Lecture 8: Bochner-Riesz Means IV.

In these lectures we'll give a second proof of the Bochner-Riesz conjecture using a different set of tools.
In particular, we will introduce :-

i) The bilinear method

ii) Induction-on-scales (Bourgain-Guth method / "broad/narrow analysis").

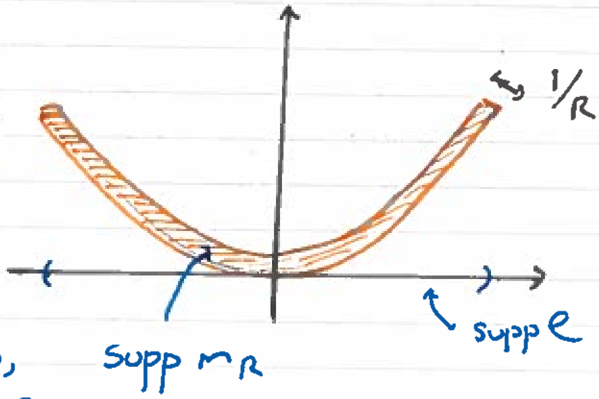
By our earlier reductions, the problem is to bound the operator T_R associated to the Fourier multiplier

$$m_R(\xi) := \chi(R(\xi_2 - \xi_1^2/2))\, e(\xi_1)$$

with a suitable dependence on R in the estimate for $\|T_R\|_{p\to p}$.

Bilinear Bochner-Riesz.

The first step is to prove 'bilinear' estimates for the Bochner-Riesz multipliers, which take the following form.



Proposition 1 (Bilinear estimate) :- Let $4 \le p \le \infty$, $L > 0$ and $R \geq 1$. For all $\varepsilon > 0$, the inequality

$$\Big\| \prod_{j=1}^{2} |T_R f_j|^{1/2} \Big\|_{L^p(\mathbb{R}^2)} \lesssim_\varepsilon L^{-1/p} R^{\frac{1}{2} - 2/p + \varepsilon} \prod_{j=1}^{2} \|f_j\|^{1/2}_{L^p(\mathbb{R}^2)}$$

holds whenever $\operatorname{supp} \hat{f_j} \subseteq I_j \times \mathbb{R}$ for $j = 1, 2$, where $I_1, I_2 \subseteq [-2, 2]$ are L-separated intervals.

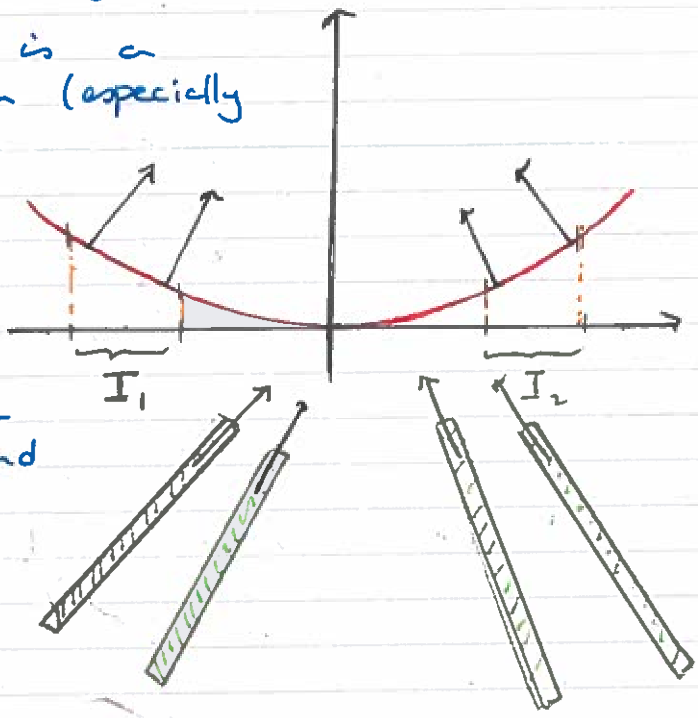
Remark :- • If we were allowed to drop the separation hypothesis and take $f_1 = f_2$, then we would immediately obtain the Carleson–Sjölin theorem (ie. linear bounds for T_R)

- On the other hand, Carleson–Sjölin implies the bilinear estimate via Hölder's inequality / Cauchy–Schwarz.

