Lecture 25: Decoupling for the moment curve II

Last time we established the following key lemma.

Lemma (Key step, general k) For $1 \le \ell \le k-1$ and

$$0 \le a_1 \le \frac{k-\ell+1}{\ell} a_2$$

we have

$$\mathcal{B}_{k,\ell}(\delta; a_1, a_2) \lesssim \qquad (1)$$
$$\mathcal{D}_{\ell,p_\ell}(\delta^{\frac{k-\ell+1}{\ell} a_2})^{p_\ell/p_k} \cdot \mathcal{B}_{k,\ell}(\delta; \tfrac{k-\ell+1}{\ell} a_2, a_2).$$

Throughout this lecture we will work with $k \geq 2$ and assume a

Dimensional induction hypothesis:- For all $1 \le \ell \le k-1$, $\varepsilon > 0$,
$$\mathcal{D}_{\ell,p_\ell}(\delta) \lesssim_\varepsilon \delta^{-\varepsilon}$$

Thus, (1) becomes

$$\mathcal{B}_{k,\ell}(\delta; a_1, a_2) \lesssim_\varepsilon \delta^{-a_2\varepsilon}\, \mathcal{B}_{k,\ell}(\delta; \tfrac{k-\ell+1}{\ell} a_2, a_2). \qquad (2)$$

As in the $k=2$ case from Lecture 23, the next step is to apply Hölder's inequality in order to obtain an estimate amenable to iteration.

In general, the iteration scheme is rather complicated, so we begin by discussing some low dimensional cases as a warm up.

The case $k=3$:-

For $k=3$ the key step provides two inequalities:

- $$\mathcal{B}_{3,1}(\delta; a_1, a_2) \lesssim_\varepsilon \delta^{-a_2\varepsilon}\, \mathcal{B}_{3,1}(\delta; 3a_2, a_2) \qquad (1)$$
  for $0 \le a_1 \le 3a_2 \le 1$

- $$B_{3,2}(\delta; a_1, a_2) \lesssim_\varepsilon \delta^{-\varepsilon a_2} B_{3,2}(\delta; a_2, a_2) \qquad (2)$$

$$\text{for} \quad 0 \le a_1 \le a_2 \le 1$$

These inequalities can be used in conjunction after relating the $B_{3,2}(\delta; a_1, a_2)$ using Hölder's inequality.

In particular, recall $p_1 = 2$, $p_2 = 6$, $p_3 = 12$ and so the 'bilinear' expressions of interest are

$$B_{3,1}(\delta; a_1, a_2) := \int_{\mathbb{R}^3} |f_{I_1}|^2 \cdot |f_{I_2}|^{10}$$

$$B_{3,2}(\delta; a_1, a_2) := \int_{\mathbb{R}^3} |f_{I_1}|^6 |f_{I_2}|^6 \qquad (3)$$

and the linear decoupling for $k = 3$ has critical exponent $p_3 = 12$:

$$D_{3,12}(\delta) := \int_{\mathbb{R}^3} |f_{I_2}|^{12}$$

- Note that

$$\int_{\mathbb{R}^3} |f_{I_1}|^2 |f_{I_2}|^{10} = \int_{\mathbb{R}^3} \left(|f_{I_1}|^6 |f_{I_2}|^6\right)^{\frac{1}{3}} \left(|f_{I_2}|^{12}\right)^{\frac{2}{3}}$$

and

$$\int_{\mathbb{R}^3} |f_{I_1}|^6 |f_{I_2}|^6 = \int_{\mathbb{R}^3} \left(|f_{I_1}|^2 |f_{I_2}|^{10}\right)^{\frac{1}{2}} \left(|f_{I_1}|^{10} |f_{I_2}|^2\right)^{\frac{1}{2}}$$

Thus, by Hölder, (and linear rescaling)

$$B_{3,1}(\delta; \vec{a}) \lesssim B_{3,2}(\delta; \vec{a})^{1/3} D_{3,12}(\delta^{1-a_2})^{2/3} \qquad (4)$$

$$B_{3,2}(\delta; \vec{a}) \lesssim B_{3,1}(\delta; a_1, a_2)^{1/2} B_{3,1}(\delta; a_2, a_1)^{1/2} \qquad (5)$$

Finally, note that the exponents in (3) are symmetric, so that

$$B_{3,2}(\delta; a_1, a_2) = B_{3,2}(\delta; a_2, a_1). \quad (6)$$

We have 5 inequalities (1), (2), (4), (5) and (6) we can use to relate the $B_{n,k}(\delta; \vec{a})$ in this case.

