Lecture 9: Bochner-Riesz Means V.

Last time we proved a bilinear variant of the Carleson-Sjölin theorem:-

Proposition:- (Bilinear estimate) Let $4 \leq p \leq \infty$, $L > 0$ and $R \geq 1$. For all $\varepsilon > 0$, the inequality

$$\Big\| \prod_{j=1}^{2} |T_R f_j|^{1/2} \Big\|_{L^p(\mathbb{R}^2)} \lesssim_\varepsilon L^{-1/p} R^{1/2 - 2/p + \varepsilon} \prod_{j=1}^{2} \|f_j\|^{1/2}_{L^p(\mathbb{R}^2)}$$

holds whenever $\operatorname{supp} \hat{f}_j \subseteq I_j \times \mathbb{R}$ for $j = 1, 2$ where $I_1, I_2 \subseteq [-2, 2]$ are L-separated.

It remains to show how such bilinear estimates, with the additional separation / transversality hypothesis, can be used to prove linear estimates.

This is achieved using "induction-on-scales" or, more precisely, what is known as the Bourgain-Guth method or a "broad-narrow analysis".

We begin with an elementary result which allows one to compare "linear" and "bilinear" expressions.

Let $\mathcal{I}$ denote the collection of all dyadic subintervals of $[-2, 2]$, say, and for $\lambda \in 2^{\mathbb{Z}}$ let

$$\mathcal{I}(\lambda) := \{ I \in \mathcal{I} : \ell(I) = \lambda \}.$$

($\ell(I)$ = length of I).

Suppose $(g_I)_{I \in \mathcal{I}}$ is a collection of functions

$$g_I : \Omega \to \mathbb{R}^n \quad \text{satisfying}$$

$$g_I = \sum_{\substack{J \in \mathcal{I}(\lambda) \\ J \subseteq I}} g_J \quad \text{for all } \lambda \leq \ell(I),\ I \in \mathcal{I}.$$

Lemma. (Bourgain-Guth decomposition) With the above set up,

$$|g(x)| \leq 3 \cdot \max_{I \in \mathcal{I}(\lambda)} |g_I(x)| + \sum_{\substack{I_1, I_2 \in \mathcal{I}(\lambda) \\ \mathrm{dist}(I_1, I_2) \geq \lambda}} \prod_{i=1}^{2} |g_{I_i}(x)|^{1/2},$$

where $g := g_{[-2,2)} = g_{[-2,0]} + g_{[0,2]}$.

The term $3 \cdot \max_{I \in \mathcal{I}(\lambda)} |g_I(x)|$

is called the "narrow part". It dominates the sum in situations where the main contributions to the sum

$$\sum_{I \in \mathcal{I}(\lambda)} g_I(x)$$

"cluster around a single interval I".

The term $\displaystyle\sum_{\substack{I_1, I_2 \in \mathcal{I}(\lambda) \\ \mathrm{dist}(I_1, I_2) \geq \lambda}} \prod_{i=1}^{2} |g_{I_i}(x)|^{1/2}$

is called the "broad part". It dominates the sum when the contributions to

$$\sum_{I \in \mathcal{I}(\lambda)} g_I(x)$$

have some "spread" over the I.

Proof (of Lemma):- The decomposition is trivial.

Let $I_x \in \mathcal{I}(\lambda)$ be a choice of interval satisfying

$$|g_{I_x}(x)| = \max_{I \in \mathcal{I}(\lambda)} |g_I(x)|$$

so that, by the triangle inequality,

$$|g(x)| \leq \sum_{I \in \mathcal{I}(\lambda)} |g_I(x)|$$

$$= \sum_{\substack{I \in \mathcal{I}(\lambda) \\ \mathrm{dist}(I, I_x) < \lambda}} |g_I(x)| + \sum_{\substack{I \in \mathcal{I}(\lambda) \\ \mathrm{dist}(I, I_x) \geq \lambda}} |g_I(x)|. \quad (1$$

The first sum is only over 3 terms:- I_x itself and the two intervals adjacent to I_x.

Thus, it can be bounded by

$$3 \cdot |g_{I_x}(x)|$$

which corresponds to the narrow term.