- In particular, the bilinear estimate is a 'weak variant' of the linear estimate with an additional separation hypothesis.

- It turns out this is a significant simplification (especially in high dimensional situations).

The reason for this is, roughly, that the wave packets from f_1 are transverse to those from f_2 and can therefore only interact in a reasonably simple fashion.



- Indeed, whilst the Bochner–Riesz conjecture is wide open in dimensions $n \geq 3$, an appropriate multilinear generalisation of Proposition 1 is known in all dimensions due to Bennett–Carbery–Tao (with various other proofs later discovered by Guth), together with a direct generalisation of the argument presented below.

The key ingredient in the proof of Proposition 1 is the following "bilinear restriction estimate":–

Proposition 2: (Bilinear restriction):– Let $4 \leq p \leq \infty$, $L > 0$, $R \geq 1$. Then

$$\| \prod_{j=1}^{2} |G_j|^{1/2} \|_{L^p(\mathbb{R}^2)} \lesssim L^{-1/p} R^{-1/2} \prod_{j=1}^{2} \|G_j\|^{1/2}_{L^2(\mathbb{R}^2)}$$

holds whenever the G_j satisfy

$$\operatorname{supp} \widehat{G_j} \subseteq \{ \xi \in \widehat{\mathbb{R}}^2 : \xi_1 \in I_j,\ |\xi_2 - \xi_1^2/2| < R^{-1} \}$$

where $I_1, I_2 \subseteq [-2, 2]$ are L-separated intervals.

We will postpone the proof of Proposition 2 and first show how it can be applied to deduce Proposition 1.

<u>Prop$^{\underline{n}}$ 2 $\Rightarrow$ Prop$^{\underline{n}}$ 1</u> :- Applying Prop$^{\underline{n}}$ 2 directly to the $T_R f_j$ yields

$$\Big\| \prod_{j=1}^{2} |T_R f_j|^{1/2} \Big\|_{L^p(\mathbb{R}^2)} \lesssim L^{-1/p} R^{-1/2} \prod_{j=1}^{2} \| T_R f_j \|_{L^2(\mathbb{R}^2)}^{1/2}$$

$$\lesssim L^{-1/p} R^{-1/2} \prod_{j=1}^{2} \| f_j \|_{L^2(\mathbb{R}^2)}^{1/2} \qquad (1)$$

Since $\| T_R f_j \|_{L^2(\mathbb{R}^2)} \leq \| m_R \|_\infty \| f_j \|_{L^2(\mathbb{R}^2)}$ by Plancherel.

The problem now is that there is an L^2-norm on the RHS when we want L^p. The trick here is to use a 'pseudo local' property of T_R which allows one to localise the L^2-norm and apply Hölder's inequality to lift it to L^p.

<u>Properties of the kernel</u> Note that the kernel K_R of T_R satisfies

$$K_R(x) = \int_{\widehat{\mathbb{R}}^2} e^{2\pi i \langle x, \xi \rangle} m_R(\xi)\, d\xi$$

$$= R^{-1} \chi^\vee (R^{-1} x_2) (d\sigma)^\vee (x)$$

and it follows from the decay properties of $(d\sigma)^\vee$ that

$$|K_R(x)| \lesssim_N R^{-1} (1 + R^{-1}|x_1| + R^{-1}|x_2|)^{-N}$$

$$\lesssim (1 + R^{-1}|x|)^{-N} \quad \text{for all } N \in \mathbb{N}$$

so K_R is essentially concentrated on $B(0, R)$. and

$$\| K_R \|_1 \lesssim R^2$$

N.B. These estimates are quite rough (i.e. far from sharp) but will suffice for our purposes.

Localisation:- Given $\varepsilon > 0$, let $\mathcal{Q}$ be a collection of essentially disjoint cubes in $\mathbb{R}^2$ which are parallel to the coordinate axes and have side length $R^{1+\varepsilon/2}$.

