Lecture 7 : Bochner-Riesz means III

Last time we proved the Córdoba-Fefferman square function bound

$$\| T_R f \|_{L^4(\mathbb{R}^2)} \lesssim \| (\sum_{\theta : \text{slab}} | T_{R,\theta} f |^2)^{1/2} \|_{L^4(\mathbb{R}^2)} .$$

We saw how to conclude the proof of the Carleson-Sjölin theorem under the additional hypothesis that each $T_{R,\theta} f$ is given by a single wave packet.

Here we consider the general case where the $T_{R,\theta} f$ are made up of many parallel wave packets.

We argue via duality:-

$$\| (\sum_{\theta : \text{slab}} | T_{R,\theta} f |^2)^{1/2} \|^2_{L^4(\mathbb{R}^2)} = \| \sum_{\theta : \text{slab}} | T_{R,\theta} f |^2 \|_{L^2(\mathbb{R}^2)}$$

$$= \int_{\mathbb{R}^2} \sum_{\theta : \text{slab}} | T_{R,\theta} f |^2 \cdot g \quad \text{for some } g \in L^2(\mathbb{R}^2) \text{ with } \|g\|_{L^2(\mathbb{R}^2)} = 1 .$$

Each $T_{R,\theta} f = K_{R,\theta} * f$ where the kernel $K_{R,\theta} = (m_{R,\theta})^\vee$ is essentially L^1 normalised:-

$$\| K_{R,\theta} \|_{L^1(\mathbb{R}^2)} \lesssim 1 . \qquad (1)$$

Recall, each multiplier $m_{R,\theta}(\xi) = m_R(\xi) \zeta(R^{1/2}\xi_1 - k_\theta)$ is supported on a strip $[R^{-1/2}(k_\theta - 1), R^{-1/2}(k_\theta + 1)] \times \mathbb{R}$

We can fix another bump function ζ_0 satisfying

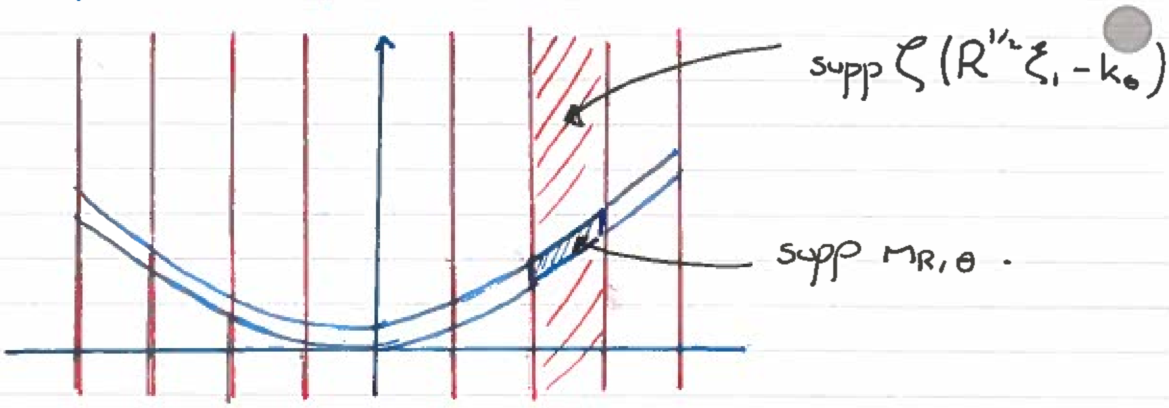
$$\zeta_0(s) = 1 \text{ for } |s| \leq 1 , \quad \zeta_0(s) = 0 \text{ if } |s| \geq 2$$

so that

$$m_{R,\theta}(\xi) = m_{R,\theta}(\xi) \cdot \zeta_0(R^{1/2}\xi_1 - k_\theta)$$

Thus, if $(P_\theta f)^\wedge(\xi) = \zeta_0(R^{1/2}\xi_1 - k_\theta) \hat{f}(\xi)$, then

