Lecture 6: Bochner-Riesz means II

Recall:- • $m^\alpha(\xi) := (\xi_n - |\xi'|^2/2)_+^\alpha \chi_0(\xi)$

where $\chi_0(\xi) := \tilde{\chi}(\xi_n - |\xi'|^2/2)\, e(\xi')$

• $B^\alpha f(x) := \int_{\hat{\mathbb{R}}^n} e^{2\pi i \langle x, \xi\rangle} m^\alpha(\xi) \hat{f}(\xi)\, d\xi$.

Theorem (Carleson-Sjölin) Let $n = 2$. If $1 \leq p \leq \infty$ and $\alpha > \alpha(p) := \max\{2|\frac{1}{p} - \frac{1}{2}| - 1/2, 0\}$, then

$$\|B^\alpha\|_{L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2)} < \infty.$$

In this lecture we'll give a proof of this result following an argument of Cordoba.

By the reductions of the previous lecture, it suffices to consider the $p = 4$ case only; that is, we want to show

$$\|B^\alpha\|_{L^4(\mathbb{R}^2) \to L^4(\mathbb{R}^2)} < \infty \quad \text{whenever } \alpha > 0.$$

The B^α are mollified versions of the ball multiplier S_1 from lectures 1 & 3. It therefore makes sense to let Fefferman's argument from lecture 3 guide our analysis.

Step 1: "All the action happens on/near the boundary of P^1."

We decompose the multiplier dyadically according to the distance from P_0^1.

Let $\chi(u) := |u|^\alpha \cdot (\tilde{\chi}(u) - \tilde{\chi}(2u))$.

So that $\chi \in C_c^\infty(\mathbb{R})$ and

$$m^\alpha(\xi) = \sum_{k=0}^{\infty} 2^{-k\alpha} \chi\left(\frac{\xi_n - |\xi'|^2/2}{2^{-k}}\right) e(\xi') \quad . \quad (1).$$

Defining $m_R(\xi) := \chi(R(\xi_n - |\xi'|^2/2))\, e(\xi')$

and $T_R f(x) := \int_{\mathbb{R}^2} e^{2\pi i \langle x, \xi \rangle} m_R(\xi) \hat{f}(\xi) \, d\xi$,

it therefore suffices to show

$$\| T_R \|_{L^4(\mathbb{R}^2) \to L^4(\mathbb{R}^2)} \lesssim_\varepsilon R^\varepsilon \quad \text{for all } \varepsilon > 0. \quad (2)$$

Indeed, by (1) and the triangle inequality

$$\| B^\alpha \|_{4 \to 4} \lesssim \sum_{k=0}^{\infty} 2^{-k\alpha} \| T_{2^k} \|_{4 \to 4}$$

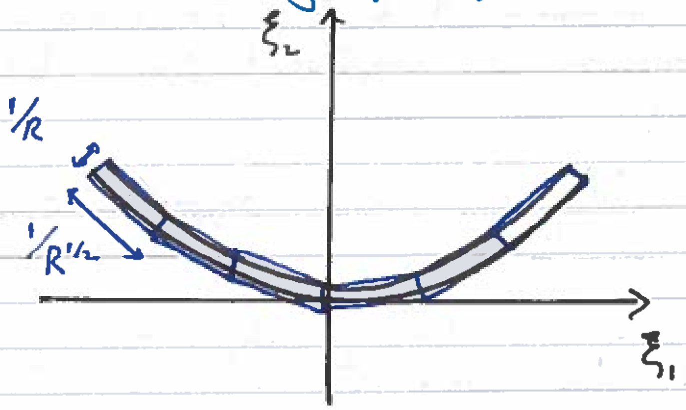
so if (2) holds with $\varepsilon = \alpha/2$, we conclude that

$$\| B^\alpha \|_{4 \to 4} \lesssim \sum_{k=0}^{\infty} 2^{-k\alpha/2} < \infty,$$

as required.

Step 2 :- "Wave packets".

- m_R is supported on a $1/R$ neighbourhood of the parabola.
- We want to break up this support into rectangular pieces we can more readily understand via the uncertainty principle.



We cover the $1/R$ neighbourhood of the paraboloid with $R^{-1} \times R^{-1/2}$ slabs θ and form a partition of unity of the frequency support so that

$$m_R = \sum_{\theta : \text{slab}} m_{R,\theta}$$

where each $m_{R,\theta}$ is supported in θ.

An explicit way to carry out this decomposition is to take

$\zeta \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} \zeta \subseteq (-1,1)$ such that

$$\sum_{k \in \mathbb{Z}} \zeta(s-k) = 1$$

(that such a function exists follows as an easy consequence of the Riemann-Lebesgue lemma).