For any $f : \mathbb{R}^2 \to \mathbb{C}$ write

$$f_Q := \chi_Q \cdot f \quad ; \quad f_Q^* := \sum_{\substack{Q' \in \mathcal{Q} \\ Q' \subseteq 3Q}} f_{Q'}$$

Here $3 \cdot Q$ is the cube concentric to Q but with $3\times$ side length.

Thus $T_R f_j = T_R f_{j,Q}^* + T_R (f_j - f_{j,Q}^*)$.

and
$$\prod_{j=1}^{2} T_R f_j = \prod_{j=1}^{2} T_R f_{j,Q}^* + \sum_{\substack{S \subseteq \{1,2\} \\ S \neq \emptyset}} \prod_{j \in S} T_R (f_j - f_{j,Q}^*) \prod_{j \notin S} T_R f_{j,Q}^* \qquad (1)$$

Error terms:- If $x \in Q$ and $y \in \operatorname{supp} f_j - f_{j,Q}^* \subseteq 3 \cdot Q^c$ then $|x-y| \geq R^{1+\varepsilon/2}$ so

$$|K_R(x-y)| \lesssim_N (1 + R^{\varepsilon/2})^{-N} (1 + R^{-1}|x-y|)^{-10}$$
$$\lesssim R^{-100} E_R(x-y)$$

provided N is chosen sufficiently large; here

$E_R(x) := (1 + R^{-1}|x|)^{-10}$.

Thus, for $x \in Q$,

$$|T_R (f_j - f_{j,Q}^*)(x)| \leq \int_{\mathbb{R}^2} |K_R(x-y)| \, |(f_j - f_{j,Q}^*)(y)| \, dy$$
$$\lesssim R^{-100} E_R * |(f_j - f_{j,Q}^*)| (x)$$
$$\lesssim R^{-100} E_R * |f_j| (x)$$

Combining this with 2, it is easy to see

$$\Big\| \prod_{j=1}^{2} |T_R f_j|^{1/2} \Big\|_{L^p(\mathbb{R}^2)} = \Big(\sum_{Q \in \mathcal{Q}} \Big\| \prod_{j=1}^{2} |T_R f_j|^{1/2} \Big\|_{L^p(Q)}^p \Big)^{1/p}$$

$$\lesssim \Big(\sum_{Q \in \mathcal{Q}} \Big\| \prod_{j=1}^{2} |T_R f_{j,Q}^*|^{1/2} \Big\|_{L^p(Q)}^p \Big)^{1/p} + R^{-50} \cdot \prod_{j=1}^{2} \|f_j\|_{L^p(\mathbb{R}^2)}^{1/2}$$

On the other hand, applying 1 to the first term in the above display yields

$$\Big(\sum_{Q \in \mathcal{Q}} \Big\| \prod_{j=1}^{2} |T_R f_{j,Q}^*|^{1/2} \Big\|_{L^p(\mathbb{R}^2)}^p \Big)^{1/p}$$

$$\lesssim L^{-1/2p} R^{-1/2} \Big(\sum_{Q \in \mathcal{Q}} \prod_{j=1}^{2} \|f_{j,Q}^*\|_{L^2(\mathbb{R}^2)}^{p/2} \Big)^{1/p} \qquad 3$$

Since $\operatorname{supp} f_{j,Q}^* \subseteq 3 \cdot Q$, Hölder's inequality implies

$$\|f_{j,Q}^*\|_{L^2(\mathbb{R}^2)} \lesssim |Q|^{\frac{1}{2} - 1/p} \|f_{j,Q}^*\|_{L^p(\mathbb{R}^2)}$$
$$\lesssim R^{2(\frac{1}{2} - 1/p) + \varepsilon} \|f_{j,Q}\|_{L^p(\mathbb{R}^2)}$$

Substituting this into 3 and applying Cauchy-Schwarz,

$$\lesssim L^{-1/p} R^{\frac{1}{2} - 2/p + \varepsilon} \prod_{j=1}^{2} \Big(\sum_{Q \in \mathcal{Q}} \|f_{j,Q}^*\|_{L^p(\mathbb{R}^2)}^p \Big)^{1/2p}.$$

Finally, since the $f_{j,Q}^*$ have finitely overlapping supports,

$$\sum_{Q \in \mathcal{Q}} \|f_{j,Q}^*\|_p^p = \Big\| \Big(\sum_{Q \in \mathcal{Q}} |f_{j,Q}^*|^p \Big)^{1/p} \Big\|_p^p$$
$$\lesssim \|f\|_{L^p(\mathbb{R}^2)}^p$$

and combining these observations concludes the proof. $\square$

We now turn to the proof of Prop^{n} 2.

