Lecture 24: Decoupling for the moment curve. I

It remains to adapt the method used in the previous lecture to prove the decoupling theorem for the moment curve in general dimensions.

Recall the statement:-

Theorem (Decoupling for the moment curve) For $\gamma_0 : [-1,1] \to \mathbb{R}^k$; $\gamma_0(t) := (t, t^2/2, \dots, t^k/k!)$, given $\varepsilon > 0$ and $2 \leq p \leq p_k := k(k+1)$, the inequality

$$\Big\| \sum_{J \in \mathcal{D}(\delta)} f_J \Big\|_{L^p(\mathbb{R}^k)} \lesssim_\varepsilon \delta^{-\varepsilon} \Big(\sum_{J \in \mathcal{D}(\delta)} \| f_J \|^2_{L^p(\mathbb{R}^k)} \Big)^{1/2}$$

holds whenever $(f_J)_{J \in \mathcal{D}(\delta)}$ satisfies $\operatorname{supp} \hat{f}_J \subseteq \Theta_J$.

Here the Θ_J are the $\delta \times \delta^2 \times \dots \times \delta^k$ parallelopipeds defined with respect to γ_0 (i.e. with centre $\gamma_0(c_I)$ and sides in the directions $\gamma_0'(c_I), \dots, \gamma_0^{(k)}(c_I)$).

- For $k=1$, $p_1 = 2$ and the result follows from Plancherel's theorem.
- For $k=2$, $p_2 = 6$ and this corresponds to the decoupling theorem for the parabola from the previous lecture.

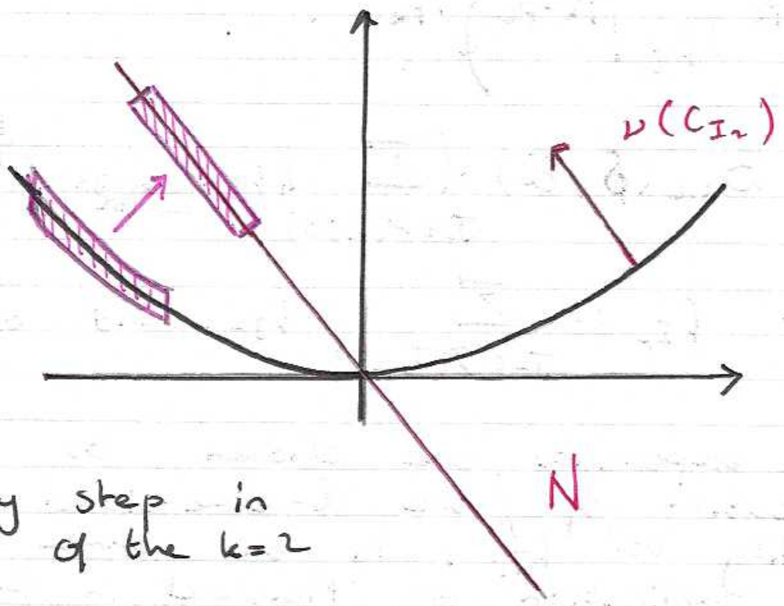


Fig 1

The key step in the proof of the $k=2$ case.

The key step in the proof of the $k=2$ case relied on using Plancherel's theorem (corresponding to the decoupling theorem when $k=1$) in a 1-dimensional subspace N :- see Figure 1.

The decoupling for the moment curve uses a similar key step, but now rather than just projecting down to a 1-dimensional subspace we project down to various subspaces of dimension $1 \le l \le k-1$. Rather than applying Plancherel's theorem ("1-dimensional" decoupling") in the subspace, we apply the l-dimensional decoupling theorem.

Thus, the proof proceeds by induction on dimension.

For $k \geq 1$ and $0 \le l \le k$ define

$$\mathcal{B}_{k,l}(\delta;\vec{a}) := \mathcal{B}_{k,p_k}^{\left(\frac{p_l}{p_k},\, 1-p_l/p_k\right)}(\delta;\vec{a})$$

for all $\vec{a} \in [0,1]^2$.

Note that $\mathcal{B}_{2,1}(\delta;\vec{a}) = \mathcal{B}_1(\delta;\vec{a})$ for $\mathcal{B}_1(\delta;\vec{a})$ as defined in the previous lecture.

