Lecture 4: Kakeya Maximal Estimates

Recall from the previous lecture:-

Key Observation: For any $M \geq 1$ there exists some $R \geq 1$ and a family $\mathbb{T}$ of disjoint $R^{1/2} \times \dots \times R^{1/2} \times R$ tubes in $\mathbb{R}^n$ with $R^{-1/2}$-separated directions such that

$$\Big|\bigcup_{T \in \mathbb{T}} T\Big| \leq M^{-1} \sum_{T \in \mathbb{T}} |T| \qquad (1)$$

Natural question:- "How bad can these examples get?"

More precisely, for a fixed $R \geq 1$ how large can we make $M = M(R)$ in (1)?

Answer:- If $n = 2$, then "not too big."

Theorem 1: Let $\mathbb{T}$ be a collection of $R^{1/2} \times R$ tubes in $\mathbb{R}^2$ with $R^{-1/2}$-separated directions. Then

$$\Big|\bigcup_{T \in \mathbb{T}} T\Big| \gtrsim (\log R)^{-1} \sum_{T \in \mathbb{T}} |T|.$$

Although the key observation tells us that the $T \in \mathbb{T}$ can have "unbounded pile up", Theorem 1 tells us that the pile up is nevertheless strictly controlled - it is "logarithmic".

Theorem 1 follows from a stronger bound on the multiplicity function.

Theorem 2 (Cordoba):- For $\mathbb{T}$ as above,

$$\Big\|\sum_{T \in \mathbb{T}} \chi_T\Big\|_{L^2(\mathbb{R}^2)} \lesssim (\log R)^{1/2} \Big(\sum_{T \in \mathbb{T}} |T|\Big)^{1/2}.$$

To see how Theorem 2 $\Rightarrow$ Theorem 1, we use a simple Cauchy-Schwarz argument similar to that appearing in the proof of Fefferman's theorem.

Assuming Theorem 2,

$$\sum_{T \in \mathbb{T}} |T| = \int_{\bigcup_{T \in \mathbb{T}} T} \sum_{T \in \mathbb{T}} \chi_T \leq \Big|\bigcup_{T \in \mathbb{T}} T\Big|^{1/2} \Big\|\sum_{T \in \mathbb{T}} \chi_T\Big\|_{L^2(\mathbb{R}^2)}$$

by Cauchy-Schwarz. Applying Theorem 2,

$$\sum_{T\in\mathbb{T}} |T| \lesssim (\log R)^{1/2} \Big|\bigcup_{T\in\mathbb{T}} T\Big|^{1/2} \Big(\sum_{T\in\mathbb{T}} |T|\Big)^{1/2}$$

and rearranging this bound gives the desired result.

It remains to prove Theorem 2.

Proof (of Theorem 2):- Write

$$\Big\|\sum_{T\in\mathbb{T}} \chi_T\Big\|^2_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \Big(\sum_{T\in\mathbb{T}} \chi_T\Big)^2$$

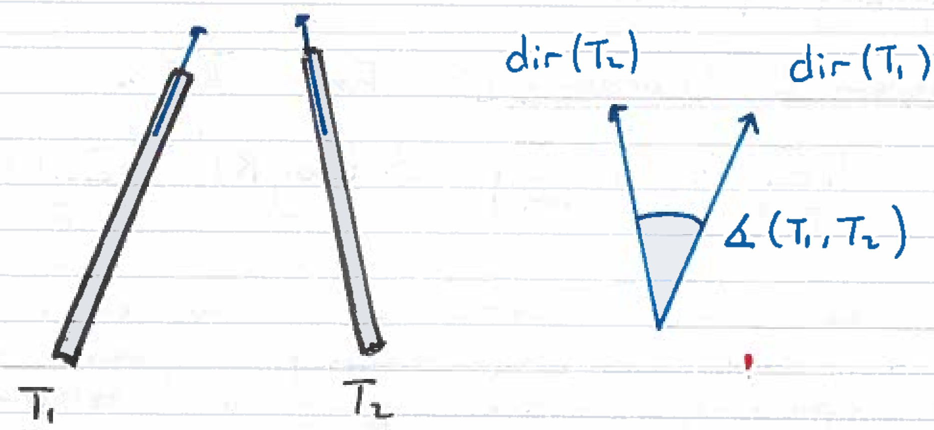
$$= \int_{\mathbb{R}^2} \sum_{T_1,T_2\in\mathbb{T}} \chi_{T_1}\cdot\chi_{T_2}$$

$$= \sum_{T_1\in\mathbb{T}} \sum_{T_2\in\mathbb{T}} |T_1\cap T_2|. \qquad (2)$$

The key geometric observation is given by:-

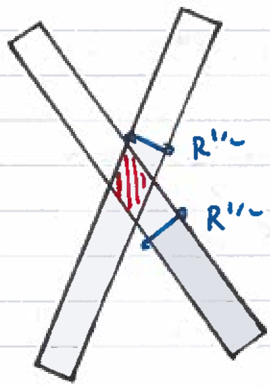
Claim:- $|T_1\cap T_2| \lesssim \dfrac{R}{R^{-1/2} + \measuredangle(T_1,T_2)}$ for $T_1,T_2\in\mathbb{T}$.