Proof (of Prop^{n} 2):- Consider the vertical translates of the parabola

$$\gamma_\eta(s) := \begin{pmatrix} s \\ \frac{s^2}{2} + \eta \end{pmatrix} \qquad \text{for } \eta \in \mathbb{R}.$$

By the Fourier support condition and Fubini, one may write

$$G_j(x) = \int_{\widehat{\mathbb{R}}^2} e^{2\pi i \langle x, \xi \rangle} \widehat{G}_j(\xi)\, d\xi$$

$$= \int_{-R^{-1}}^{R^{-1}} \int_{I_j} e^{2\pi i (x_1 s + x_2(\frac{s^2}{2} + \eta))} \widehat{G}_j \circ \gamma_\eta(s)\, ds\, d\eta$$

$$= \int_{-R^{-1}}^{R^{-1}} E_\eta g_{j,\eta}(x)\, d\eta$$

where $g_{j,\eta} : [-2,2] \longrightarrow \mathbb{C}$; $g_{j,\eta}(s) := \chi_{I_j}(s)\, \widehat{G}_j \circ \gamma_\eta(s)$

and

$$E_\eta g(x) := \int_{-2}^{2} e^{2\pi i \langle x, \gamma_\eta(s) \rangle} g(s)\, ds$$

for $g \in L^1([-2,2])$ is the "extension operator" associated to γ_s.

Now,

$$\Big\| \prod_{j=1}^{2} |G_j|^{1/2} \Big\|_{L^p(\mathbb{R}^2)} = \Big\| \prod_{j=1}^{2} \int_{-R^{-1}}^{R^{-1}} E_{\eta_j} g_{j,\eta_j}(\cdot)\, d\eta_j \Big\|_{L^{p/2}(\mathbb{R}^2)}^{1/2}$$

$$= \Big\| \int_{-R^{-1}}^{R^{-1}} \int_{-R^{-1}}^{R^{-1}} \prod_{j=1}^{2} E_{\eta_j} g_{j,\eta_j}\, d\eta_1 d\eta_2 \Big\|_{p/2}^{1/2}$$

$$\leq \left(\int_{-R^{-1}}^{R^{-1}} \int_{-R^{-1}}^{R^{-1}} \Big\| \prod_{j=1}^{2} |E_{\eta_j} g_{j,\eta_j}| \Big\|_{p/2}\, d\eta_1 d\eta_2 \right)^{1/2}$$

$$= \left(\int_{-R^{-1}}^{R^{-1}} \int_{-R^{-1}}^{R^{-1}} \Big\| \prod_{j=1}^{2} |E g_{j,\eta_j}|^{1/2} \Big\|_p^2\, d\eta_1 d\eta_2 \right)^{1/2} \qquad (4)$$

where $E := E_0$. The key claim is the following:-

Proposition 3:– (Bilinear extension) Let $4 \leq p \leq \infty$ and $L > 0$. Then

$$\Big\| \prod_{j=1}^{2} |Eg_j|^{1/2} \Big\|_{L^p(\mathbb{R}^2)} \lesssim L^{-1/p} \prod_{j=1}^{2} \| g_j \|^{1/2}_{L^2([-2,2])}$$

holds whenever $g_j \in L^1([-2,2])$ satisfy $\operatorname{supp} g_j \subseteq I_j$ for $j = 1, 2$ where $I_1, I_2 \subseteq [-2,2]$ are L-separated intervals.

Temporarily assuming this, we can complete the proof of Prop$^{\text{n}}$ 2 as follows.