Define
$$m_{R,\theta}(\xi) := m_R(\xi) \cdot \zeta(R^{1/2}\xi_1 - k_\theta)$$

so that $R^{-1/2}k_\theta \in R^{-1/2}\mathbb{Z}$ is the projection of the slab θ onto the ξ_1-axis.

Let
$$(T_{R,\theta} f)^\wedge(\xi) = m_{R,\theta}(\xi)\,\hat{f}(\xi) \quad \text{so that :-}$$

- $T_R f = \sum_{\theta : \text{slab}} T_{R,\theta} f$
- Each $T_{R,\theta} f$ is frequency supported in θ.

By the uncertainty principle, $T_{R,\theta} f$ is essentially constant on translates of the dual rectangle θ^*.

One precise manifestation of this is the following lemma.

Lemma (Uncertainty principle) :- There exists some non-negative function η_{θ^*} such that :-

i) If $\operatorname{supp} \hat{g} \subseteq \theta$, then

$$|g| \leq \sum_{T \parallel \theta^*} c_T \chi_T \lesssim |g| * \eta_{\theta^*} \qquad (3)$$

where the sum is taken over a family of translates T of θ^* which tessellate $\mathbb{R}^2$ and

$$c_T := \sup_{x \in T} |g(x)|$$

ii) $\|\eta_{\theta^*}\|_1 \lesssim 1$.

iii) η_{Θ^*} rapidly decays away from Θ^* in the sense that

$$\sup_{x \in T} \eta_{\Theta^*}(x) \lesssim_N \frac{1}{N} \operatorname{dist}(T, \Theta^*)^{-N} \quad \text{for all } N \in \mathbb{N}$$

whenever $T \parallel \Theta^*$. Here $\operatorname{dist}(T, \Theta^*)$ is the (e.g.) Hausdorff distance between T and Θ^*.

Proof :- The bound

$$|g| \leq \sum_{T \parallel \Theta^*} c_T \chi_T$$

follows by definition.

Let $\phi_\Theta \in \mathcal{S}(\mathbb{R}^n)$ satisfy $\hat{\phi}_\Theta(\xi) = 1$ for $\xi \in \Theta$ and $\hat{\phi}_\Theta(\xi) = 0$ for $\xi \notin 2 \cdot \Theta$. Thus,

$$g = g * \phi_\Theta. \qquad (4)$$

Define $\displaystyle \eta_{\Theta^*}(x) := \sup_{\omega \in x + 10 \cdot \Theta^*} |\phi_\Theta(\omega)|$.

It is easy to see η_{Θ^*} satisfies ii) and iii) and it remains to show the second inequality in i) holds.

In particular, given any $T \parallel \Theta^*$ and any $z \in T$ it suffices to show

$$|g(z)| \lesssim |g| * \eta_{\Theta^*}(x). \quad \text{for any } x \in T$$

By (4),

$$|g(z)| \leq \int_{\mathbb{R}^n} |g(y)| \, |\phi_\Theta(z - y)| \, dy.$$

Writing

$$|\phi_\Theta(z-y)| = |\phi_\Theta(x - y + (z - x))|$$
$$\leq |\eta_{\Theta^*}(x - y)|, \qquad x \in T,$$

since $x, z \in T$ implies $z - x \in \theta^*$

Combining these observations concludes the proof. $\square$.

Step 3: Destructive interference :-

Each $T_{R,\theta} f$ carries oscillation - we need to understand how the different pieces interact.

This is achieved via a 'reverse square function' estimate.

Theorem (Fefferman - Cordoba) Suppose $\operatorname{supp} g \subseteq N_{1/R} P'$ and write

$$g = \sum_\theta g_\theta$$

where $\operatorname{supp} \widehat{g_\theta} \subseteq \theta$ for each $R^{-1} \times R^{-1/2}$ slab θ.
Then

$$\|g\|_{L^4(\mathbb{R}^2)} \lesssim \Big\| \Big(\sum_{\theta:\text{slab}} |g_\theta|^2 \Big)^{1/2} \Big\|_{L^4(\mathbb{R}^2)} .$$

This represents a $\sqrt{\cdot}$ gain (on average) over the trivial triangle inequality bound

$$|g| \leq \sum_{\theta:\text{slab}} |g_\theta| .$$

The Fefferman - Cordoba Theorem is the most important piece of the proof. Before continuing, let's sketch how it can be used to bound the operator $T_R f$.

Applying the theorem to $T_R f$ together with the locally constant property

$$\|T_R f\|_{L^4(\mathbb{R}^2)} \lesssim \Big\| \Big(\sum_{\theta:\text{slab}} |T_{R,\theta} f|^2 \Big)^{1/2} \Big\|_{L^4(\mathbb{R}^2)}$$

$$\lesssim \Big\| \sum_{\theta:\text{slab}} \sum_{T \parallel \theta^*} |c_T|^2 \chi_T \Big\|^{1/2}_{L^2(\mathbb{R}^2)} .$$

The last line looks like our Kakeya

maximal function from Lecture 4, except now we have many parallel tubes in each direction T.

