Lecture 23: Decoupling for the parabola.

Here we will combine the tools discussed in the previous lectures to prove the $k=2$ case of the Bourgain - Demeter - Guth theorem (which originally appeared in earlier work of Bourgain - Demeter).

Recall the statement :-

Theorem (Decoupling for the parabola). For $\gamma_0 : [-1,1] \to \mathbb{R}^2$, $\gamma_0(t) := (t, t^2/2)$, given $\varepsilon > 0$ and $2 \le p \le 6$, the inequality

$$\Big\| \sum_{J \in \mathcal{P}(\delta)} f_J \Big\|_{L^p(\mathbb{R}^2)} \lesssim_\varepsilon \delta^{-\varepsilon} \Big(\sum_{J \in \mathcal{P}(\delta)} \| f_J \|^2_{L^p(\mathbb{R}^2)} \Big)^{1/2}$$

holds whenever $(f_J)_{J \in \mathcal{P}(\delta)}$ satisfies $\operatorname{supp} \hat{f}_J \subseteq \Theta_J$.

Here the Θ_J are the $\delta \times \delta^2$ parallelograms defined with respect to γ_0 (ie. with centre $\gamma_0(c_I)$ and sides in directions $\gamma_0'(c_I)$, $\gamma_0''(c_I)$).

Recall the $p=2$ case follows by Plancherel. We consider the critical exponent $p=6$; the remaining cases follow by interpolation.

We will first work with the asymmetric bilinear decouplings introduced in the previous lecture. In particular, let

$$B_1(\delta; \vec{a}) := \mathcal{B}^{(\frac{1}{3}, \frac{2}{3})}_{2,6}(\delta; \vec{a})$$

for $\vec{a} = (a_1, a_2) \in [0,1]^2$ so that :-

- For all $I_r \in \mathcal{P}(\delta^{a_r})$, $r = 1, 2$, $\operatorname{dist}(I_1, I_2) \geq \frac{1}{4}$,

$$\Big(\int_{\mathbb{R}^2} |f_{I_1}|^2 |f_{I_2}|^4 \Big)^{1/6} \le B_1(\delta; \vec{a}) \Big(\sum_{J_1 \in \mathcal{P}(I_1;\delta)} \|f_{J_1}\|_6^2 \Big)^{1/6} \Big(\sum_{J_2 \in \mathcal{P}(I_2;\delta)} \|f_{J_2}\|_6^2 \Big)^{1/3} \qquad (1)$$

where $f_{I_r} := \sum_{J_r \in \mathcal{P}(I_r;\delta)} f_{J_r}$.

The choice of $\vec{\alpha} = (\frac{1}{3}, \frac{2}{3})$ is motivated by the observations from Lecture 21. Recall, Plancherel's theorem can be combined with the uncertainty principle to show -

Lemma (Asymmetric L^2-orthogonality) :- Suppose $f, g \in L^1(\mathbb{R}^d)$ satisfy

- $f = \sum_{j=1}^{K} f_j$ where $f_j \in L^1(\mathbb{R}^d)$ with $\mathcal{N}_{1/R}(\operatorname{supp} \hat{f}_j)$ finitely-overlapping
- $\operatorname{supp} \hat{g} \subseteq B_{1/R}$, some $1/R$ ball in $\mathbb{R}^d$.

Then
$$\int_{\mathbb{R}^d} |f|^2 \cdot |g|^4 \lesssim \int_{\mathbb{R}^d} \Big(\sum_{j=1}^{K} |f_j|^2 \Big)^{1/2} |g|^4 .$$

Remark :- In the discussion in Lecture 21 we focussed on $d = 2$, but the same argument works in all dimensions.

The $\vec{\alpha} = (\frac{1}{3}, \frac{2}{3})$ exponent ensures we are working with a favourable expression
$$\int_{\mathbb{R}^2} |f_{I_1}|^2 \cdot |f_{I_2}|^4$$
on the left-hand side of (1). It is also necessary to ensure suitable Fourier localisation of the f_{I_1}, f_{I_2}. To this end, suppose
$$0 \leq a_1 \leq 2 a_2 \qquad (2)$$

The $\hat{f}_{I_1}, \hat{f}_{I_2}$ are supported on $\theta_{I_1}, \theta_{I_2}$, respectively, as pictured in fig 1 below.

Let $\nu : [-1, 1] \to S^1 \cap (\mathbb{R} \times [0, \infty))$ denote the 'inward' Gauss map associated to γ_0 so that $\langle \gamma_0'(s), \nu(s) \rangle = 0$ for all $s \in [-1, 1]$.