Here $\measuredangle(T_1,T_2)$ denotes the angle between the tubes T_1, T_2:-



The proof of the claim is a simple trigonometry exercise. We can get a feel for the numerology, however, by considering the extreme cases:-

- If $\measuredangle(T_1, T_2) \sim 1$, then $T_1 \cap T_2$ is contained in roughly an $R^{1/2} \times R^{1/2}$ box.



"transverse" case - large angle

Hence, here $|T_1 \cap T_2| \lesssim R$.

- On the other extreme, if $\measuredangle(T_1, T_2) \lesssim R^{-1/2}$ then T_1 and T_2 can essentially overlap completely. Hence the best we can say in this case is

$$|T_1 \cap T_2| \leq |T_1| \lesssim R^{3/2}.$$

"narrow angle" case

We see both extremes agree with the claim and the claim "interpolates" between them.

Returning to the proof of Cordoba's theorem, recall we want to estimate

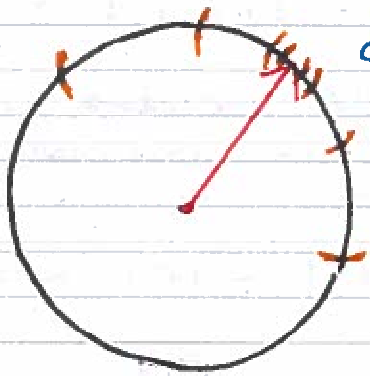
$$\sum_{T_1 \in \mathbb{T}} \sum_{T_2 \in \mathbb{T}} |T_1 \cap T_2| \qquad (3)$$

By the claim, it makes sense to partition the $T_2 \in \mathbb{T}$ according to the size of $\measuredangle(T_1, T_2)$. In particular, we write (3) as

$$\sum_{T_1 \in \mathbb{T}} \Big(\sum_{k=0}^{\overline{K}} \sum_{\substack{T_2 \in \mathbb{T} \\ \measuredangle(T_1,T_2) \sim 2^k R^{-1/2}}} |T_1 \cap T_2| + |T_1| \Big) \tag{4}$$

where the innermost sum is over all $T_2 \in \mathbb{T}$ such that
$$2^k R^{-\frac{1}{2}} \leq \measuredangle(T_1, T_2) < 2^{k+1} R^{-1/2}.$$

and $\overline{K}$ satisfies $2^{\overline{K}} R^{-1/2} \sim 1$ so that $\overline{K} \sim \log R$.



Dyadically decomposing S^1 into arcs, around $\mathrm{dir}(T_1)$.

For each fixed k we can bound the corresponding sum in (4) using the claim:–

$$\sum_{\substack{T_2 \in \mathbb{T} \\ \measuredangle(T_1,T_2) \sim 2^k R^{-1/2}}} |T_1 \cap T_2| \lesssim \frac{R^{3/2}}{2^k} \# \{ T_2 \in \mathbb{T} : \measuredangle(T_1, T_2) \sim 2^k R^{-1/2} \}$$
$$\lesssim R^{3/2}.$$

The second inequality follows due to the angular separation.

Summing everything together, on finds (4) (and hence (3)) is bounded by

$$\overline{K} \cdot \sum_{T_1 \in \mathbb{T}} R^{3/2} \sim (\log R) \cdot \sum_{T_1 \in \mathbb{T}} |T_1|$$

Plugging this into (2) yields

$$\Big\|\sum_{T\in\mathbb{T}} \chi_T\Big\|^2_{L^2(\mathbb{R}^2)} \lesssim \log R \cdot \sum_{T\in\mathbb{T}} |T|$$

as desired. $\square$.

The same argument can be carried out in higher dimensions, but here one obtains a power loss in R in the L^2 bound for $n \geq 3$. This power loss is sharp and prevents one from obtaining an 'almost disjoint' result of the kind in Theorem 1 in higher dimensions.

The 'correct' multiplicity bound for general n appears to be

Conjecture (Kakeya maximal conjecture) If $\mathbb{T}$ is a collection of $R^{1/2}\times\cdots\times R^{1/2}\times R$ - tubes in $\mathbb{R}^n$ with $R^{-1/2}$ - separated directions, $n \geq 2$, then

$$\Big\|\sum_{T\in\mathbb{T}} \chi_T\Big\|_{L^{n/n-1}(\mathbb{R}^n)} \lesssim (\log R) \Big(\sum_{T\in\mathbb{T}} |T|\Big)^{\frac{n-1}{n}}$$

- This corresponds with Cordoba's theorem for $n=2$ but constitutes a major open problem for $n \geq 3$.

We finish this discussion by noting Theorem 2 easily implies a version of itself with coefficients.

