Lecture 5: Bochner-Riesz means. I

- L^p convergence fails for the spherical summation method whenever $n \geq 2$ and $p \neq 2$ (Fefferman's theorem).
- However, at least for $n=2$, this failure is 'marginal' – the blow up in R is logarithmic. (Kakeya maximal theorem).

Question:– Can we mollify our operator to obtain some 'surrogate' for the spherical Fourier summation method?

Example:– If $\chi \in C_c^\infty(\mathbb{R}^n)$ with $\chi(0)=1$ (think of this as roughly a 'smooth' approximation to $\chi_{B(0,1)}$), then trivially

$$\tilde{S}_R f(x) := \int_{\hat{\mathbb{R}}^n} e^{2\pi i \langle x, \xi \rangle} \chi(R^{-1}\xi)\, \hat{f}(\xi)\, d\xi.$$

satisfies $\tilde{S}_R f \to f$ in L^p as $R \to \infty$, whenever $f \in L^p(\mathbb{R}^n)$, for all $1 \leq p \leq \infty$.

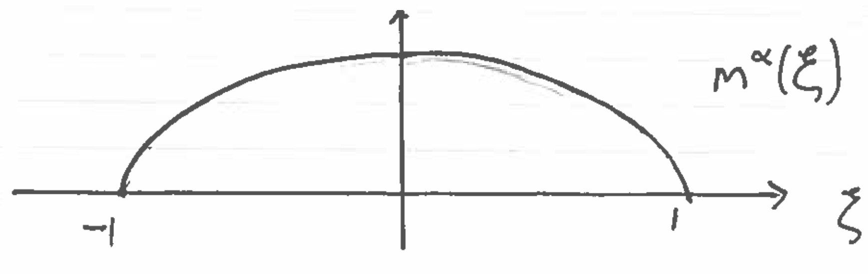
Indeed, $\tilde{S}_R f = [R^n \check{\chi}(R\,\cdot)] * f =: K_R * f$
and
$$\|K_R\|_1 \lesssim 1.$$

Question:– What can we say in the 'intermediate' case – i.e. when the multiplier has a limit degree of smoothness?

Defn:– For $\alpha \geq 0$ define the (spherical) Bochner–Riesz multiplier of order α to be the function

$$m^\alpha(\xi) := (1-|\xi|^2)_+^\alpha \quad \text{for } \xi \in \hat{\mathbb{R}}^n.$$

Here $(t)_+ := \begin{cases} t & \text{if } t \geq 0 \\ 0 & \text{otherwise}. \end{cases}$



- m^α is smooth on $\mathbb{R}^n \setminus S^{n-1}$
- If $\alpha = 0$, then $m^\alpha = \chi_{B(0,1)}$ is the ball multiplier.
- If $\alpha > 0$, then m^α is continuous with a limited degree of smoothness on S^{n-1}. In particular, m^α satisfies a Hölder condition in the radial direction on S^{n-1} with exponent α.

Def^{n}:- Given $\alpha \geq 0$ define the (spherical) Bochner-Riesz means of order α by

$$B_R^\alpha f(x) := \int_{\hat{\mathbb{R}}^n} e^{2\pi i \langle x, \xi \rangle} m^\alpha(\xi/R) \hat{f}(\xi)\, d\xi \qquad (1)$$

for $R \geq 1$.

Problem:- Given $\alpha \geq 0$ determine the range of p for which

$$B_R^\alpha f \to f \quad \text{in } L^p \quad \text{as } R \to \infty.$$

Generalising the argument used to study the ball multiplier (ie $\alpha = 0$ case) in earlier lectures, this is equivalent to the following:-

Problem:- Given $\alpha \geq 0$ determine the range of p for which

$$\| B^\alpha \|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} < \infty \qquad (2)$$

where $B^\alpha := B_1^\alpha$.

We will actually study a closely related parabolic variant of this operator, which shares the essential features of B^α (at least for our purposes), but is somewhat cleaner to analyse.

Def^{n}:- Let $\tilde{\chi} \in C_c^\infty(\mathbb{R})$, $\varrho \in C_c^\infty(\mathbb{R}^{n-1})$ satisfy

- $0 \leq \varrho, \tilde{\chi} \leq 1$
- $\operatorname{supp} \tilde{\chi} \subseteq [-1,1]$, $\operatorname{supp} \varrho \subseteq [-1,1]^{n-1}$
- $\tilde{\chi}(t) = \varrho(u) = 1$ for $|t|, |u| \leq 1/2$.

and define $m^\alpha(\xi) := [\xi_n - |\xi'|^2/2]_+^\alpha \, \chi_0(\xi)$

where $\chi_0(\xi) := \tilde{\chi}(\xi_n - |\xi'|^2/2)\, \varrho(\xi')$.

We call the function m^α the (parabolic) Bochner-Riesz multiplier of order α.

We define the Bochner-Riesz means as in (1) with this new multiplier.

Necessary Conditions:-

We can deduce a simple necessary condition for (2) to hold as follows.