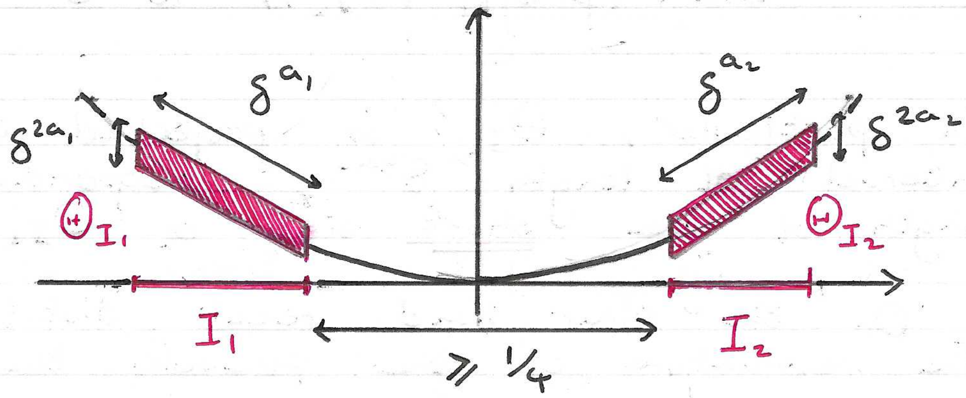

Fig 1

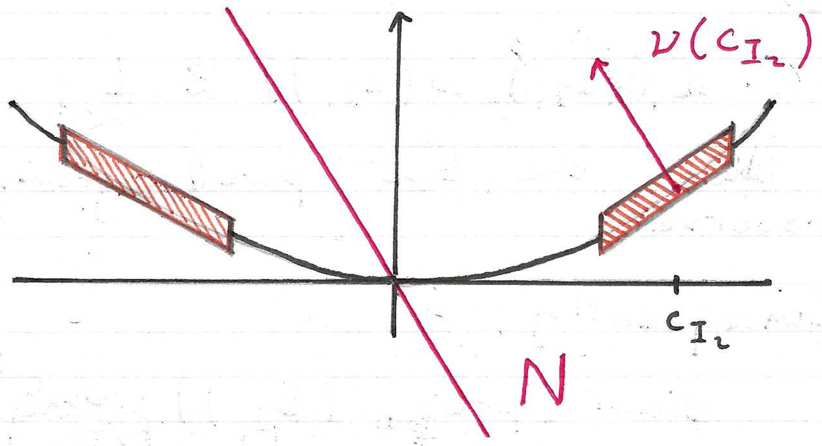

Fig 2.

For c_{I_2} the centre of I_2, let

$$N := \text{span}\{\nu(c_{I_2})\}$$

be the subspace of $\mathbb{R}^2$ spanned by $\nu(c_{I_2})$; see fig 2.

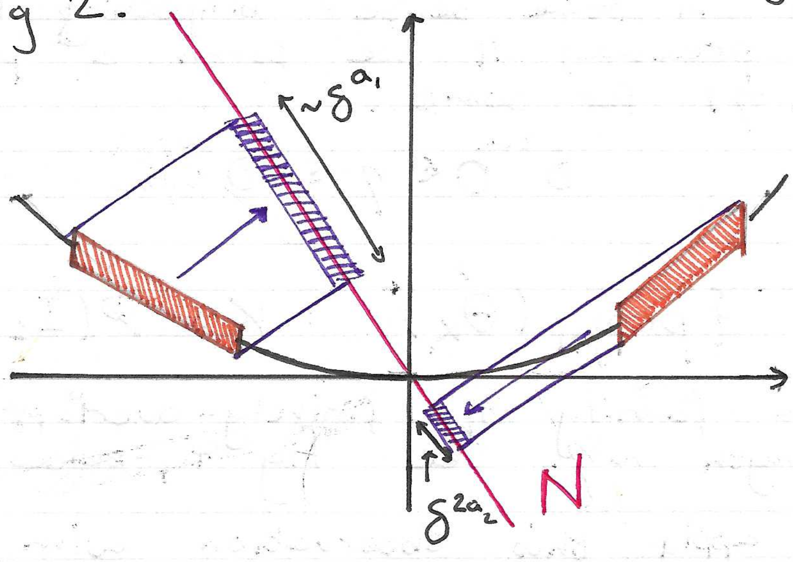

Fig 3.

