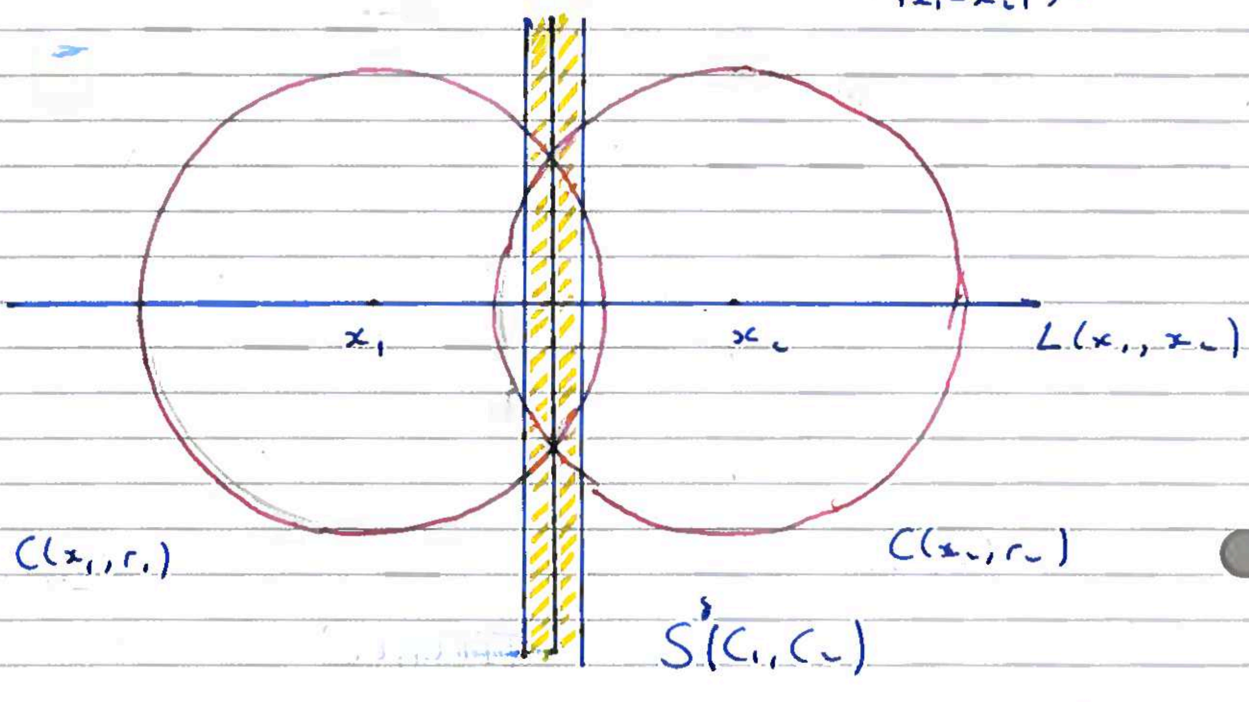
If $y \in C^\delta(x_1, r_1) \cap C^\delta(x_2, r_2)$, then

$$|y-x_1|^2 = r_1^2 + O(\delta) \; ; \; |y-x_2|^2 = r_2^2 + O(\delta)$$

(recall, we are assuming $(x_j, r_j) \in [-1/2, 1/2]^d \times [1,2]$). Carrying out the argument above, we see

$$\left\langle y - \frac{x_1+x_2}{2}, \frac{x_2-x_1}{|x_2-x_1|} \right\rangle = \frac{r_1^2-r_2^2}{2|x_2-x_1|} + O\left(\frac{\delta}{|x_2-x_1|}\right).$$

Thus, $C^\delta(x_1,r_1) \cap C^\delta(x_2, r_2)$ lies in a strip around $\Pi(C_1, C_2)$ of thickness $O\left(\frac{\delta}{|x_1-x_2|}\right)$.



Let's call this strip $S^\delta(C_1, C_2)$.