$$T_{R,\theta} f = K_{R,\theta} * P_\theta f.$$



By Cauchy–Schwarz and (1),

$$|T_{R,\theta} f|^2 \lesssim |K_{R,\theta}| * |P_\theta f|^2$$

so that, combining all the above observations,

$$\|T_R f\|_{L^4(\mathbb{R}^2)}^2 \lesssim \sum_{\theta : \text{slab}} \int_{\mathbb{R}^2} |K_{R,\theta}| * |P_\theta f|^2 \, |g|$$

$$= \sum_{\theta : \text{slab}} \int_{\mathbb{R}^2} |P_\theta f|^2 \, |K_{R,\theta}| * |g|$$

$$\leq \int_{\mathbb{R}^2} \Big(\sum_{\theta : \text{slab}} |P_\theta f|^2\Big) \max_{\theta : \text{slab}} |K_{R,\theta}| * |g|$$

$$\leq \Big\| \sum_{\theta : \text{slab}} |P_\theta f|^2 \Big\|_{L^2(\mathbb{R}^2)} \|\tilde{M}_R g\|_{L^2(\mathbb{R}^2)}$$

where $\tilde{M}_R g(x) := \max_{\theta : \text{slab}} |K_{R,\theta}| * |g|$.

Thus, it suffices to show:-

Lemma 1:- For $2 \leq p \leq \infty$,

$$\Big\| \Big(\sum_{\theta : \text{slab}} |P_\theta f|^2\Big)^{1/2} \Big\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}.$$

Lemma 2:-

$$\|\tilde{M}_R g\|_{L^2(\mathbb{R}^2)} \lesssim \log R \, \|g\|_{L^2(\mathbb{R}^2)}$$

Indeed, assuming these lemmata,

$$\|{}^{\circ}T_R f\|^2_{L^4(\mathbb{R}^2)} \lesssim \Big\|\Big(\sum_{\theta : \text{slab}} |P_\theta f|^2\Big)^{1/2}\Big\|^2_{L^4(\mathbb{R}^2)} \, \|\tilde{M}_R g\|_{L^2(\mathbb{R}^2)}$$

$$\lesssim \|f\|^2_{L^4(\mathbb{R}^2)} \cdot \log R \; \|g\|_{L^2(\mathbb{R}^2)}$$

$$= (\log R) \cdot \|f\|^2_{L^4(\mathbb{R}^2)},$$

as required. □.

<u>Proof (of Lemma 1)</u>:– By rescaling it suffices to consider the case $R=1$.

For $k \in \mathbb{Z}$ write $(P_k f)^\wedge(\xi) = \zeta_0(\xi_1 - k)\,\hat{f}(\xi)$ so it suffices to show for all $2 \le p \le \infty$,

$$\Big\|\Big(\sum_{k\in\mathbb{Z}} |P_k f|^2\Big)^{1/2}\Big\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}.$$

- If $p=2$, then the result follows from Plancherel and the almost disjoint supports of the multiplier.

- By interpolation, it suffices to prove the $p=\infty$ case.

Note that $P_k f(x) = [\zeta_0(\cdot - k)]^\vee * f(\cdot, x_2)(x_1)$ where the convolution is in the first variable only so that,

$$P_k f(x) = \int_{\mathbb{R}} e^{2\pi i (x_1 - y_1)k}\, \zeta_0^\vee(x_1 - y_1)\, f(y_1, x_2)\, dy_1$$

$$= \sum_{n\in\mathbb{Z}} \int_0^1 e^{2\pi i (x_1 + n - y_1)k}\, \zeta_0^\vee(x_1 + n - y_1) f(y_1 - n, x_2)\, dy_1.$$

By the triangle inequality

$$\Big(\sum_{k\in\mathbb{Z}} |P_k f(x)|^2\Big)^{1/2} \le \sum_{n\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}} |(F_{x,n})^\wedge(k)|^2\Big)^{1/2}$$

where $F_{x,n}(y_1) := \zeta_0^\vee(x_1 + n - y_1)\, f(y_1 - n, x_2)$.