For convenience, let's compile all the estimates:-

(a) $B_{3,1}(\delta; a_1, a_2) \lesssim_\varepsilon \delta^{-a_2\varepsilon} B_{3,1}(\delta; 3a_2, a_1)$

for $0 \le a_1 \le 3a_2 \le 1$

(b) $B_{3,2}(\delta; a_1, a_2) \lesssim_\varepsilon \delta^{-a_2\varepsilon} B_{3,2}(\delta; a_2, a_2)$

for $0 \le a_1 \le a_2 \le 1$

(c) $B_{3,1}(\delta; \vec{a}) \lesssim B_{3,2}(\delta; \vec{a})^{1/3} D_{3,12}(\delta^{1-a_2})^{2/3}$

(d) $B_{3,2}(\delta; \vec{a}) \le B_{3,1}(\delta, a_1, a_2)^{1/2} B_{3,1}(\delta; a_2, a_1)^{1/2}$

(e) $B_{3,2}(\delta; a_1, a_2) = B_{3,2}(\delta; a_2, a_1)$

These inequalities can be combined to produce an estimate amenable to iteration:-

Let $0 \le b \le 3b \le 9b \le 1$. Then

$$\begin{aligned}
B_{3,1}(\delta, 3b, b) &\underset{(c)}{\lesssim} B_{3,2}(\delta; 3b, b)^{1/3} D_{3,12}(\delta^{1-b})^{2/3} \\
&\underset{(e)}{=} B_{3,2}(\delta; b, 3b)^{1/3} D_{3,12}(\delta^{1-b})^{2/3} \\
&\underset{(b)}{\lesssim_\varepsilon} \delta^{-b\varepsilon/2} B_{3,2}(\delta; 3b, 3b)^{1/3} D_{3,12}(\delta^{1-b})^{2/3} \\
&\underset{(d)}{\le} \delta^{-b\varepsilon/2} B_{3,1}(\delta; 3b, 3b)^{1/3} D_{3,12}(\delta^{1-b})^{2/3} \\
&\underset{(a)}{\lesssim_\varepsilon} \delta^{-b\varepsilon} B_{3,1}(\delta; 9b, 3b)^{1/3} D_{3,12}(\delta^{1-b})^{2/3}
\end{aligned}$$

So in the end we arrive at the iterative step:-

Lemma ($k=3$, iterative step) If $0 \le 9b \le 1$,

$$B_{3,1}(\delta; 3b, b) \lesssim_\varepsilon \delta^{-b\varepsilon} B_{3,1}(\delta; 9b, 3b)^{1/3} D_{3,12}(\delta^{1-b})^{2/3}$$

We can now procede as in the $k=2$ case. Supposing $3^{N+1} b \le 1$ and applying this N times

$$B_{3,1}(\delta; 3b, b) \lesssim_\varepsilon \delta^{-b\varepsilon} B_{3,1}(\delta; 3^{N+1} b, 3^N b)^{1/3^N} \cdot \prod_{j=1}^{N} D_{3,12}(\delta^{1-3^{j-1}b})^{2/3^j}$$

We will fix $b = 3^{-(N+1)}$ and choose N to be a large parameter, depending only on ε.

As before, we have a trivial decoupling inequality

$$B_{3,1}(\delta; \vec{a}) \lesssim \delta^{-1/2}$$

coming from Cauchy-Schwarz. If N is sufficiently large, then

$$\frac{1}{3^N} \le \frac{Nb}{200} \cdot \varepsilon$$

and so

$$B_{3,1}(\delta; 3b, b) \lesssim_\varepsilon \delta^{-\frac{Nb}{100}\varepsilon} \prod_{j=1}^{N} D_{3,12}(\delta^{1-3^{j-1}b})^{2/3^j}$$