Thus, if $I_r \in \mathcal{P}(\delta^{a_r})$, $r=1,2$, $\mathrm{dist}(I_1,I_2) \geq \frac{1}{4}$, then

$$\left(\int_{\mathbb{R}^k} |f_{I_1}|^{p_l}\, |f_{I_2}|^{p_k-p_l}\right)^{1/p_k} \qquad (1)$$

$$\le \mathcal{B}_{k,l}(\delta,\vec{a}) \cdot \Big(\sum_{J_1 \in \mathcal{P}(I_1;\delta)} \|f_{J_1}\|^2_{L^{p_k}(\mathbb{R}^k)}\Big)^{\frac{p_l}{2p_k}} \Big(\sum_{J_2 \in \mathcal{P}(I_2;\delta)} \|f_{J_2}\|^2_{L^{p_k}(\mathbb{R}^k)}\Big)^{\frac{p_k-p_l}{2p_k}}$$

where $f_{I_r} := \sum_{J_r \in \mathcal{P}(I_r;\delta)} f_{J_r}$ and $\mathrm{supp}\, \hat{f}_{J_r} \subseteq \Theta_{J_r}$.

The exponents are chosen to facilitate application of the l-dimensional decoupling inequality to f_{I_1}, in the same manner as the application of Plancherel in the $k=2$ case.

Recall from last time:-

<u>Lemma (key step: $k=2$)</u> If $0 \leq a_1 \leq 2a_2$, then

$$B_{2,1}(\delta; a_1, a_2) \lesssim B_{2,1}(\delta; 2a_2, a_1).$$

The general version of this result is as follows.

<u>Lemma (Key step: general k)</u> For $1 \leq \ell \leq k-1$ and

$$0 \leq a_1 \leq \frac{k-\ell+1}{\ell} a_2 \qquad (2)$$

we have

$$B_{k,\ell}(\delta; a_1, a_2) \lesssim \qquad (3)$$

$$D_{\ell, p_\ell}(\delta^{\frac{k-\ell+1}{\ell}a_2})^{p_\ell/p_k} \cdot B_{k,\ell}(\delta; \tfrac{k-\ell+1}{\ell} a_2, a_2).$$

<u>Remarks:-</u> • If $k=2$ and $\ell=1$, then we recover the result from lecture 23. In particular,

$\frac{k-\ell+1}{\ell} = 2$ and $D_{\ell, p_\ell}(\delta^{\frac{k-\ell+1}{\ell}a_2}) = D_{1,2}(\delta^{2a_2})$.

Since decoupling in $\mathbb{R}^1$ just corresponds to Plancherel's theorem, we know $D_{1,2}(\delta^{2a_2}) \leq 1$.

• The inequality (3) will facilitate an induction-on-dimension. In particular, as part of the induction hypothesis it will be assumed that

$$D_{\ell, p_\ell}(\delta^{\frac{k-\ell+1}{\ell}a_2}) \lesssim_\varepsilon \delta^{-\varepsilon a_2}$$

and so (3) takes the form

$$B_{k,\ell}(\delta; a_1, a_2) \lesssim_\varepsilon \delta^{-\varepsilon} B_{k,\ell}(\delta; \tfrac{k-\ell+1}{\ell} a_2, a_2).$$

• The condition $a_1 \leq \frac{k-\ell+1}{\ell} a_2$ in (2) arises naturally from geometric considerations, as shall be described below. In particular, it corresponds to the sizes of certain projections of the parallelopipeds θ_I for $I \in \mathcal{P}(\delta^{a_2})$.

The proof of the key step (3) directly generalises that of the $k=2$, $\ell=1$ case seen in the previous lecture.

Let $k \geq 1$, $1 \leq \ell \leq k-1$ and suppose $\vec{a} = (a_1, a_2)$ satisfies (2). Fix $I_r \in \mathcal{P}(\delta^{a_r})$ for $r = 1, 2$ with $\mathrm{dist}(I_1, I_2) \geq 1/4$.

For c_{I_2} the centre of I_2, consider the space

$$V^{k-\ell} = V^{k-\ell}(c_{I_2}) = \langle \gamma_0^{(1)}(c_{I_2}), \dots, \gamma_0^{(k-\ell)}(c_{I_2}) \rangle$$

of dimension $k-\ell$. Let

$$N^\ell = N^\ell(c_{I_2}) := (V^{k-\ell})^\perp$$

denote the orthogonal complement.