By Plancherel's theorem on the torus,

$$\Big(\sum_{h\in\mathbb{Z}} |P_h f(x)|^2\Big)^{1/2} \lesssim \sum_{n\in\mathbb{Z}} \|F_{x,n}\|_{L^2(\mathbb{T})}$$

$$= \sum_{n\in\mathbb{Z}} \Big(\int_0^1 |\zeta_0^\vee(x_1+n-y_1)|^2 \, |f(y_1-n, x_2)|^2 \, dy_1\Big)^{1/2}$$

$$\lesssim \|f\|_\infty \Big(\sum_{n\in\mathbb{Z}} \int_0^1 |\zeta_0^\vee(x_1+n-y_1)|^2 \, dy_1\Big)^{1/2}$$

$$\lesssim \|f\|_\infty,$$

as required, where the convergence of the sum is due to the rapid decay of $\zeta_0^\vee$. $\square$

Remark:- Alternatively, one may appeal here to a general square function bound of Rubio de Francia which applies to arbitrary decompositions of $\mathbb{R}$ into intervals (not just equal length intervals) with rough cut offs.

We turn to the proof of Lemma 2.

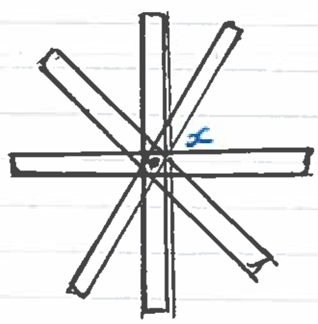
Since $K_{R,\theta} = (m_{R,\theta})^\vee$, it follows that $K_{R,\theta}$ is rapidly decaying away from the dual slab θ^*:-

$$|K_{R,\theta}(x)| \lesssim_N \frac{1}{|\theta^*|} \sum_{k\in\mathbb{N}_0} 2^{-kN} \chi_{2^k\theta^*}(x) \qquad \text{for all } N\in\mathbb{N}$$

By the triangle inequality and rescaling, it suffices to bound

$$M_R g(x) := \sup_{T\ni x} \fint_T |g| \qquad (2)$$

where the supremum is taken over all $R^{1/2}\times R$ tubes centred at x:-



M is referred to as a 'Nikodym' maximal operator.

Recall from Lecture 4 the estimate

$$\Big\| \sum_{T \in \mathbb{T}} a_T \chi_T \Big\|_{L^2(\mathbb{R}^2)} \lesssim \log R \cdot \Big(\sum_{T \in \mathbb{T}} |a_T|^2 |T| \Big)^{1/2}$$

for $\mathbb{T}$ a collection of $R^{1/2} \times R$ tubes pointing in $R^{-1/2}$-separated directions.

This can be used to bound a 'dual' version of the operator M_R from (2). In particular, let

$$K_R g(\omega) := \sup_{T \parallel \omega} \fint_T |g|, \qquad \omega \in S^1, \qquad (3)$$

where the supremum is taken over all $R^{1/2} \times R$ tubes T pointing in the direction ω.

K_R is referred to as a 'Kakeya' maximal function. Comparing (2) and (3) we see the rôles of the position x and direction ω have been 'swapped'.