As in the $k=2$ case, we combine this with the bilinear reduction

$$D_{3,12}(\delta) \lesssim_\varepsilon \delta^{-\frac{Nb}{100}\varepsilon} B_{3,12}(\delta)$$

and the crude comparison between symmetric and asymmetric decoupling constants

$$\mathcal{B}_{3,12}(\delta) \lesssim \delta^{-O(b)} \mathcal{B}_{3,1}(\delta; 3b, b)$$
$$\leq \delta^{-\frac{Nb}{100}\varepsilon} \mathcal{B}_{3,1}(\delta; 3b, b)$$

to conclude that

$$\mathcal{D}_{3,12}(\delta) \lesssim_\varepsilon \delta^{-\frac{Nb}{10}\varepsilon} \prod_{j=1}^{N} \mathcal{D}_{3,12}(\delta^{1-3^{j-1}b})^{2/3^j} . \quad (7)$$

Using the above estimate, one may conclude the proof via a simple induction-on-scale.

As before, the base case

$$1 \geq \delta \geq \delta_0(\varepsilon)$$

follows trivially.

<u>Induction hypothesis:-</u> Suppose whenever $1 \geq \delta' \geq 2\delta$ we have

$$\mathcal{D}_{3,12}(\delta') \leq \overline{C}_\varepsilon (\delta')^{-\varepsilon}$$

for some fixed constant $\overline{C}_\varepsilon \geq 1$.

Applying the induction hypothesis directly to (7), using the fact that

$$2\sum_{j=1}^{N} 3^{-j} = 1 - 3^{-N} \leq 1,$$

$$\mathcal{D}_{3,12}(\delta) \leq C_\varepsilon \delta^{-\frac{Nb}{10}\varepsilon} \overline{C}_\varepsilon \prod_{j=1}^{N} \delta^{-2\cdot 3^{-j}(1-3^{j-1}b)\varepsilon}$$

The δ-exponent arising from the product is

$$-2\sum_{j=1}^{N} 3^{-j}\cdot\varepsilon + 2\cdot\frac{1}{3}\cdot Nb\cdot\varepsilon$$

$$= -\varepsilon + 3^{-N}\varepsilon + \frac{2Nb}{3}\cdot\varepsilon$$

Thus, altogether we have

$$\mathcal{D}_{3,12}(\delta) \le \left(C_\varepsilon \delta^{+\frac{Nb}{2}\varepsilon}\right) \overline{C}_\varepsilon \delta^{-\varepsilon}$$

and provided $\delta \le \delta_0(\varepsilon)$ it follows that

$$C_\varepsilon \delta^{\frac{Nb}{2}\varepsilon} \le 1$$

and the induction closes.

The case $k = 4$:-

The cases $k=2$ and $k=3$ treated above are fairly similar and have a straightforward iteration scheme. Complications arise when $k \ge 4$ as we shall see presently.

For $k=4$ the key step provides three inequalities:-

(a) $B_{4,1}(\delta; a_1, a_2) \lesssim_\varepsilon \delta^{-a_2\varepsilon} B_{4,1}(\delta; 4a_2, a_2)$

for $0 \le a_1 \le 4a_2 \le 1$

(b) $B_{4,2}(\delta; a_1, a_2) \lesssim_\varepsilon \delta^{-a_2\varepsilon} B_{4,2}(\delta; \frac{3}{2}a_2, a_2)$

for $0 \le a_1 \le \frac{3}{2}a_2 \le 1$

(c) $B_{4,3}(\delta; a_1, a_2) \lesssim_\varepsilon \delta^{-a_2\varepsilon} B_{4,3}(\delta; \frac{2}{3}a_2, a_2)$

for $0 \le a_1 \le \frac{2}{3}a_2 \le 1$.

Also, by Hölder we can show

(d) $B_{4,1}(\delta; a_1, a_2) \lesssim B_{4,3}(\delta; a_2, a_1)^{1/4} \mathcal{D}_{4,p_4}(\delta^{1-a_2})^{3/4}$

(e) $B_{4,2}(\delta; a_1, a_2) \le B_{4,2}(\delta; a_2, a_1)^{1/3} B_{4,1}(\delta; a_1, a_2)^{2/3}$

(f) $B_{4,3}(\delta; a_1, a_2) \le B_{4,1}(\delta; a_2, a_1)^{1/2} B_{4,2}(\delta; a_1, a_2)^{1/2}$