Remark :- For $k=2$ and $\ell = 1$

$$V^1 = \langle \gamma_0'(c_{I_2}) \rangle \quad , \quad N^1 = (V^1)^\perp$$

denote the tangent and normal spaces to the parabola at $\gamma_0(c_{I_2})$, respectively.

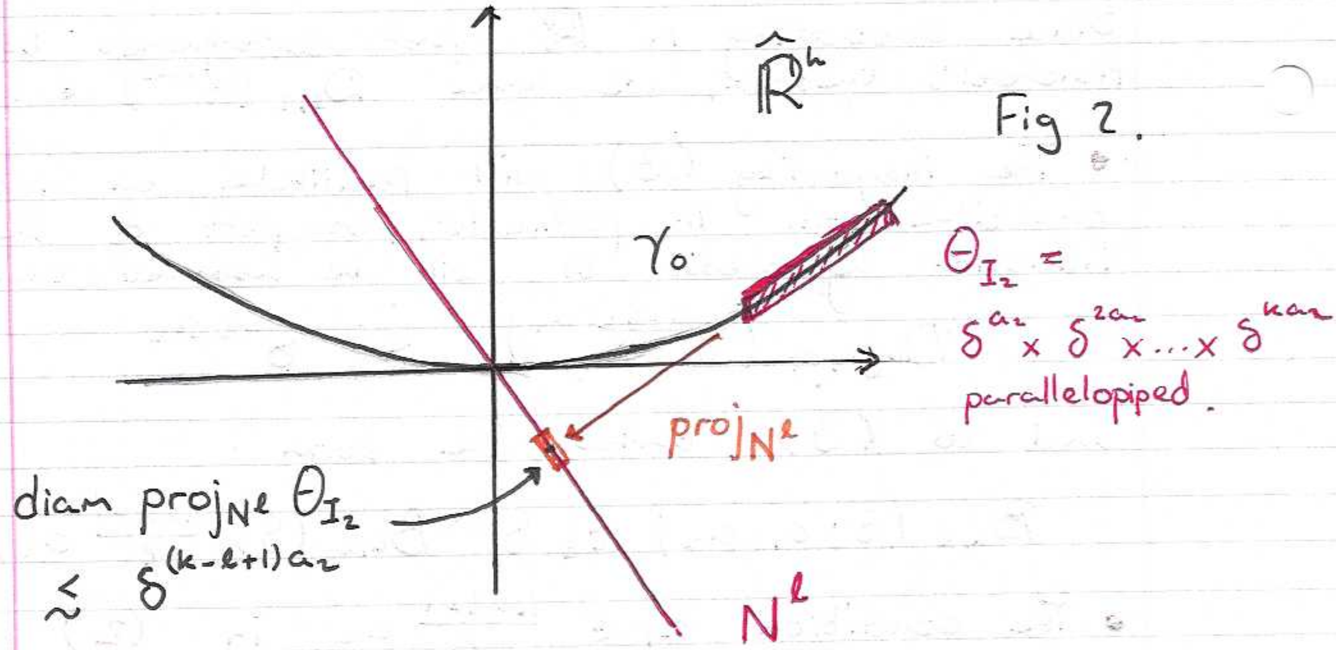


Fig 2.

It follows that the projection of Θ_{I_2} onto the subspace N^ℓ is contained in a ball of radius $\lesssim \delta^{(k-\ell+1)a_2}$. In particular, by projecting onto N^ℓ we "ignore" the ℓ longest sides lying in directions in $V^{k-\ell}$.

Thus, if $\mathrm{proj}_{N^\ell} : \mathbb{R}^k \to N^\ell$ denotes orthogonal projection onto N^ℓ, then

$$\mathrm{diam}\ \mathrm{proj}_{N^\ell} \Theta_{I_2} \lesssim \delta^{(k-\ell+1)a_2},$$

see figure 2. This partially accounts for the numerology in (2).

On the other hand, consider the N^ℓ-projection of the portion of the moment curve over I_1 :-

$$\gamma_{0,\ell}(t) := \mathrm{proj}_{N^\ell} \gamma_0(t) \quad , \quad t \in I_1 .$$

The following geometric observation is crucial.

<u>Lemma (Non-degeneracy)</u> The curve $\gamma_{0,\ell} : I_1 \to N^\ell$, ($N^\ell \cong \mathbb{R}^\ell$) is non-degenerate in the sense that

$$\Big| \bigwedge_{j=1}^{\ell} \gamma_{0,\ell}^{(j)}(t) \Big| \gtrsim_k 1 \qquad \text{for all } t \in I_1 .$$

(That is, it is of non-vanishing torsion).