Let $\text{proj}_N : \mathbb{R}^2 \to N$ denote the orthogonal projection mapping.

It follows that

$$\mathrm{diam}\,(\mathrm{proj}_N\,(\Theta_{I_2})) \lesssim \delta^{2a_2}; \qquad (3)$$

in particular, Θ_{I_2} is projected onto an interval of length δ^{2a_2}, corresponding to the length of the short side of Θ_{I_2}. See Fig 3.

On the other hand, recall $\mathrm{dist}\,(I_1, I_2) \geq 1/4$. The curvature of the parabola therefore implies

$$\measuredangle\,(\nu(c_{I_1}), \nu(c_{I_2})) \gtrsim 1;$$

so that the two parallelograms $\Theta_{I_1}, \Theta_{I_2}$ are 'transverse'.

Consequently,

$$\mathrm{diam}\,(\mathrm{proj}_N\,(\Theta_{I_1})) \sim \delta^{a_1};$$

in particular, Θ_{I_1} is projected onto an interval of length δ^{a_1}, corresponding to the length of the long side of Θ_{I_1}. See fig 3.

Moreover, the projection mapping on Θ_{I_1} is in some sense "bilipschitz". To be more precise, if we form a partition $\mathcal{P}(I_1; \eta)$ for some

$$0 < \eta \leq \delta^{a_1}, \qquad (4)$$

then

$$\{\mathrm{proj}_N\,(\Theta_{K_1}) \;:\; K_1 \in \mathcal{P}(I_1; \eta)\}$$

is a family of finitely-overlapping intervals of length $\sim \eta$. See Fig 4, below.

We apply this observation with $\eta := \delta^{2a_2}$. Note that this satisfies (4) in view of our hypothesis (2). The choice is motivated by (3).

For $K_1 \in \mathcal{P}(I_1; \delta^{2a_2})$ write

$$f_{K_1} := \sum_{J \in \mathcal{P}(K_1; \delta)} f_{J_1} \quad \text{so that}$$

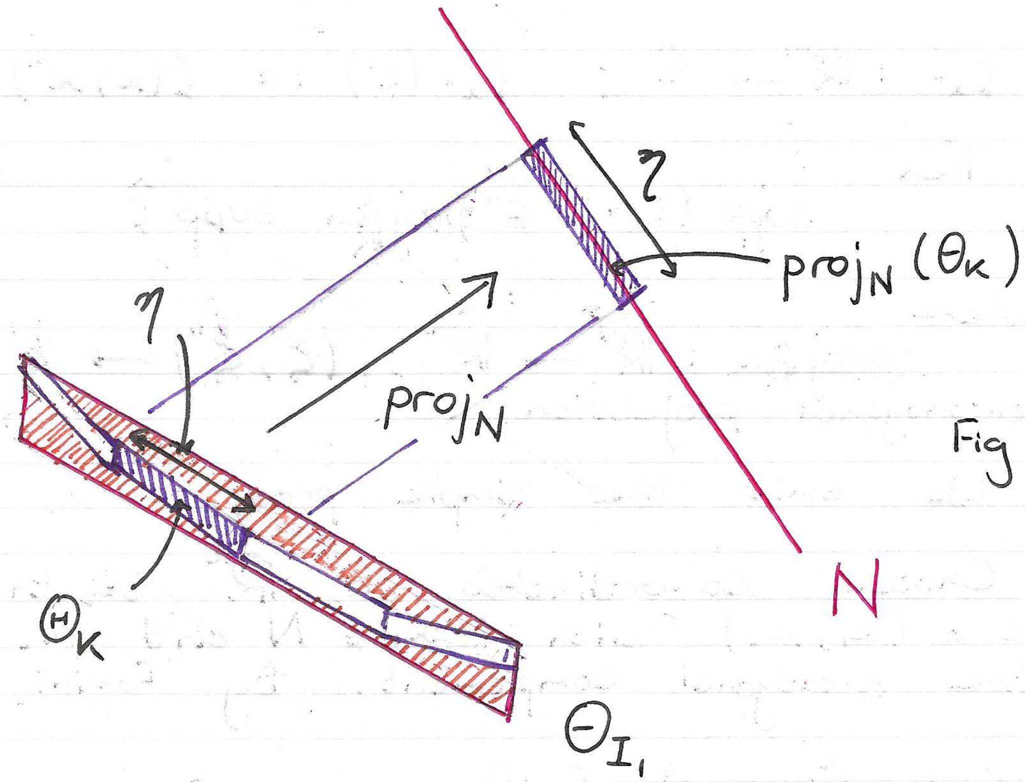

Fig 4.