Applying :- • (d) then (c)

• (e) then (b) and (a)

• (f) then (a) and (b)

yields :-

(I) $B_{4,1}(\delta; 4b, b) \lesssim_\varepsilon \delta^{-\varepsilon b} B_{4,3}(\delta; \tfrac{8}{3}b; 4b)^{1/4} D_{4,p_4}(\delta^{1-b})^{3/4}$

(II) $B_{4,2}(\delta; \tfrac{3}{2}b, b) \lesssim_\varepsilon \delta^{-\varepsilon b} B_{4,2}(\delta; \tfrac{9}{4}b, \tfrac{3}{2}b)^{1/3} B_{4,1}(\delta; 4b, b)^{2/3}$

(III) $B_{4,3}(\delta; \tfrac{2}{3}b; b) \lesssim_\varepsilon \delta^{-\varepsilon b} B_{4,1}(\delta; \tfrac{8}{3}b; \tfrac{2}{2}b)^{1/2} B_{4,2}(\delta; \tfrac{3}{2}b, b)^{1/2}$

Let's try to use these inequalities to cook up an iteration scheme...

Combining (I) and (III) yields

$$B_{4,1}(\delta; 4b, b) \le B_{4,1}(\delta; 4\cdot\tfrac{8}{3}b, \tfrac{8}{3}b)^{1/8} B_{4,2}(\delta; 6b, 4b)^{1/8} \cdot D_{4,p_4}(\delta^{1-b})^{3/4};$$

here, for simplicity, we are ignoring the constant terms in the inequalities.

The above inequality can be iterated to give

$$B_{4,1}(\delta; 4b, b) \le B_{4,1}\left(\delta; 4\cdot\left(\tfrac{8}{3}\right)^N b, \left(\tfrac{8}{3}\right)^N b\right)^{1/8^N} \cdot \prod_{j=1}^{N} B_{4,2}\left(\delta; 6\cdot\left(\tfrac{8}{3}\right)^{j-1} b; 4\cdot\left(\tfrac{8}{3}\right)^{j-1} b\right)^{1/8^j} \cdot \prod_{j=1}^{N} D_{4,p_4}\left(\delta^{1-(8/3)^{j-1} b}\right)^{\frac{3}{4}\cdot 1/8^{j-1}} \qquad (8)$$

This looks similar to what we had in the $k=2,3$ cases, but now we have additional $B_{4,2}$ factors.

To deal with the $B_{4,2}$ factors, we note we can directly iterate the estimate (II) to give

$$B_{4,2}(\delta;\tfrac{3}{2}b,b) \le B_{4,2}\left(\delta;\left(\tfrac{3}{2}\right)^{N+1}b;\left(\tfrac{3}{2}\right)^{N}b\right)^{1/3^N} \prod_{j=1}^{N} B_{4,1}\left(\delta;4\cdot\left(\tfrac{3}{2}\right)^{j-1}b;\left(\tfrac{3}{2}\right)^{j-1}b\right)^{2/3^j}. \quad (9).$$

It is clear that the full iteration scheme will require repeated applications of (8) and (9). This process will eventually lead to a suitable inequality, since at each step the powers of the $B_{4,1}$ and $B_{4,2}$ are reduced, so eventually these terms can be bounded using trivial estimates.

However, writing out this "nested" iteration procedure is clearly complicated and messy. Furthermore, in the $k=4$ case we only encounter 2 iteration schemes in our testing ((8) and (9)); in general there will be many such schemes.

It is necessary to keep track of the exponents, at least those appearing in the linear decoupling terms, in order to ensure the induction-on-scale argument carries through after the iteration procedure.

For the reasons discussed above, it is desirable to find a cleaner method for carrying out the induction-on-scale argument, which does not rely so explicitly on the iteration procedure.

The $k=3$ case revisited

We return to the $k=3$ case and describe an alternative approach to the iteration/induction-on-scale argument from Guo-Li-Yung-Zorin-Kranich, which readily extends to larger k values.