For the sake of discussion, let's pretend that for each θ there is only 1 tube $T_\theta \parallel \theta^*$ such that $c_T \neq 0$. Then we can apply the Kakeya maximal estimate:-

$$\| {}^{\circ}T_R f \|_{L^4(\mathbb{R}^2)} \lesssim \Big\| \sum_{\theta : \text{slab}} |c_{T_\theta}|^2 \chi_{T_\theta} \Big\|^{1/2}_{L^2(\mathbb{R}^2)}$$

$$\lesssim (\log R)^{1/2} \Big(\sum_{\theta : \text{slab}} |c_{T_\theta}|^4 |T_\theta| \Big)^{1/4}$$

$$= (\log R)^{1/2} \Big(\sum_{\theta : \text{slab}} \int_{\mathbb{R}^2} |c_{T_\theta}|^4 \chi_{T_\theta} \Big)^{1/4}$$

$$\lesssim (\log R)^{1/2} \Big(\sum_{\theta : \text{slab}} \| {}^{\circ}T_{R,\theta} f * \eta_{\theta^*} \|^4_{L^4(\mathbb{R}^2)} \Big)^{1/4}$$

$$\lesssim (\log R)^{1/2} \Big(\sum_{\theta : \text{slab}} \| {}^{\circ}T_{R,\theta} f \|^4_{L^4(\mathbb{R}^2)} \Big)^{1/4}$$

since $\|\eta_{\theta^*}\|_1 \lesssim 1$. Thus, to conclude the proof it suffices to show

<u>Claim</u>:- For $2 \leq p \leq \infty$,

$$\Big(\sum_{\theta : \text{slab}} \| {}^{\circ}T_{R,\theta} f \|^p_{L^p(\mathbb{R}^2)} \Big)^{1/p} \lesssim \|f\|_{L^p(\mathbb{R}^2)}$$

<u>Proof</u>:- For $p = 2$ this follows from Plancherel's theorem & the disjoint support of the $({}^{\circ}T_{R,\theta} f)^\wedge$.

For $p = \infty$, this follows since each ${}^{\circ}T_{R,\theta}$ has a kernel $K_{R,\theta}$ with $\|K_{R,\theta}\|_1 \lesssim 1$.

Interpolation gives the full range of p. $\square$

N.B. In practice, this argument does not work in general, since we may have many parallel copies of T_0.

A variant of this approach can be used, however, which exploits additional 'duality': see lecture 7.

Proof (of Cordoba – Fefferman):- Write

$$\|g\|_4^4 = \| |\sum_\theta g_\theta|^2 \|_2^2 = \| \sum_{\theta,\theta'} g_\theta \bar{g}_{\theta'} \|_2^2 .$$

We write $\begin{cases} \theta \sim \theta' & \text{if } \operatorname{dist}(\theta,\theta') \lesssim R^{-1/2} \\ \theta \not\sim \theta' & \text{otherwise.} \end{cases}$

The pairs (θ,θ') such that $\theta \sim \theta'$ are called 'adjacent'.

Thus,

$$\|g\|_4^4 \lesssim \| \sum_{\theta\sim\theta'} g_\theta \bar{g}_{\theta'} \|_2^2 + \| \sum_{\theta\not\sim\theta'} g_\theta \bar{g}_{\theta'} \|_2^2$$

Since for any fixed θ

$$\#\{\theta' : \theta \sim \theta'\} = O(1)$$

it follows by a 2-fold application of Cauchy-Schwarz that

$$|\sum_{\theta\sim\theta'} g_\theta \bar{g}_{\theta'}| \lesssim \sum_\theta |g_\theta|^2$$

and so $$\| \sum_{\theta\sim\theta'} g_\theta \bar{g}_{\theta'} \|_2^2 \lesssim \| \sum_\theta |g_\theta|^2 \|_2^2 = \| (\sum_\theta |g_\theta|^2)^{1/2} \|_4^4 .$$

It therefore remains to bound the contributions from non-adjacent pairs.

By Plancherel,

$$\Big\|\sum_{\theta\neq\theta'} g_\theta \overline{g_{\theta'}}\Big\|_2^2 = \Big\|\sum_{\theta\neq\theta'} \hat{g}_\theta * \hat{\overline{g}}_{\theta'}\Big\|_2^2$$

where $\operatorname{supp} \hat{g}_\theta * \hat{\overline{g}}_{\theta'} \subseteq \theta - \theta'$.