$$f := \sum_{K_1 \in \mathcal{P}(I_1; \delta^{2a_2})} f_{K_1} = \sum_{J_1 \in \mathcal{P}(I_1; \delta)} f_{J_1} .$$

Note that

- $\mathcal{N}_{\delta^{2a_2}}(\mathrm{proj}_N \operatorname{supp} \hat{f}_{K_1})$, $K_1 \in \mathcal{P}(I_1; \delta^{2a_2})$

are finitely-overlapping

- $g := f_{I_2}$ satisfies

$$\mathrm{proj}_N \operatorname{supp} \hat{g} \subseteq B_{\delta^{2a_2}}$$

for some δ^{2a_2}-ball.

Thus, "after taking projections" we are in a situation amenable to the Asymmetric L^2-Orthogonality Lemma.

To take account of the projections, we use the following elementary property of the Fourier transform:-

Lemma (Fourier slice): Let $f \in \mathcal{S}(\mathbb{R}^d)$ and write $x = (x', x'') \in \mathbb{R}^m \times \mathbb{R}^{d-m}$. For each $x'' \in \mathbb{R}^{d-m}$ consider the "slice"

$$f_{x''} : \mathbb{R}^m \to \mathbb{C} \; ; \quad f_{x''}(x') := f(x', x'')$$

Then

$$\operatorname{supp} (f_{x''})^{\wedge} \subseteq \operatorname{proj}_{\hat{\mathbb{R}}^m} \operatorname{supp} \hat{f}$$

where

$\operatorname{proj}_{\hat{\mathbb{R}}^m} : \hat{\mathbb{R}}^d \to \hat{\mathbb{R}}^m \; ; \; (\xi', \xi'') \mapsto \xi'$ denotes orthogonal projection.

We omit the simple proof.

Choose co-ordinates on $\mathbb{R}^2$ so that $x = (x_1, x_2)$ with $x_1 \in N$ and $x_2 \in N^\perp$, the orthogonal complement. By Fubini,

$$\int_{\mathbb{R}^2} |f_{I_1}|^2 \cdot |f_{I_2}|^4 = \int_{N^\perp} \int_N |f_{I_1, x_2}(x_1)|^2 |f_{I_2, x_2}(x_1)|^4 \, dx_1 \, dx_2$$

For each fixed $x_2 \in N^\perp$ we can apply the Asymmetric L^2-Orthogonality Lemma to conclude

$$\int_N |f_{I_1, x_2}(x_1)|^2 |f_{I_2, x_2}(x_1)|^4 \, dx_1$$

$$\lesssim \sum_{K_1 \in \mathcal{P}(I_1; \delta^{2a_2})} \int_N |f_{K_1, x_2}(x_1)|^2 \, |f_{I_2, x_2}(x_1)|^4 \, dx_1$$

Integrating both sides of the above inequality with respect to x_2 and again applying Fubini, one concludes

$$\int_{\mathbb{R}^2} |f_{I_1}|^2 |f_{I_2}|^4 \lesssim \sum_{K_1 \in \mathcal{P}(I_1; \delta^{2a_2})} \int_{\mathbb{R}^2} |f_{K_1}|^2 \cdot |f_{I_2}|^4 \qquad (5)$$

This provides a 'partial decoupling' of the expression

$$\Big(\int_{\mathbb{R}^2} |f_{I_1}|^2 |f_{I_2}|^4 \Big)^{1/6}$$

on the left-hand side of (1).

For each $\underline{K}_1 \in \mathcal{P}(I_1; \delta^{2a_2})$ one may decouple

$$\int_{\mathbb{R}^2} |f_{\underline{K}_1}|^2 |f_{I_2}|^4 \leq B_1(\delta; 2a_2, a_2)^6 \cdot \Big(\sum_{J_1 \in \mathcal{P}(\underline{K}_1; \delta)} \|f_{J_1}\|_6^2 \Big) \cdot \Big(\sum_{J_2 \in \mathcal{P}(I_2; \delta)} \|f_{J_2}\|_6^2 \Big)^2$$