First recall the estimates (a) – (e) for the $k=3$ case. We combine these inequalities to give

(I) $B_{3,1}(\delta; 3b, b) \lesssim_\varepsilon \delta^{-b\varepsilon} B_{3,2}(\delta; 3b, 3b)^{1/3} \, D_{3,12}(\delta^{1-b})^{2/3}$

(II) $B_{3,2}(\delta; b, b) \lesssim_\varepsilon \delta^{-b\varepsilon} B_{3,1}(\delta; 3b, b)$

In general, the key step can be combined with Hölder to give:-

Lemma:- ("General iterative steps")

Let $1 \leq \ell \leq k-1$ and $\varepsilon > 0$. For all $b \in [0,1]$ satisfying

$$\frac{\ell+1}{k-\ell} \cdot \frac{k-\ell+1}{\ell} b \leq 1 \quad \text{and} \quad \frac{k-\ell+2}{\ell-1} b \leq 1 \quad \text{if } \ell \neq 1,$$

we have

$$B_{k,\ell}\left(\delta; \frac{k-\ell+1}{\ell} b; b\right) \lesssim_\varepsilon \delta^{-\varepsilon b} \cdot B_{k,k-\ell}\left(\delta; \frac{\ell+1}{k-\ell} \cdot \frac{k-\ell+1}{\ell} b, \frac{k-\ell+1}{\ell} b\right)^{\frac{1}{k-\ell+1}} \cdot B_{k,\ell-1}\left(\delta; \frac{k-\ell+2}{\ell-1} b, b\right)^{\frac{k-\ell}{k-\ell+1}}$$

We recast the decoupling problem as follows:-

Let η denote the infimum over all $\varepsilon > 0$ for which the inequality

$$D_{k,p_k}(\delta) \lesssim_\varepsilon \delta^{-\varepsilon} \quad \text{holds for all } 0 < \delta \ll 1.$$

Thus, the new goal is to show

$$\eta = 0.$$

For $k=3$, we recast (I) and (II) as follows:-

- Let $A_0(b)$ denote the infimum over all $A > 0$ for which

$$B_{3,0}(\delta; *, b) \lesssim_A \delta^{-A} \quad \text{holds for all } 0 < \delta \ll 1$$

- Let $A_1(b)$ denote the infimum over all $A > 0$

for which

$$B_{3,1}(\delta;\, 3b, b) \lesssim_A \delta^{-A} \quad \text{holds for all } 0<\delta\ll 1$$

- Let $A_2(b)$ denote the infimum over all $A>0$ for which

$$B_{3,2}(\delta;\, b, b) \lesssim_A \delta^{-A} \quad \text{holds for all } 0<\delta\ll 1.$$

- By linear rescaling, we have

$$A_0(b) = \eta(1-b) \qquad (10)$$

- By (I) it follows that

$$A_1(b) \leq \frac{1}{3} A_2(3b) + \frac{2}{3} A_0(b)$$

$$= \frac{1}{3} A_2(3b) + \frac{2}{3}\cdot \eta \cdot (1-b) \qquad (11)$$

- By (II) it follows that

$$A_2(b) \leq A_1(b). \qquad (12)$$

In general, we define $A_\ell(b)$ to be the infimum over all $A>0$ for which

$$B_{k,\ell}\left(\delta;\, \frac{k-\ell+1}{\ell} b, b\right) \lesssim_A \delta^{-A} \quad \text{holds for all } 0<\delta\ll 1.$$

The general iterative steps then imply

$$A_\ell(b) \leq \frac{1}{k-\ell+1} A_{k-\ell}\left(\frac{k-\ell+1}{\ell} b\right) + \frac{k-\ell}{k-\ell+1} A_{\ell-1}(b)$$

$$\text{for } 1 \leq \ell \leq k-1 \qquad (13)$$

From the formula (10), it follows that

$$\eta = \frac{\eta - A_0(b)}{b}$$

and so, in particular,

$$\eta = \liminf_{b \to 0+} \frac{\eta - A_0(b)}{b} =: A_0 \qquad (10')$$

In general, we define

$$A_\ell := \liminf_{b \to 0+} \frac{\eta - A_\ell(b)}{b}$$

Using the bilinear reduction and crude comparison between symmetric and asymmetric decoupling constants, one can prove rough bounds relating η with the $A_\ell(b)$ which are enough to show

Claim:- $A_\ell < \infty$ for $0 \le \ell \le k-1$.