On the other hand, the second sum in (1) can be bounded by

$$\sum_{\substack{I \in \mathcal{I}(\lambda) \\ \mathrm{dist}(I, I_x) \geq \lambda}} |g_I(x)|^{1/2} |g_{I_x}(x)|^{1/2}$$

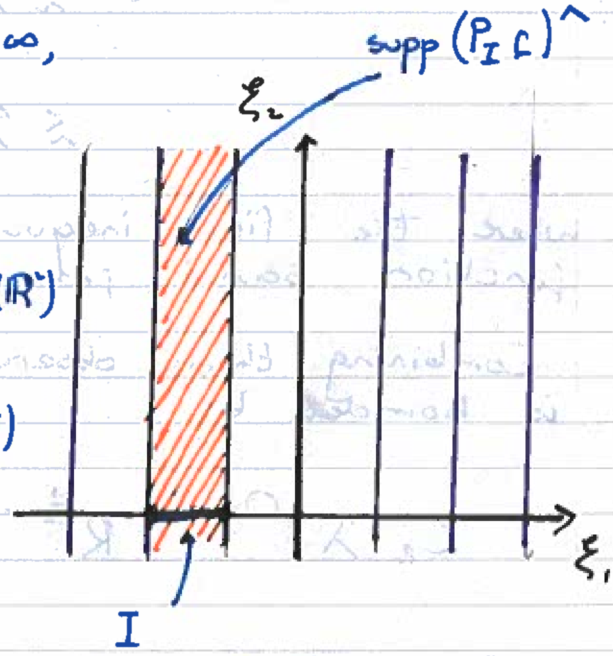
which is clearly bounded by the broad term. □

We may decompose $f \in \mathcal{S}(\mathbb{R}^2)$ as a sum

$$f = \sum_{I \in \mathcal{I}(\lambda)} f_I \quad \text{for each } \lambda \in 2^{\mathbb{Z}}$$

where $f_I = P_I f$ is a smooth frequency projection onto the strip $I \times \mathbb{R}$.

Thus, given $4 \leq p < \infty$,



$$\|T_R f\|_{L^p(\mathbb{R}^2)} \leq$$

$$3 \cdot \| \max_{I \in \mathcal{I}(\lambda)} |T_R f_I| \|_{L^p(\mathbb{R}^2)}$$

$$+ \sum_{\substack{I_1, I_2 \in \mathcal{I}(\lambda) \\ \mathrm{dist}(I_1, I_2) \geq \lambda}} \| \prod_{j=1}^{2} |T_R f_{I_j}|^{1/2} \|_{L^p(\mathbb{R}^2)}$$

Since the $T_R f_I$ oscillate, the use of the triangle inequality is rather brutal here, but we shall nevertheless avoid essential losses by working with a coarse scale λ.

We begin by bounding the broad term.

By the bilinear estimate,

$$\Big\| \prod_{j=1}^{2} |T_R f_{I_j}|^{1/2} \Big\|_{L^p(\mathbb{R}^2)} \lesssim_\varepsilon \lambda^{-1/p} R^{\frac{1}{2}-\frac{2}{p}+\varepsilon} \prod_{j=1}^{2} \|f_{I_j}\|^{1/2}_{L^p(\mathbb{R}^2)}$$

whenever $I_1, I_2 \in \mathcal{I}(\lambda)$ satisfy $\mathrm{dist}(I_1, I_2) \geq \lambda$.

Summing over all such interval pairs, and then dropping the separation hypothesis, the broad term is bounded by

$$C_\varepsilon \lambda^{-1/p} R^{\frac{1}{2}-2/p+\varepsilon} \sum_{I_1, I_2 \in \mathcal{I}(\lambda)} \prod_{j=1}^{2} \|f_{I_j}\|^{1/2}_{L^p(\mathbb{R}^2)}$$

The double sum can be written as

$$\Big(\sum_{I \in \mathcal{I}(\lambda)} \|f_I\|^{1/2}_{L^p(\mathbb{R}^2)} \Big)^2 \leq [\#\mathcal{I}(\lambda)]^{2-1/p} \Big(\sum_{I \in \mathcal{I}(\lambda)} \|f_I\|^p_{L^p(\mathbb{R}^2)} \Big)^{\frac{1}{p}}$$

$$\sim \lambda^{-(2-1/p)} \Big\| \Big(\sum_{I \in \mathcal{I}} |f_I|^p \Big)^{1/p} \Big\|_{L^p(\mathbb{R}^2)}$$

$$\lesssim \lambda^{-2(1-1/p)} \Big\| \Big(\sum_{I \in \mathcal{I}} |f_I|^2 \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^2)}$$

$$\lesssim \lambda^{-2(1-1/p)} \|f\|_{L^p(\mathbb{R}^2)},$$

where the final inequality is by the square function bound from Lecture 7.