Key geometric observation :- The sets $\theta - \theta'$ are finitely-overlapping. In particular,

$$\#\{(\theta,\theta') : \theta \neq \theta' \text{ and } \xi \in \theta - \theta'\} = O(1) \qquad (5)$$

for all $\xi \in \hat{\mathbb{R}}^2$.

Assuming (5) and applying Cauchy-Schwarz

$$\Big|\sum_{\theta\neq\theta'} \hat{g}_\theta * \hat{\overline{g}}_{\theta'}\Big|^2 \lesssim \sum_{\theta\neq\theta'} |\hat{g}_\theta * \hat{\overline{g}}_{\theta'}|^2$$

so that, by another application of Plancherel,

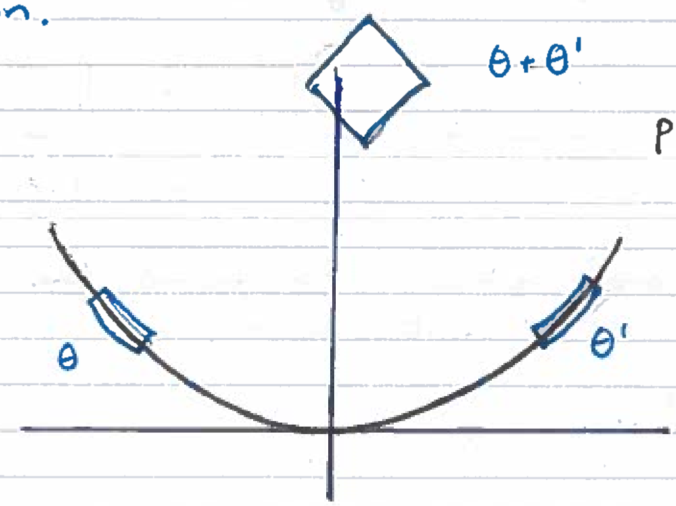
$$\Big\|\sum_{\theta\neq\theta'} g_\theta \overline{g_{\theta'}}\Big\|_2^2 \lesssim \sum_{\theta\neq\theta'} \|\hat{g}_\theta * \hat{\overline{g}}_{\theta'}\|_2^2$$

$$= \sum_{\theta\neq\theta'} \|g_\theta \overline{g_{\theta'}}\|_2^2$$

$$\leq \Big\|\Big(\sum_\theta |g_\theta|^2\Big)^{1/2}\Big\|_4^4$$

as required.

Thus, it remains to show the key geometric observation.



The curvature of P' implies the sum sets $\theta + \theta'$ are essentially disjoint boxes - Similar considerations hold for the difference sets $\theta - \theta'$.

Proof (of key geometric observation) Fixing (θ_1, θ_1') with $\theta_1 \neq \theta_1'$ it suffices to show

$$\#\{(\theta_2, \theta_2') : \theta_1 - \theta_1' \cap \theta_2 - \theta_2' \neq \emptyset\} = O(1).$$

Suppose $y_i \in \theta_i$, $y_i' \in \theta_i'$ are such that

$$y_1 - y_1' - y_2 + y_2' = 0.$$

Since $\theta_i, \theta_i' \subseteq N_{1/R} P^1$, it follows that ~~there~~ exist $t_i, t_i' \in [-2, 2]$ such that

$$\begin{cases} |t_1 - t_1' - t_2 + t_2'| \lesssim R^{-1} & (a) \\ |t_1^2 - (t_1')^2 - t_2^2 + (t_2')^2| \lesssim R^{-1} & (b) \end{cases}$$

with $|t_1 - t_1'| \gtrsim R^{-1/2}$ by the non-adjacent hypothesis.

Now,

$$t_1^2 - (t_1')^2 - t_2^2 + (t_2')^2 = (t_1 - t_1')(t_1 + t_1') - (t_2 - t_2')(t_2 + t_2')$$

$$= (t_1 - t_1')(t_1 + t_1' - t_2 - t_2') + (t_2 + t_2')(t_1 - t_1' - t_2 + t_2').$$

Consequently, (a) and (b) imply

$$|t_1 - t_1'|\,|t_1 + t_1' - t_2 - t_2'| \lesssim R^{-1}$$

and so by the seperation condition,

$$|t_1 + t_1' - t_2 - t_2'| \lesssim R^{-1/2} \qquad (c).$$

Combining a) and c),

$$|t_1 - t_2| \lesssim R^{-1/2} \quad \text{and} \quad |t_1' - t_2'| \lesssim R^{-1/2}.$$

This means that

$$\mathrm{dist}(\theta_1, \theta_2) \lesssim R^{-1/2}, \quad \mathrm{dist}(\theta_1', \theta_2') \lesssim R^{-1/2}$$

so there can be only $O(1)$ choices of (θ_2, θ_2') for any fixed (θ_1, θ_1'). $\square$