This lemma relies heavily on the separation hypothesis

$$\mathrm{dist}\,(I_1, I_2) \geqslant 1/4 .$$

The proof is an exercise in manipulating certain Vandermonde-type expressions, and is omitted. One may consult Guo-Li-Yung-Zorin-Kranich for details. We mention the key underlying estimate:-

<u>Lemma:</u> For $0 \le \ell \le k$ and $t_1, t_2 \in \mathbb{R}$,

$$\Big| \bigwedge_{j=1}^{\ell} \gamma_0^{(j)}(t_1) \wedge \bigwedge_{j=1}^{k-\ell} \gamma_0^{(j)}(t_2) \Big| \gtrsim_k |t_1 - t_2|^{\ell(k-\ell)} .$$

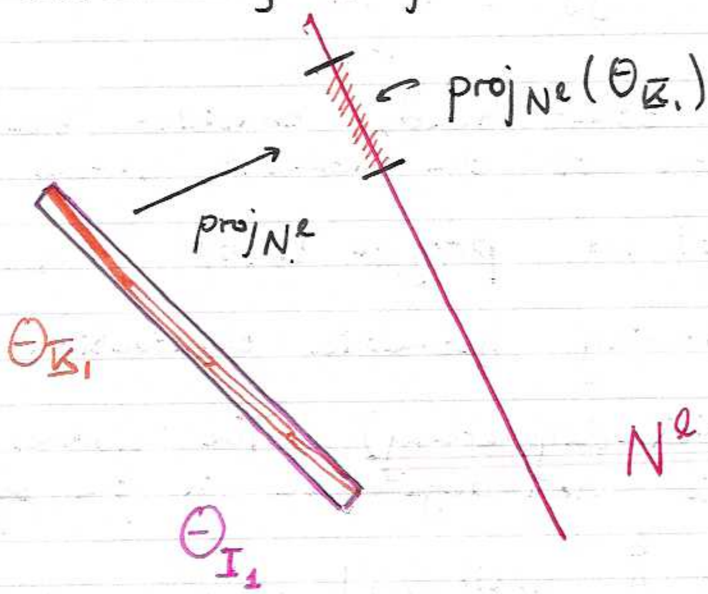
The projection proj_{N^ℓ} is also in some sense "bilipschitz" on Θ_{I_1}, as in the $k=2$ case. More precisely, given

$$0 < \eta \lesssim \delta^{a_1} \qquad (4)$$

the sets

$$\{ \mathrm{proj}_{N^\ell}(\Theta_{\underline{K}_1}) \quad : \quad \underline{K}_1 \in \mathcal{P}(I_1;\eta) \}$$

are essentially disjoint.



Moreover, the $\mathrm{proj}_{N^\ell}(\Theta_{\underline{K}_1})$ essentially form $\eta \times \eta^2 \times \dots \times \eta^\ell$ parallelopipeds around the projected curve $\gamma_{0,\ell}$, in the sense that

$$\mathrm{proj}_{N^\ell}(\Theta_{\underline{K}_1}) \subseteq C \cdot \Theta_{\gamma_{0,\ell}, \underline{K}_1}$$

for an appropriate constant C (depending only on k).

In particular,

$$\Big\| \sum_{\underline{K}_1 \in \mathcal{P}(I_1;\eta)} F_{\underline{K}_1} \Big\|_{L^{p_\ell}(\mathbb{R}^\ell)} \lesssim D_{\ell,p_\ell}(\eta) \Big(\sum_{\underline{K}_1 \in \mathcal{P}(I_1;\eta)} \| F_{\underline{K}_1} \|^2_{L^{p_\ell}(\mathbb{R}^\ell)} \Big)^{\frac{1}{2}} \qquad (5)$$

holds whenever $\mathrm{supp}\, \widehat{F}_{\underline{K}_1} \subseteq \mathrm{proj}_{N^\ell}(\Theta_{\underline{K}_1})$.

Here we have used the non-degeneracy lemma together with the result from Lecture 20 (Pramanik - Seeger iteration) which allows us to pass from decoupling for the moment curve to decoupling for general curves of non-vanishing torsion.