<u>Lemma 3</u>

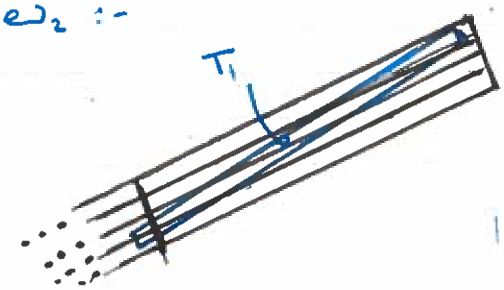
$$\| K_R g \|_{L^2(S^1)} \lesssim (\log R) R^{-1} \| g \|_{L^2(\mathbb{R}^2)}$$

<u>Proof</u> :- The first step is to note that K_R satisfies a 'locally constant' property which allows one to discretize the operator. In particular,

if $|\omega_1 - \omega_2| \lesssim C \cdot R^{-1/2}$, $\omega_1, \omega_2 \in S^1$, then

$$K_R g(\omega_1) \sim_C K_R g(\omega_2).$$

This follows since any tube T_1 in the direction ω_1 can be covered by $O_C(1)$ tubes $\{T_2\}$ in the direction ω_2 :-



Let Ω be an $R^{-1/2}$-net in S^1 so that

$$\|K_R g\|_{L^2(S^1)} \lesssim \Big(\sum_{\omega \in \Omega} \|K_R g\|^2_{L^2(B(\omega, R^{-1/2}) \cap S^1)} \Big)^{1/2}$$

$$\lesssim R^{-1/4} \Big(\sum_{\omega \in \Omega} |K_R g(\omega)|^2 \Big)^{1/2}$$

by the locally constant property.

For each $\omega \in \Omega$ we can find some $T_\omega \parallel \omega$ such that

$$K_R g(\omega) \leq 2 \cdot \fint_{T_\omega} |g|$$

Thus, by duality,

$$\|K_R g\|_{L^2(S^1)} \lesssim R^{-1/4 - 3/2} \Big(\sum_{\omega \in \Omega} \Big(\int_{T_\omega} |g| \Big)^2 \Big)^{1/2}$$

$$= R^{-1/4 - 3/2} \sum_{\omega \in \Omega} a_\omega \int_{T_\omega} |g|$$

$$= R^{-1/4 - 3/2} \int_{\mathbb{R}^2} \Big(\sum_{\omega \in \Omega} a_\omega \chi_{T_\omega} \Big) \cdot |g|$$

$$\leq R^{-1/4 - 3/2} \Big\| \sum_{\omega \in \Omega} a_\omega \chi_{T_\omega} \Big\|_2 \|g\|_2$$

for some sequence $(a_\omega)_{\omega \in \Omega}$ with $\|a_\omega\|_{\ell^2(\Omega)} = 1$.

Applying the bound from Lecture 4,

$$\|K_R g\|_{L^2(S^1)} \lesssim R^{-1/4 - 3/2} \log R \Big(\sum_{\omega \in \Omega} |a_\omega|^2 |T_\omega| \Big)^{1/2} \|g\|_2$$

$$\lesssim (\log R) R^{-1} \|g\|_2 ,$$

as required. $\square$.

We can use Lemma 3 to bound the Nikodym maximal function

<u>Lemma 4</u>:- $\|M_R g\|_{L^2(\mathbb{R}^2)} \lesssim \log R \, \|g\|_{L^2(\mathbb{R}^2)}$.

We will sketch the proof; further details can be found in the referenced paper of Tao.

Proof (sketch).

Initial reductions

i) Let M^δ, K^δ denote the operators defined in the same manner as M_R, K_R but with the $R^{1/2} \times R$ tubes replaced with $\delta \times 1$ tubes.

By rescaling, it suffices to show

$$\|M^\delta g\|_{L^2(\mathbb{R}^2)} \lesssim |\log \delta| \, \|g\|_{L^2(\mathbb{R}^2)}, \quad 0 < \delta \ll 1. \tag{4}$$

Lemma 3 implies

$$\|K^\delta g\|_{L^2(S^1)} \lesssim |\log \delta| \, \|g\|_{L^2(\mathbb{R}^2)}, \quad 0 < \delta \ll 1. \tag{5}$$

We will show (5) implies (4).