We will now carefully analyse the area of intersection between a sphere and a strip. By translation, rotation and dilation, it suffices to consider

$$\mathrm{Area}^{d-1}\big(C(0;1) \cap \underbrace{([a, a+\lambda] \times \mathbb{R}^{n-1})}_{S :=}\big)$$

where $0 \leq a < a+\lambda \leq 1$. We compute this area by chopping the sphere $C(0,1)$ into slices $C(0,1) \cap H_t$ where $H_t := \{t\} \times \mathbb{R}^{n-1}$. In particular,

$$\mathrm{Area}^{d-1}(C(0,1) \cap S) = \int_a^{a+\lambda} \mathrm{Area}^{d-2}(C(0,1) \cap H_t) \frac{dt}{(1-t^2)^{1/2}}$$

Note that $C(0,1) \cap H_t$ is a $(d-2)$-dimensional sphere of radius $(1-t^2)^{1/2}$ and so we have

$$\mathrm{Area}^{d-1}(C(0,1) \cap S) \approx \int_a^{a+\lambda} (1-t^2)^{\frac{d-2}{2} - \frac{1}{2}} dt$$

$$= \int_a^{a+\lambda} (1-t^2)^{\frac{d-3}{2}} dt.$$

If $d \geq 3$, then $(1-t^2)^{\frac{d-3}{2}} \lesssim 1$ for all $t \in [-1, 1]$ and so

$$\mathrm{Area}^{d-1}(C(0,1) \cap S) \lesssim \lambda ;$$

that is, the area is simply bounded by the width of the strip.

From this computation, we see

$$|C^\delta(x_1, r_1) \cap C^\delta(x_2, r_2)| \lesssim \delta \cdot \mathrm{Area}^{d-1}(C(x_1, r_1) \cap S(C_1, C_2))$$

$$\lesssim \delta \cdot \frac{\delta}{|x_1 - x_2|} \approx \frac{\delta}{1 + \delta^{-1}|x_1 - x_2|}$$

if $|x_1 - x_2| \gtrsim \delta$. If $|x_1 - x_2| \lesssim \delta$, then the desired estimate is trivial, so this completes the proof. □

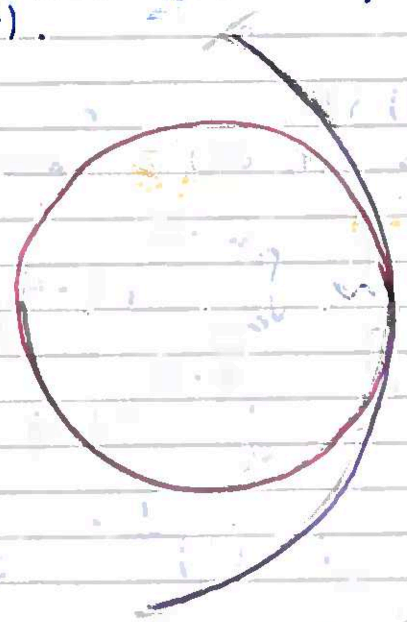
What goes wrong when $d = 2$?

If we consider the above argument in the $d = 2$ case, we see that the proof breaks down since

$$(1-t^2)^{\frac{d-3}{2}} = (1-t^2)^{-1/2} \qquad (7)$$

is no longer a bounded function on $[-1, 1]$. What is going on here?

The issue arises when we have tangent pairs of circles, corresponding to t close to ± 1 in (7).



Roughly speaking, if C_1 and C_2 are tangent, then we can expect $C^\delta(x_1, r_1)$ and $C^\delta(x_2, r_2)$ to intersect along an arc of length $\delta^{1/2}$ when $|x_1 - x_2| \sim 1$ and so we get

$$|C^\delta(x_1, r_1) \cap C^\delta(x_2, r_2)| \sim \delta^{3/2}$$

This is much larger than δ^2, corresponding to the numerology in the $d \geq 3$ case.