We note that (5) can be strengthened to a local estimate

$$\Big\|\sum_{\mathbb{K}_1 \in \mathcal{P}(I_1;\eta)} F_{\mathbb{K}_1}\Big\|_{L^{p_\ell}(B_{\eta^{-\ell}})} \lesssim D_{\ell,p_\ell}(\eta)\cdot\Big(\sum_{\mathbb{K}_1\in\mathcal{P}(I_1;\eta)} \|F_{\mathbb{K}_1}\|^2_{L^{p_\ell}(w_{B_{\eta^{-\ell}}})}\Big)^{1/2} \tag{6}$$

where $B_{\eta^{-\ell}} \subseteq \mathbb{R}^\ell$ is an arbitrary ball of radius $\eta^{-\ell}$ and $w_{B_{\eta^{-\ell}}}$ is a rapidly decaying weight function concentrated on $B_{\eta^{-\ell}}$. For simplicity, henceforth we will work under the heuristic that $w_{B_{\eta^{-\ell}}} = \chi_{B_{\eta^{-\ell}}}$ is a sharp cutoff.

Remark:- The estimate (6) should be contrasted with the local L^2 decoupling discussed in Lecture 21. In the L^2 case, one may pass down to much "stronger" localised estimates, in the sense that the localisation occurs at much smaller $B_{\eta^{-1}}$ balls, rather than $B_{\eta^{-\ell}}$ balls.

The local version (6) follows from the global version (5) via the uncertainty principle. Introducing a smooth cutoff $\psi_{B_{\eta^{-\ell}}} \in \mathcal{S}(\mathbb{R}^\ell)$ satisfying

- $|\psi_{B_{\eta^{-\ell}}}(x)| \gtrsim 1$ for $x \in B_{\eta^{-\ell}}$
- $\operatorname{supp}(\psi_{B_{\eta^{-\ell}}})^\wedge \subseteq B(0, \eta^\ell)$,

the result follows by applying (5) to the functions

$$\widetilde{F}_{\mathbb{K}_1} := F_{\mathbb{K}_1} \cdot \chi_{B_{\eta^{-\ell}}} ,$$

noting the $\widetilde{F}_{\mathbb{K}_1}$ essentially satisfy the required Fourier support condition by virtue of the fact that

$$(\widetilde{F}_{\mathbb{K}_1})^\wedge = \widehat{F}_{\mathbb{K}_1} * (\chi_{B_{\eta^{-\ell}}})^\wedge$$

and $\operatorname{supp} \widehat{F}_{\mathbb{K}_1} \subseteq C \cdot \Theta_{\gamma_{0,\ell}, \mathbb{K}_1}$; $\operatorname{supp}(\chi_{B_{\eta^{-\ell}}})^\wedge \subseteq B(0, \eta^\ell)$, where η^ℓ is the smallest length of $\Theta_{\gamma_{0,\ell}, \mathbb{K}_1}$.

We will choose $\eta := \delta^{\frac{k-\ell+1}{\ell} a_2}$. With this definition, (4) holds by our hypothesis (2). Moreover, the spatial localisation in (6) occurs at scale

$$\delta^{-(k-\ell+1)a_2} = \eta^{-\ell}. \qquad (7)$$

On the other hand, going back to I_2, recall $\operatorname{proj}_{N^\ell} \Theta_{I_2}$ is contained in a ball of radius

$$\lesssim \delta^{(k-\ell+1)a_2}.$$

Consequently, any function $G \in L^1(N^\ell)$ satisfying $\operatorname{supp} \widehat{G} \subseteq \operatorname{proj}_{N^\ell}(\Theta_{I_2})$ will be locally constant at scale $\delta^{-(k-\ell+1)a_2}$, matching (7).

These observations will allow us to argue as in Lecture 21 and Lecture 23, but with the local decoupling inequality (6) playing the rôle of the local L^2-orthogonality lemma used previously.