By (4), for $4 \leq p \leq \infty$,

$$\Big\| \prod_{j=1}^{2} |G_j|^{1/2} \Big\|_{L^p(\mathbb{R}^2)} \leq \Big(\int_{-R^{-1}}^{R^{-1}} \int_{-R^{-1}}^{R^{-1}} \Big\| \prod_{j=1}^{2} |Eg_{j,\eta_j}|^{1/2} \Big\|_p^2 \, d\eta_1 d\eta_2 \Big)^{1/2}$$

$$\lesssim L^{-1/p} \Big(\int_{-R^{-1}}^{R^{-1}} \int_{-R^{-1}}^{R^{-1}} \prod_{j=1}^{2} \| g_{j,\eta_j} \|_{L^2([-2,2])} \, d\eta_1 d\eta_2 \Big)^{1/2}$$

$$= L^{-1/p} \Big(\prod_{j=1}^{2} \int_{-R^{-1}}^{R^{-1}} \| g_{j,\eta_j} \|_{L^2} \, d\eta_j \Big)^{1/2}.$$

By Cauchy–Schwarz,

$$\int_{-R^{-1}}^{R^{-1}} \| g_{j,\eta} \|_2 \, d\eta \lesssim R^{-1/2} \Big(\int_{-R^{-1}}^{R^{-1}} \| \hat{G}_j \circ \gamma_\eta \|_2^2 \, d\eta \Big)^{1/2}$$

$$\lesssim R^{-1/2} \| \hat{G}_j \|_{L^2(\hat{\mathbb{R}}^2)} = R^{-1/2} \| G_j \|_{L^2(\mathbb{R}^2)}.$$

Combining these observations,

$$\Big\| \prod_{j=1}^{2} |G_j|^{1/2} \Big\|_{L^p(\mathbb{R}^2)} \lesssim L^{-1/p} R^{-1/2} \prod_{j=1}^{2} \| G_j \|^{1/2}_{L^2(\mathbb{R}^2)},$$

as required.

Proof (of Prop$^{\text{n}}$ 3) Write

$$\prod_{j=1}^{2} Eg_j(x) = \int_{I_2} \int_{I_1} e^{2\pi i \langle x, \gamma(s_1) + \gamma(s_2) \rangle} \prod_{j=1}^{2} g_j(s_j) \, ds_1 ds_2$$

where the range of integration is localised to I_1 and I_2 due to the support hypothesis on the g_j.

Apply a change of variables $\xi = \gamma(s_1) + \gamma(s_2)$ so that

$$\left|\det \frac{\partial \xi}{\partial s}\right| = \left|\det \begin{pmatrix} 1 & 1 \\ s_1 & s_2 \end{pmatrix}\right| = |s_2 - s_1| \gtrsim L$$

for $s_j \in I_j$, $j = 1, 2$. Thus,

$$\prod_{j=1}^{2} Eg_j(x) = \iint_D e^{2\pi i \langle x, \xi \rangle} \prod_{j=1}^{2} g_j \circ s_j(\xi) \, |s_2(\xi) - s_1(\xi)|^{-1} d\xi$$

so that

$$\left\| \prod_{j=1}^{2} |Eg_j|^{1/2} \right\|_4 = \left\| \prod_{j=1}^{2} Eg_j \right\|_2^{1/2}$$

$$= \left(\iint_D \left| \prod_{j=1}^{2} g_j \circ s_j(\xi) \right|^2 |s_2(\xi) - s_1(\xi)|^{-2} d\xi \right)^{1/4}$$

by Plancherel. Changing back the variables, we obtain

$$\left\| \prod_{j=1}^{2} |Eg_j|^{1/2} \right\|_4 = \left(\iint_{I_2 I_1} \prod_{j=1}^{2} |g_j(s_j)|^2 \, |s_2 - s_1|^{-1} ds_1 ds_2 \right)^{1/4}$$

$$\lesssim L^{-1/4} \prod_{j=1}^{2} \|g_j\|_{L^2(I_j)}^{1/2} \qquad (5)$$

On the other hand,

$$\left\| \prod_{j=1}^{2} |Eg_j|^{1/2} \right\|_\infty \leq \prod_{j=1}^{2} \|g_j\|_{L^1(I_j)}^{1/2}$$

$$\lesssim \prod_{j=1}^{2} \|g_j\|_{L^2(I_j)}^{1/2} \qquad (6)$$

since $\|Eg\|_\infty \leq \|g\|_1$.

Interpolating (5) and (6) via Hölder's inequality concludes the proof. $\square$.