Corollary:- Let $\mathbb{T}$ be as in the statement of Theorem 2 and $(a_T)_{T\in\mathbb{T}}$ be a complex sequence. Then

$$\Big\|\sum_{T\in\mathbb{T}} a_T \chi_T\Big\|_{L^2(\mathbb{R}^2)} \lesssim (\log R)\cdot\Big(\sum_{T\in\mathbb{T}} |a_T|^2 |T|\Big)^{1/2}$$

Proof:- By homogeneity we may assume wlog that

$$\sum_{T\in\mathbb{T}} |a_T|^2 = 1.$$

Consequently, we can write $\mathbb{T}$ as a finite disjoint union

$$\mathbb{T} = \bigcup_{k=0}^{K} \mathbb{T}_k \cup \mathbb{T}_{>K}$$

where $\mathbb{T}_k := \{ T \in \mathbb{T} : 2^{-k-1} < |a_T| \leq 2^{-k} \}$

and

$$\mathbb{T}_{>K} := \bigcup_{k=K+1}^{\infty} \mathbb{T}_k .$$

for K a large integer (we will choose $K \sim \log R$).

Then $\| \sum_{T \in \mathbb{T}} a_T \chi_T \|_{L^2(\mathbb{R}^2)}$

$$\leq \sum_{k=0}^{K} \| \sum_{T \in \mathbb{T}_k} |a_T| \chi_T \|_{L^2(\mathbb{R}^2)} + \| \sum_{T \in \mathbb{T}_{>K}} |a_T| \chi_T \|_{L^2(\mathbb{R}^2)}$$

$$\lesssim \sum_{k=0}^{K} 2^{-k} \| \sum_{T \in \mathbb{T}_k} \chi_T \|_{L^2(\mathbb{R}^2)} + 2^{-K} \| \sum_{T \in \mathbb{T}_{>K}} \chi_T \|_{L^2(\mathbb{R}^2)} .$$

Applying Theorem 2, this is bounded by

$$(\log R)^{1/2} \Big[\sum_{k=0}^{K} 2^{-k} \Big(\sum_{T \in \mathbb{T}_k} |T| \Big)^{1/2} + 2^{-K} \Big(\sum_{T \in \mathbb{T}_{>K}} |T| \Big)^{1/2} \Big]$$

To estimate the left-hand sum, note that

$$\sum_{k=0}^{K} 2^{-k} \Big(\sum_{T \in \mathbb{T}_k} |T| \Big)^{1/2} = \sum_{k=0}^{K} \Big(\sum_{T \in \mathbb{T}_k} 2^{-2k} |T| \Big)^{1/2}$$

$$\sim \sum_{k=0}^{K} \Big(\sum_{T \in \mathbb{T}_k} |a_T|^2 |T| \Big)^{1/2}$$

$$\lesssim K^{1/2} \Big(\sum_{k=0}^{\infty} \sum_{T \in \mathbb{T}_k} |a_T|^2 |T| \Big)^{1/2}$$

$$= K^{1/2} \Big(\sum_{T \in \mathbb{T}} |a_T|^2 |T| \Big)^{1/2} .$$

On the other hand,

$$2^{-K} \Big(\sum_{T \in \mathbb{T}_{>K}} |T| \Big)^{1/2} = 2^{-K} \Big(\sum_{T \in \mathbb{T}_{>K}} |T| \Big)^{1/2} \Big(\sum_{T \in \mathbb{T}} |a_T|^2 \Big)^{1/2}$$

$$= 2^{-\overline{k}} [\# \mathbb{T}_{>\overline{k}}]^{1/2} \Big(\sum_{T \in \mathbb{T}} |a_T|^2 |T| \Big)^{1/2}$$

by the ℓ^2 normalisation.

Since $\# \mathbb{T}_{>\overline{k}} \leq \# \mathbb{T} \lesssim R^{1/2}$ (since the tubes have $R^{-1/2}$-separated dirs),

it follows that this term is

$$\lesssim \Big(\sum_{T \in \mathbb{T}} |a_T|^2 |T| \Big)^{1/2}$$

provided $\overline{k} \sim \log R$.

Combining these two bounds concludes the proof. $\square$.

Remark. Both the proof of Theorem 2 and the corollary involve examples of 'dyadic pigeonholing'. In particular, a parameter taking values in logarithmically many relevant scales can be assumed to take values at only a single scale, at the cost of a logarithmic factor in the inequality.

For example, in Theorem 2 the angle $\angle(T_1, T_2)$ lies at scale $2^k R^{-1/2}$ for only $\sim \log R$ values of k (other scales are "empty").

In the corollary the $|a_T|$ can take values in infinitely many scales 2^{-k} $k \geq 0$. However, many of these scales do not contribute to the problem significantly – in particular, if k is very large then the coefficients a_T satisfying $|a_T| \sim 2^{-k}$ are too tiny to play a major role in the analysis. In particular, we can reduce to studying the problem for $0 \leq k \lesssim \log R$ only.

In both cases, dyadic pigeonholing then allows one to analyse the problem with the parameter fixed at a certain scale.

These arguments appear frequently in the theory and can be quite powerful, in the sense they afford significant simplifications.