Let's put the pieces together, as in Lecture 23. Recall, we want to bound

$$\mathrm{Br}_{k,\ell}(\delta; a_1, a_2)$$

so let's fix a tuple of functions

$$(f_J)_{J \in \mathcal{P}(I_1;\delta) \cup \mathcal{P}(I_2;\delta)} \quad \text{with } \operatorname{supp} \widehat{f}_J \subseteq \Theta_J,$$

for I_r, $r=1,2$, as above. We will write

$$f_{I_r} := \sum_{J_r \in \mathcal{P}(I_r;\delta)} f_{J_r}$$

so the expression appearing on the left-hand side of the norm inequality (1) defining $\mathcal{B}_{k,\ell}(\delta;\vec{a})$ is given by

$$\Big(\int_{\mathbb{R}^k} |f_{I_1}|^{p_\ell} \, |f_{I_2}|^{p_k - p_\ell} \Big)^{1/p_k} .$$

For $a_2' := \frac{k-\ell+1}{\ell} a_2$, define

$$f_{K_1} := \sum_{J_1 \in \mathcal{P}(K_1;\delta)} f_{J_1} \quad \text{for} \quad K_1 \in \mathcal{P}(I_1, \delta^{a_2'})$$

so that
$$f_{I_1} = \sum_{K_1 \in \mathcal{P}(I_1;\delta^{a_2'})} f_{K_1} .$$

Choose coordinates on $\mathbb{R}^k$ so that $x = (x_N, x_V)$ with $x_N \in N^\ell$ and $x_V \in V^{k-\ell}$. By Fubini,

$$\int_{\mathbb{R}^k} |f_{I_1}|^{p_\ell} \, |f_{I_2}|^{p_k - p_\ell} = \int_{V^{k-\ell}} \int_{N^\ell} |f_{I_1, x_V}(x_N)|^{p_\ell} \, |f_{I_2, x_V}(x_N)|^{p_k - p_\ell} \, dx_N \, dx_V$$

where $f_{x_V}(x_N) := f(x_N, x_V)$ for $f : \mathbb{R}^k \to \mathbb{C}$.

By the Fourier slice lemma, we know

$$f_{I_1, x_V} = \sum_{K_1 \in \mathcal{P}(I_1, \delta^{a_2'})} f_{K_1, x_V}$$

with $\mathrm{supp}\,(f_{K_1, x_V})^\wedge \subseteq \mathrm{proj}_{N^\ell}\, \Theta_{K_1} \subseteq C \cdot \Theta_{\gamma_{0,\ell}, K_1}$

whilst

$$\mathrm{supp}\,\big(f_{I_2, x_V}\big)^\wedge \subseteq \mathrm{proj}_{N^\ell}\, \Theta_{I_2}$$

has diameter $\lesssim \delta^{(k-\ell+1)a_2}$.

Thus, $|f_{I_2,x_v}|$ is locally constant at scale $\eta^{-\ell} = \delta^{-(k-\ell+1)a_2}$ and so one may write

$$\int_{N^\ell} |f_{I_1,x_v}|^{p_\ell} \cdot |f_{I_2,x_v}|^{p_k-p_\ell}$$

$$\sim \sum_{B_{\eta^{-\ell}} \in \mathcal{B}_{\eta^{-\ell}}} \left(\int_{B_{\eta^{-\ell}}} |f_{I_1,x_v}|^{p_\ell} \right) \cdot |f_{I_2,x_v}(y_{B_{\eta^{-\ell}}})|^{p_k-p_\ell} \qquad (8)$$

where
- $\mathcal{B}_{\eta^{-\ell}}$ is a covering of $\mathbb{R}^n$ by finitely-overlapping $\eta^{-\ell}$-balls
- Each $y_{B_{\eta^{-\ell}}}$ is a choice of (arbitrary) point in $B_{\eta^{-\ell}}$.

The equation (8) is somewhat heuristic, but can be made precise using standard rigorous interpretations of the uncertainty principle. The reader is directed to Guo–Li–Yung–Zorin-Kranich for details.

By the frequency support conditions, we may use (6) to decouple

$$\int_{B_{\eta^{-\ell}}} |f_{I_1,x_v}|^{p_\ell} \lesssim D_{\ell,p_\ell}(\delta^{a_i'})^{p_\ell} \Big(\sum_{K_1 \in \mathcal{P}(I_1;\, \delta^{a_i'})} \|f_{K_1,x_v}\|^2_{L^{p_\ell}(B_{\eta^{-\ell}})} \Big)^{\frac{p_\ell}{2}}$$

and so

$$\left(\int_{B_{\eta^{-\ell}}} |f_{I_1,x_v}|^{p_\ell} \right) \cdot |f_{I_2,x_v}(y_{B_{\eta^{-\ell}}})|^{p_k-p_\ell}$$

$$\lesssim D_{\ell,p_\ell}(\delta^{a_i'})^{p_\ell} \Big[\sum_{K_1 \in \mathcal{P}(I_1;\delta^{a_i'})} \Big(\int_{B_{\eta^{-\ell}}} |f_{K_1,x_v}|^{p_\ell} |f_{I_2,x_v}|^{p_k-p_\ell} \Big)^{2/p_\ell} \Big]^{\frac{p_\ell}{2}},$$

again using the locally-constant property of the $|f_{I_2,x_v}|$ functions.