Combining these observations, the broad term is bounded by

$$C_\varepsilon \lambda^{-O(1)} R^{\frac{1}{2}-2/p+\varepsilon} \|f\|_{L^p(\mathbb{R}^2)}$$

The $O(1)$ exponent is explicitly given by

$2 - 1/p$

but we use this notation ~~as~~ it will turn out the exact power of λ^{-1} is not important here.

For the narrow part, we bound

$$\|\max_{I \in \mathcal{I}(\lambda)} |T_R f_I|\|_{L^p(\mathbb{R}^2)} \le \|(\sum_{I \in \mathcal{I}(\lambda)} |T_R f_I|^p)^{1/p}\|_{L^p(\mathbb{R}^2)}$$

$$= \left(\sum_{I \in \mathcal{I}(\lambda)} \|T_R f_I\|^p_{L^p(\mathbb{R}^2)}\right)^{1/p}.$$

Thus, altogether

$$\|T_R f\|_{L^p(\mathbb{R}^2)} \le 3 \cdot \left(\sum_{I \in \mathcal{I}(\lambda)} \|T_R f_I\|^p_{L^p(\mathbb{R}^2)}\right)^{1/p} \qquad (2)$$

$$+ C_\varepsilon^b \lambda^{-O(1)} R^{1/2 - 2/p + \varepsilon} \|f\|_{L^p(\mathbb{R}^2)}.$$

Note :

- Provided $\lambda \sim_\varepsilon 1$, the second term on RHS is good for us.
- On the other hand for each of the $T_R f_I$ we have gained some (small) additional Fourier localisation in the ξ_1 direction.
- If $\lambda \sim_\varepsilon 1$, then the $T_R f_I$ is only localised rather coarsely and so we don't gain much directly from (2).
- The key is that we can iterate (2) over and over until we obtain a fine scale localisation.

The iteration procedure mentioned above can be conveniently expressed in terms of an induction-on-scale argument, via a rescaling of the operator.

In particular, a rescaling can be used to convert the Fourier localisation in the ξ_1 direction in $T_R f_I$ to a drop in the R parameter. We can then induct on R.

Base case :- If $1 \le R \lesssim_\varepsilon 1$, then

$\|T_R f\|_{L^p(\mathbb{R}^2)} \lesssim_\varepsilon \|f\|_{L^p(\mathbb{R}^2)}$ clearly holds since the kernel K_R satisfies $\|K_R\|_1 \lesssim_\varepsilon 1$ in this case.

<u>Induction hypothesis</u>:- Let $R \gg 1$ and suppose whenever $1 \le R' \le R/2$ we have

$$\| T_{R'} f \|_{L^p(\mathbb{R}^2)} \le \bar{C}_\varepsilon (R')^{1/2 - 2/p + \varepsilon} \| f \|_{L^p(\mathbb{R}^2)}$$

Here $\bar{C}_\varepsilon$ is a fixed constant, depending only on ε and chosen to satisfy the forthcoming requirements of the proof.

<u>Lemma (Parabolic rescaling)</u>:- Under the induction hypothesis,

$$\| T_R f_I \|_{L^p(\mathbb{R}^2)} \lesssim \bar{C}_\varepsilon (\lambda^2 R)^{\frac{1}{2} - 2/p + \varepsilon} \| f_I \|_{L^p(\mathbb{R}^2)}.$$

<u>Proof</u>:- Since $T_R \circ P_I$ is a Fourier multiplier operator, its operator norm is invariant under $GL(2, \mathbb{R})$ action.