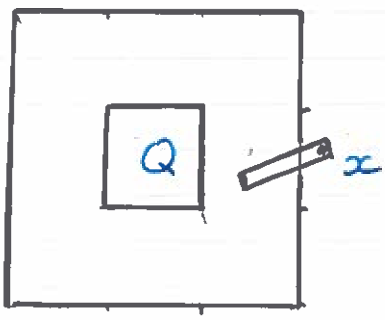
ii) The operator M^δ is local at scale 1 in the sense that

if $\operatorname{supp} f \subseteq Q$ where Q is a cube of side-length 1, then

$$Mf(x) = 0 \quad \text{if} \quad x \in \mathbb{R}^2 \setminus 3 \cdot Q.$$

By this property and 'translation invariance' it suffices to show (4) holds under the hypothesis

$$\operatorname{supp} f \subseteq [0,1]^2.$$



iii). By Fubini, it suffices to show

$$\left(\int_{\mathbb{R}} |M_\delta g(x_1, x_2)|^2 \, dx_1\right)^{1/2} \lesssim |\log \delta| \cdot \|g\|_{L^2(\mathbb{R}^2)} \; 3Q$$

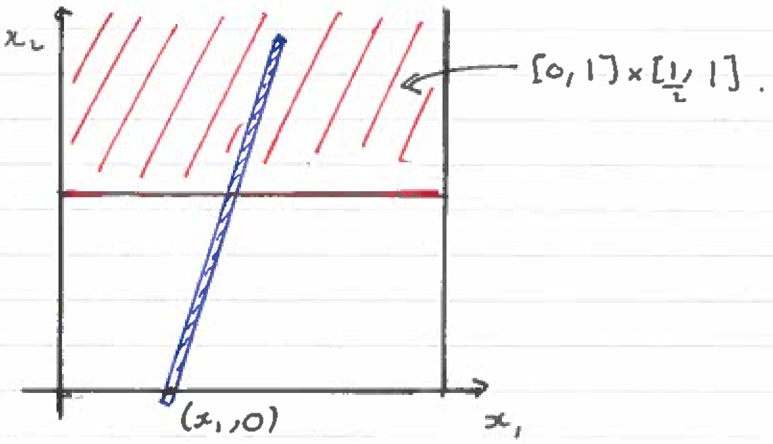
for all $0 \leq x_2 \leq 1$

By a translation argument we can further assume $x_2 = 0$.

iv) One can combine ii) and iii) with a scaling argument to reduce the problem to showing:

$$\left(\int |M^\delta g(x_1,0)|^2 \, dx_1\right)^{1/2} \lesssim |\log \delta| \, \|g\|_{L^2(\mathbb{R}^2)}$$

whenever $\operatorname{supp} g \subseteq [0,1] \times [1/2, 1]$.



Key observation:- We have a pointwise bound

$$M^\delta_\delta \, g(x_1,0) \lesssim \mathcal{K}^\delta (g \circ \phi_c)\left(\frac{(x_1,1)}{|(x_1,1)|}\right).$$

where $\phi_c(x) = \phi(C^{-1}x)$ and

$$\phi(x_1,x_2) := \left(\frac{x_1}{x_2}, \frac{1}{x_2}\right)$$

Here $C \geq 1$ is a choice of uniform constant.

To see why this might hold, we see that ϕ is a projective transformation which takes, roughly,

lines through $(x_1,0)$ in direction $(a,1)$

$\downarrow$

lines through $(a,0)$ in direction $(x_1,1)$.

Indeed, consider

$$\ell = \{(x_1 + ta, t) : \tfrac{1}{2} \leq t \leq 2\}. \text{ Then}$$

$$\begin{aligned}\phi(\ell) &= \{(a + \tfrac{1}{t}x_1, \tfrac{1}{t}) : \tfrac{1}{2} \leq \tfrac{1}{t} \leq 2\} \\ &= \{(a + s x_1, s) : \tfrac{1}{2} \leq s \leq 2\}. \qquad \square\end{aligned}$$