Summing over all balls $B_{\eta^{-\ell}} \in \mathcal{B}_{\eta^{-\ell}}$ and using Minkowski's inequality to exchange the order of the $\ell^2(\mathcal{P}(I_1;\delta^{a_2'}))$ and $\ell^{p_\ell}(\mathcal{B}_{\eta-\ell})$ norms (noting $p_\ell \geq 2$),

$$\int_{N^\ell} |f_{I_1,x_v}|^{p_\ell} \cdot |f_{I_2,x_v}|^{p_n - p_\ell}$$

$$\lesssim D_{\ell,p_\ell}(\delta^{a_2'})^{p_\ell} \Big[\sum_{K_1 \in \mathcal{P}(I_1;\delta^{a_2'})} \Big(\int_{N^\ell} |f_{K_1,x_v}|^{p_\ell} |f_{I_2,x_v}|^{p_n - p_\ell} \Big)^{2/p_\ell} \Big]^{\frac{p_\ell}{2}}.$$

Integrating over $x_v \in V^{n-\ell}$ and again using Minkowski,

$$\int_{\mathbb{R}^n} |f_{I_1}|^{p_\ell} |f_{I_2}|^{p_n - p_\ell}$$

$$\lesssim D_{\ell,p_\ell}(\delta^{a_2'})^{p_\ell} \Big[\sum_{K_1 \in \mathcal{P}(I_1;\delta^{a_2'})} \Big(\int_{\mathbb{R}^n} |f_{\underline{K}_1}|^{p_\ell} |f_{I_2}|^{p_n - p_\ell} \Big)^{2/p_\ell} \Big]^{\frac{p_\ell}{2}}.$$

Finally, by the definition of the asymmetric decoupling constant, it follows that for $\underline{K}_1 \in \mathcal{P}(I_1;\delta^{a_2'})$ and $I_2 \in \mathcal{P}(\delta^{a_2})$ we have

$$\int_{\mathbb{R}^n} |f_{\underline{K}_1}|^{p_\ell} |f_{I_2}|^{p_n - p_\ell} \lesssim \mathcal{B}_{n,\ell}(\delta; a_2', a_2)^{p_n} \cdot$$

$$\Big(\sum_{J_1 \in \mathcal{P}(\underline{K}_1;\delta)} \|f_{J_1}\|^2_{L^{p_n}(\mathbb{R}^n)} \Big)^{\frac{p_\ell}{2}} \Big(\sum_{J_2 \in \mathcal{P}(I_2;\delta)} \|f_{J_2}\|^2_{L^{p_n}(\mathbb{R}^n)} \Big)^{\frac{p_n - p_\ell}{2}}$$

Taking the ℓ^{2/p_ℓ} sum of both sides of this inequality over all $\underline{K}_1 \in \mathcal{P}(I_1;\delta^{a_2'})$ yields

$$\int_{\mathbb{R}^n} |f_{I_1}|^{p_\ell} |f_{I_2}|^{p_n - p_\ell} \lesssim D_{\ell,p_\ell}(\delta^{a_2'})^{p_\ell} \cdot \mathcal{B}_{n,\ell}(\delta; a_2'; a_2)^{p_n}$$

$$\cdot \Big(\sum_{J_1 \in \mathcal{P}(I_1;\delta)} \|f_{J_1}\|^2_{L^{p_n}(\mathbb{R}^n)} \Big)^{\frac{p_\ell}{2}} \Big(\sum_{J_2 \in \mathcal{P}(I_2;\delta)} \|f_{J_2}\|^2_{L^{p_n}(\mathbb{R}^n)} \Big)^{\frac{p_n - p_\ell}{2}}.$$

Thus, by the definition of the asymmetric decoupling constant,

$$B_{k,\ell}(\delta, a_1, a_2) \lesssim D_{\ell,p_\ell}(\delta^{a_2'})^{p_\ell/p_k} B_{k,\ell}(\delta; a_2', a_2)$$

where $a_2' = \frac{k-\ell+1}{\ell} a_2$, as required $\square$.