The key observation is that we can rescale

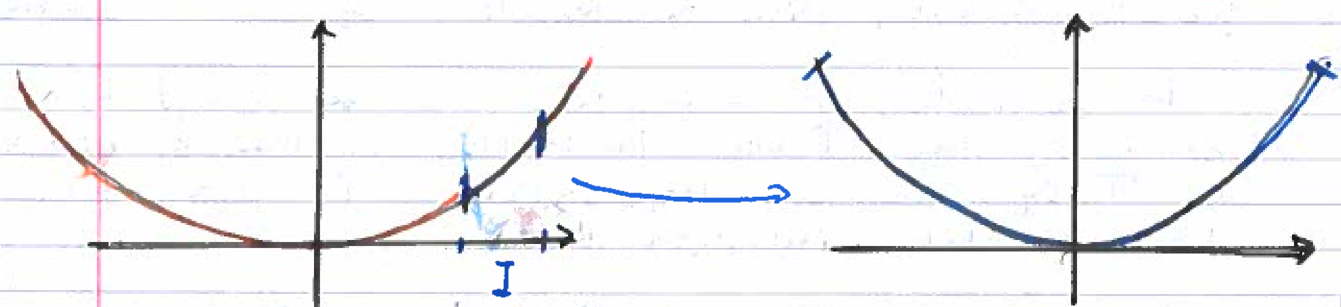
$$\chi(R(\xi_2 - \xi_1^2/2)) \, \varrho(\lambda^{-1}(\xi_1 - c_I))$$

to

$$\chi(\lambda^2 R(\xi_2 - \xi_1^2/2)) \, \varrho(\xi_1)$$

by

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \longmapsto \begin{pmatrix} \lambda & 0 \\ \lambda c_I & \lambda^2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} + \begin{pmatrix} c_I \\ c_I^2/2 \end{pmatrix}. \qquad (3)$$



Note that this map takes the small portion of the parabola over I to the whole parabola.

This special 'affine self-symmetry' is an important tool in much of our analysis.

Thus, by applying the affine rescaling (3) in the frequency variables, we can write

$$\| T_R f_I \|_{L^p(\mathbb{R}^2)} \sim \| T_{\lambda^2 R} \tilde{f}_I \|_{L^p(\mathbb{R}^2)}$$

where $\tilde{f}_I$ is a suitably rescaled and renormalised version of f_I.

Applying the induction hypothesis,

$$\| T_R f_I \|_{L^p(\mathbb{R}^2)} \lesssim \bar{C}_\varepsilon \, (\lambda^2 R)^{\frac{1}{2} - 2/p + \varepsilon} \, \| \tilde{f}_I \|_{L^p(\mathbb{R}^2)}$$

$$= \bar{C}_\varepsilon \, (\lambda^2 R)^{1/2 - 2/p + \varepsilon} \, \| f_I \|_{L^p(\mathbb{R}^2)}.$$

Going back to the narrow term in (2), we now have

$$\lesssim \bar{C}_\varepsilon \, \lambda^{1 - 4/p + 2\varepsilon} \, R^{\frac{1}{2} - 2/p + \varepsilon} \Big(\sum_{I \in \mathcal{I}(\lambda)} \| f_I \|^p_{L^p(\mathbb{R}^2)} \Big)^{1/p}$$

$$\leq C^n_\varepsilon \, \bar{C}_\varepsilon \, \lambda^{1 - 4/p + 2\varepsilon} \, R^{1/2 - 2/p + \varepsilon} \, \| f \|_{L^p(\mathbb{R}^2)}$$

by the same square function argument as used in the broad case.

If $p \geq 4$, then we can choose λ sufficiently small, depending on ε only, so that

$$C^n_\varepsilon \, \lambda^{1 - 4/p + 2\varepsilon} \leq \frac{1}{2}.$$

On the other hand, provided $\bar{C}_\varepsilon$ is chosen sufficiently large from the outset,

$$C^b_\varepsilon \, \lambda^{-O(1)} \leq \frac{\bar{C}_\varepsilon}{2},$$

where the LHS is the factor appearing in the second line in (2).

Combining these bounds, it follows that

$$\|T_R f\|_{L^p(\mathbb{R}^n)} \leq \bar{C}_\varepsilon R^{1/2 - 2/p + \varepsilon} \|f\|_{L^p(\mathbb{R}^n)}$$

which establishes the inductive step and therefore completes the proof of the theorem. $\square$