We return to the proof of the claim later. Manipulating (11) and (12) and taking liminfs,

$$\begin{aligned} A_1 &= \liminf_{b \to 0+} \frac{\eta - A_1(b)}{b} \\ &\geq \liminf_{b \to 0+} \left[\frac{1}{3} \cdot \frac{\eta - A_2(3b)}{b} + \frac{2}{3} \cdot \frac{\eta - A_0(b)}{b} \right] \\ &\geq \liminf_{b \to 0+} \frac{\eta - A_2(3b)}{3b} + \frac{2}{3} \liminf_{b \to 0+} \frac{\eta - A_0(b)}{b} \\ &= A_2 + \frac{2}{3} A_0 \end{aligned}$$

whilst

$$A_2 = \liminf_{b \to 0+} \frac{\eta - A_2(b)}{b} \geq \liminf_{b \to 0+} \frac{\eta - A_1(b)}{b} = A_1.$$

Thus, we have

$$A_1 \geq A_2 + \frac{2}{3} A_0 \qquad (11')$$

$$A_2 \geq A_1 \qquad (12')$$

Combining (11′) and (12′),

$$0 \geq A_0$$

and so $A_0 = 0$. By (10′) it follows that $\eta = 0$, as required.

In general, (13) implies

$$A_\ell \geq \frac{1}{\ell} \cdot A_{k-\ell} + \frac{k-\ell}{k-\ell+1} A_{\ell-1} \qquad (13')$$

$$\text{for } 1 \leq \ell \leq k-1$$

Sum these inequalities over all $1 \leq \ell \leq k-1$. On the left-hand side we obtain

$$\sum_{\ell=1}^{k-1} A_\ell .$$

On the right-hand side we obtain

$$\sum_{\ell=1}^{k-1} \left\{ \frac{1}{\ell} A_{k-\ell} + \frac{k-\ell}{k-\ell+1} A_{\ell-1} \right\} = \sum_{\ell=1}^{k-1} \frac{1}{k-\ell} A_\ell + \sum_{\ell=0}^{k-2} \frac{k-\ell-1}{k-\ell} A_\ell$$

$$= \sum_{\ell=1}^{k-1} A_\ell + \frac{k-1}{k} \cdot A_0 .$$

Thus, one concludes

$$0 \geq A_0$$

and so $\eta = A_0 = 0$ as in the $k=3$ case. $\square$

It remains to prove the claim that the A_ℓ are all real numbers. We carry out the argument in the $k=3$, $\ell=1$ case only since the proof is the same in other cases with only minor changes to the numerology.

Recall $A_1 := \liminf_{b \to 0+} \frac{\eta - A_1(b)}{b}$

On the one hand, a simple Hölder's inequality argument together with linear rescaling shows that

$$\mathcal{B}_{3,1}(\delta; 3b, b) \lesssim \mathcal{D}_{3,12}(\delta^{1-3b})^{1/6} \; \mathcal{D}_{3,12}(\delta^{1-b})^{5/6}$$

Consequently,

$$\begin{aligned} A_1(b) &\leq \eta(1-3b)\cdot\tfrac{1}{6} + \eta(1-b)\tfrac{5}{6} \\ &= \eta - \eta\cdot\tfrac{4b}{3}\,. \end{aligned} \qquad (14)$$

On the other hand, the bilinear reduction implies that

$$\mathcal{D}_{3,12}(\delta) \lesssim_\varepsilon \delta^{-\varepsilon}\cdot\mathcal{B}_{3,12}(\delta)$$

whilst the crude comparison between symmetric and asymmetric decoupling constants further implies that

$$\mathcal{D}_{3,12}(\delta) \lesssim_\varepsilon \delta^{-\varepsilon - Cb}\,\mathcal{B}_{3,1}(\delta; 3b, b) \qquad (15)$$

for a suitable constant C (which is an absolute constant, independent of δ, b and ε).

Thus, (15) implies that

$$\eta \leq Cb + A_1(b). \qquad (16)$$

Combining (14) and (16) we see that

$$\eta\cdot\tfrac{4}{3} \leq \frac{\eta - A_1(b)}{b} \leq C$$

and so A_1 is indeed a finite quantity, which satisfies

$$\eta \lesssim A_1 \lesssim 1\,. \qquad \square\,.$$