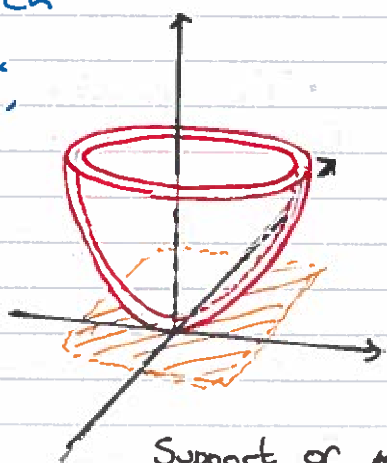
If we take $f \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\hat{f}(\xi) = 1 \quad \text{for } \xi \in \operatorname{supp} m^\alpha,$$

then $B^\alpha f = K^\alpha$ where K^α is the Bochner-Riesz kernel

$$K^\alpha(x) := (m^\alpha)^\vee(x)$$

$$= \int_{\hat{\mathbb{R}}^n} e^{2\pi i \langle x, \xi \rangle} m^\alpha(\xi)\, d\xi$$



Support of m^α

Thus, (2) $\Longrightarrow$ $\|K^\alpha\|_{L^p(\mathbb{R}^n)} < \infty$ and we can therefore obtain a necessary condition on p by determining the p for which $\|K^\alpha\|_p < \infty$.

Remark:- There is a major open problem which aims to characterise the L^p-boundedness of multipliers in terms of the finiteness of the L^p norm of the associated kernel, under the following assumptions:-

i) the multiplier is radial and compactly supported

ii) $1 < p < \frac{2n}{n-1}$ (n = ambient dimension).

This is the so-called 'Radial multiplier conjecture' - c.f. Heo, Nazarov, Seeger.

This conjecture implies many of the results discussed in the forthcoming lectures.

- For the parabolic Bochner–Riesz multiplier,

$$K^\alpha(x) = \int_{\hat{\mathbb{R}}^n} e^{2\pi i(\langle x', \xi' \rangle + x_n \xi_n)} \left(\xi_n - \tfrac{|\xi'|^2}{2}\right)_+^\alpha \tilde{\chi}\left(\xi_n - \tfrac{|\xi'|^2}{2}\right) e(\xi') \, d\xi$$

$$= \mathrm{I}(x) \cdot \mathrm{II}(x_n) \qquad \text{where}$$

$$\mathrm{I}(x) := \int_{\hat{\mathbb{R}}^{n-1}} e^{2\pi i(\langle x', \xi' \rangle + x_n |\xi'|^2/2)} \, e(\xi') \, d\xi'$$

$$\mathrm{II}(x_n) := \int_{\hat{\mathbb{R}}} e^{2\pi i x_n \xi_n} (\xi_n)^\alpha \, \tilde{\chi}(\xi_n) \, d\xi_n .$$

- The function I corresponds to the (inverse) Fourier transform of the measure

$$\int f \, d\sigma = \int_{\mathbb{R}^{n-1}} f(u, |u|^2/2) \, e(u) \, du$$

on the paraboloid $P^{n-1} := \{(u, |u|^2/2) : u \in \mathbb{R}^{n-1}\}$.

By well known stationary phase computations,

$$\mathrm{I}(x) = \frac{a(x)\, e^{-2\pi i |x'|^2/2x_n}}{(1+|x|)^{\frac{n-1}{2}}} + \varepsilon(x) \qquad (3)$$

where a is a symbol of order 0 in the sense that

$$|\partial_x^\alpha a(x)| \lesssim_\alpha (1+|x|)^{-|\alpha|} \qquad \forall \alpha \in \mathbb{N}_0^n$$

and $\varepsilon(x)$ is a rapidly decaying error term:

$$|\partial_x^\alpha \varepsilon(x)| \lesssim_{\alpha, N} (1+|x|)^{-N} \qquad \forall \alpha \in \mathbb{N}_0^n, \; N \in \mathbb{N}.$$

Moreover, a is bounded below at ∞ on an open cone in $\mathbb{R}^n$, centred around e_n.

- On the other hand,

$$\mathrm{II}(t) = \int_{\mathbb{R}} e^{2\pi i t r} (r)_+^{\alpha} \tilde{\chi}(r)\, dr$$

Here $(\cdot)_+^{\alpha}$ is a homogeneous distribution of order α which is C^∞ away from 0.

By general theory (see, e.g., Grafakos Proposition 2.4.8),

$u := \mathcal{F}^{-1}[(\cdot)_+^{\alpha}]$

is a homogeneous distribution of order $-1-\alpha$ which is C^∞ away from 0.