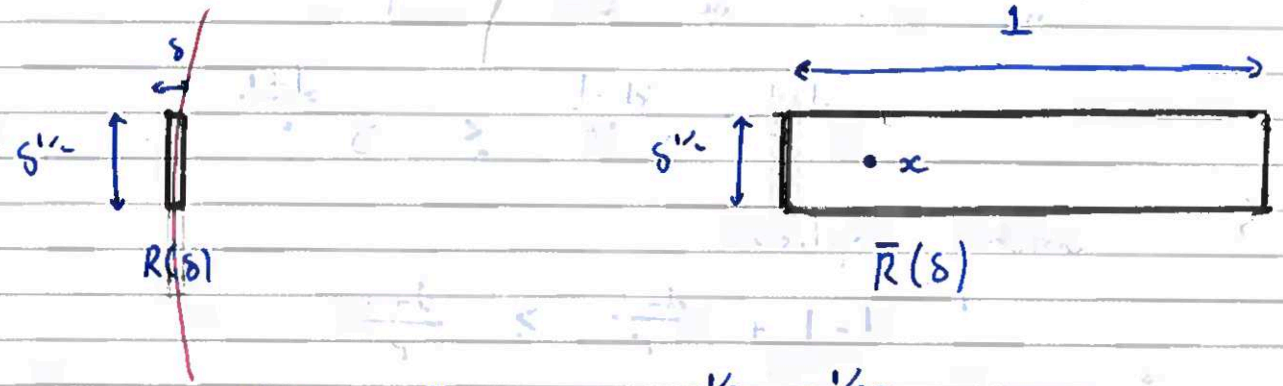
For spheres in $\mathbb{R}^d$ for $d \geq 3$ tangent intersections do not cause a problem. This is because, for $|x_1 - x_2| \sim 1$, $C^\delta(x_1, r_1)$ and $C^\delta(x_2, r_2)$ tangent will intersect along a patch of the sphere radius $\delta^{1/2}$. This is a neighbourhood of a 0-dimensional point on S^{d-1} of radius $\delta^{1/2}$ and so has area $\delta^{(d-1)/2}$. Compare this with a non-tangential intersection which occurs along a strip of width δ. This is a δ-neighbourhood of a $(d-2)$-dimensional circle on S^{d-1} and so has area δ.

Although the radius is larger in the tangential case, the dimension of the intersection is small enough so that non-tangential intersections are always at least as large.

Another sharp example :-

The rôle of tangency in the $d=2$ can also be seen by considering an alternative sharp* example for Bourgain's theorem.

Take $f := \chi_{R(\delta)}$ where $R(\delta)$ is a $\delta \times \delta^{1/2}$ rectangle. Then $Mf(x) \gtrsim \delta^{1/2}$ for all x belonging to a $1 \times \delta^{1/2}$ rectangle $\bar{R}(\delta)$



Note $\|M\chi_{R(\delta)}\|_{L^p(\mathbb{R}^2)} \gtrsim \delta^{1/2} \cdot \delta^{1/2p}$ whilst $\|\chi_{R(\delta)}\|_{L^p(\mathbb{R}^2)} \sim \delta^{3/2p}$. Combining these observations, we see that if M is (restricted) strong-type (p,p), then

$$\delta^{1+1/p} \lesssim \delta^{3/p} \quad \text{for all } 0<\delta<1$$

*Or "near sharp", since we won't rule out $p=2$ bounded-ness using this example.

However, this can hold only if $1 + 1/p \geq \frac{3}{p}$
$\Leftrightarrow$ $p \geq 2$.

This simple example was combined with a Besicovitch set construction by Seeger-Tao-Wright to rule out a restricted weak-type (2,2) inequality for the circular maximal function. (Interestingly, the spherical maximal function *is* restricted weak-type $\left(\frac{d}{d-1}, \frac{d}{d-1}\right)$ for $d \geq 3$!).

Finally, we remark that the analogue of the $\delta \times \delta^{1/2}$ rectangle is *not* a sharp example for Stein's maximal function for $d \geq$. This reflects the fact that tangent intersections are not critical in this case.

Indeed, to extend $R(\delta)$ to $\mathbb{R}^d$, we take it to be a

$$\delta \times \underbrace{\delta^{1/2} \times \dots \times \delta^{1/2}}_{(d-1)\text{-fold}} \text{-rectangle.}$$

Then $M\chi_{R(\delta)}(x) \gtrsim \delta^{\frac{d-1}{2}}$ on a set $\tilde{R}(\delta)$ of dimension $1 \times \underbrace{\delta^{1/2} \times \dots \times \delta^{1/2}}_{(d-1)\text{-fold}}$.

This leads to the condition

$$\delta^{\frac{d-1}{2}} \cdot \delta^{\frac{d-1}{2p}} \lesssim \delta^{\frac{d+1}{2p}} \quad \text{for } 0 < \delta \ll 1$$

which implies

$$d - 1 + \frac{d-1}{p} \geq \frac{d+1}{p}$$

$$\Leftrightarrow \quad p \geq \frac{2}{d-1}.$$

For $d \geq 3$ this is much weaker than the condition $p \geq \frac{d}{d-1}$ from the ball example.