Hence, if $\phi := (\tilde{\chi})^{\vee}$, then

$$\mathrm{II}(t) = u * \phi(t) = \langle u, \phi(t-\cdot)\rangle$$

Let $\eta \in C^\infty$ satisfy . $\eta(s) := \begin{cases} 1 & \text{if } |s| \leq 1/4 \\ 0 & \text{if } |s| \geq 1/2 \end{cases}$

and break up $\phi(t-\cdot)$ thus:-

$$\phi(t-s) = \phi(t-s)\,\eta(t^{-1}s) + \phi(t-s)\,(1-\eta(t^{-1}s)).$$

Since u is a distribution, there exists some $M \in \mathbb{N}$ such that

$$|\langle u, \phi(t-\cdot)\,\eta(t^{-1}\cdot)\rangle| \lesssim \sum_{j=0}^{M} \|\partial_s^j\, \phi(t-\cdot)\,\eta(t^{-1}\cdot)\|_\infty$$

$$\lesssim_N (1+|t|)^{-N}$$

by the support condition on η and the rapid decay of ϕ.

Moreover, by homogenuity,

$$\langle u, \phi(t-\cdot)(1-\eta)(t^{-1}\cdot)\rangle = t^{-\alpha}\langle u, \phi(t(1-\cdot))\,(1-\eta)\rangle$$

$$= t^{-\alpha} \int_{\mathbb{R}} u(s)\,\phi(t(1-s))\,(1-\eta(s))\, ds$$

since $1-\eta(s)$ is supported away from 0.

Note that $s \mapsto \phi(t(1-s))$ is concentrated around the interval $|s-1| \lesssim 1/|t|$ so that, combining these observations,

$$|\mathrm{II}(t)| \sim (1+|t|)^{-\alpha-1} \quad \text{for } |t| \text{ large}.$$

Putting I and II together, we have

$$K^\alpha(x) = \frac{a(x)\, e^{-2\pi i |x'|^2/2x_n}}{(1+|x|)^{\frac{n+1}{2}+\alpha}} + \mathcal{E}(x)$$

where a and ε are slight modifications of the functions appearing in (3) (which, in particular, satisfy the same properties).

Thus, $K^\alpha \in L^p$ if and only if

$$\int_{\mathbb{R}^n} (1+|x|)^{-(\frac{n+1}{2}+\alpha)p}\, dx < \infty.$$

$$\iff \left(\frac{n+1}{2}+\alpha\right)p > n$$

$$\iff \alpha > n\left(\frac{1}{p}-\frac{1}{2}\right) - \frac{1}{2}.$$

Since B^α is self-adjoint, this condition can be strengthened to

$$\alpha > n\left|\frac{1}{p}-\frac{1}{2}\right| - 1/2$$

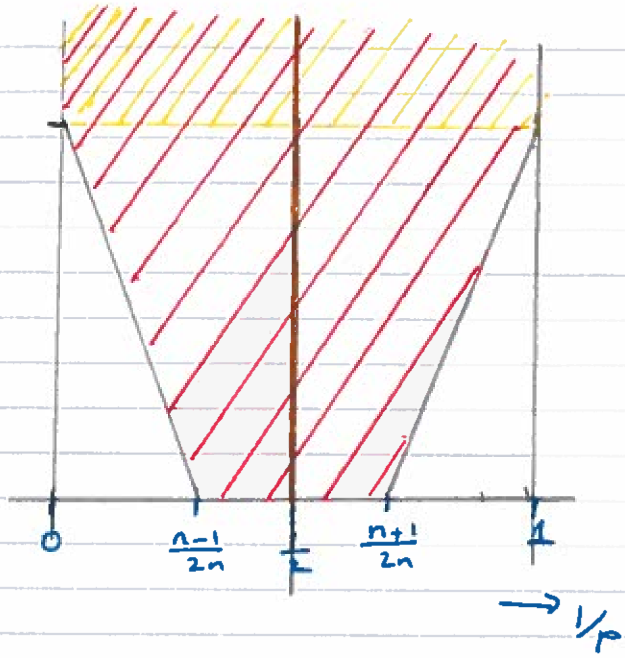
and from the Fefferman result, we obtain

$$\alpha > \max\left\{n\left|\frac{1}{p}-\frac{1}{2}\right| - 1/2,\ 0\right\} =: \alpha(p)$$

<u>Conjecture (Bochner–Riesz)</u>: For $1 \leq p \leq \infty$, if $\alpha > \alpha(p)$, then

$$\|B^\alpha\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} < \infty.$$

<u>An easy case</u>:– • If $\alpha > \frac{n-1}{2}$, then the analysis above shows $K^\alpha \in L^1$ and so



$\|B^\alpha\|_{p\to p} < \infty$ for all $1 \le p \le \infty$. In particular, this establishes the $p=1$ and $p=\infty$ cases of the conjecture.

Another easy case:- If $p=2$, then

$$\|B^\alpha\|_{p\to p} < \infty \quad \text{for}$$

all $\alpha \geq 0$ since $m^\alpha \in L^\infty$. This establishes the $p=2$ case of the conjecture.

A very hard case:- If we could show the conjecture holds for either $p = \frac{2n}{n+1}$ or $\frac{2n}{n-1}$, then the whole conjecture follows via duality and interpolation with the easy cases above.

Remarks:- • The conjecture is known for $n=2$ (due to Carleson – Sjölin).

• We will give 2 proofs of the $n=2$ case, neither of which follow the original Carleson–Sjölin argument.

• For $n \geq 3$ this problem is open, but there are a number of significant partial results (with restricted p ranges).