Summing over $\underline{K}_1 \in \mathcal{P}(I_1; \delta^{2a_2})$ and taking the $1/6$ power, it follows from (5) that

$$\Big(\int_{\mathbb{R}^2} |f_{I_1}|^2 |f_{I_2}|^4 \Big)^{1/6} \lesssim B_1(\delta; 2a_2, a_2) \cdot \Big(\sum_{J_1 \in \mathcal{P}(I_1; \delta)} \|f_{J_1}\|_6^2 \Big)^{1/6} \Big(\sum_{J_2 \in \mathcal{P}(I_2; \delta)} \|f_{J_2}\|_6^2 \Big)^{1/3}$$

Thus, by the definition of $B_1(\delta, \vec{a})$, we conclude:-

<u>Lemma (Key step)</u>:- If $0 \leq a_1 \leq 2a_2$, then

$$B_1(\delta; a_1, a_2) \lesssim B_1(\delta; 2a_2, a_2). \qquad (6)$$

□

This lemma represents the crucial "gain" which lies at the heart of the decoupling theorem.

The remainder of the argument uses the bilinear reduction and iteration / induction procedures to exploit this key lemma.

<u>The iteration procedure</u>.

The first step is obtain from (6) an inequality amenable to iteration. In order to do this, we would like to raise the right hand exponent a_2.

The idea is to swap the exponents:

$$B_1(\delta;\, 2b,\, b)$$
$$\big\downarrow \text{ swap}$$
$$B_1(\delta;\, b,\, 2b)$$
$$\big\downarrow \text{ key step (6)}$$
$$B_1(\delta;\, 4b,\, 2b)$$
$$\big\downarrow \text{ swap}$$
$$B_1(\delta;\, 2b,\, 4b)$$
$$\big\downarrow \text{ key step (6)}$$
$$B_1(\delta;\, 8b,\, 4b)$$
$$\big\downarrow$$
$$\vdots$$

Clearly this process can be repeated indefinitely.

The question is how to make sense of the swap. This is achieved using Cauchy-Schwarz:

$$\int_{\mathbb{R}^2} |f_{I_1}|^2 |f_{I_2}|^4 = \int_{\mathbb{R}^2} \left(|f_{I_2}| \cdot |f_{I_1}|^2\right) \cdot |f_{I_2}|^3$$
$$\leq \left(\int_{\mathbb{R}^2} |f_{I_2}|^2 |f_{I_1}|^4\right)^{1/2} \left(\int_{\mathbb{R}^2} |f_{I_2}|^6\right)^{1/2}.$$

Consequently,

$$B_1(\delta;\, a_1, a_2) \lesssim B_1(\delta;\, a_2, a_1)^{1/2}\, D_{2,6}(\delta^{1-a_2})^{\frac{1}{2}}$$

Thus, we can swap the exponents, but at a cost of introducing a linear decoupling constant. However, the linear decoupling constant is at a larger scale δ^{1-a_2}, by parabolic rescaling.

Thus, the combination of a swap and application of key step (6) can be formalised by :-

Lemma (Iteration)

$$B_1(\delta; 2b, b) \lesssim B_1(\delta; 4b, 2b)^{1/2} D_{2,6}(\delta^{1-b})^{1/2}.$$

Repeated application of this inequality yields

$$B_1(\delta; 2b, b) \lesssim C^N \cdot B_1(\delta; 2^{N+1} b, 2^N b)^{1/2^N} \cdot \prod_{j=1}^{N} D_{2,6}(\delta^{1-2^{j-1}b})^{1/2^j}$$

By Hölder's inequality, we always have the trivial bound

$$B_1(\delta; \vec{a}) \lesssim \delta^{-1/2}$$

which arises from the Cauchy-Schwarz estimate

$$\Big| \sum_{J_r \in \mathcal{P}(I_r;\delta)} f_{J_r} \Big| \lesssim \delta^{-1/2} \Big(\sum_{J_r \in \mathcal{P}(I_r;\delta)} |f_{J_r}|^2 \Big)^{1/2}$$

Thus, if N is chosen large, depending only on ε, so that

$$\frac{1}{2^N} \leq \frac{1}{100} \cdot \varepsilon \cdot Nb, \tag{7}$$

we have

$$B_1(\delta; 2b, b) \lesssim \delta^{-\frac{Nb}{100}\varepsilon} \prod_{j=1}^{N} D_{2,6}(\delta^{1-2^{j-1}b})^{1/2^j}.$$

for $b = 2^{-(N+1)}$. (8)

(note, it is important $b \leq 2^{-(N+1)}$ to make sense of $B_1(\delta, 2^{N+1}b, 2^N b)$).

On the other hand, from the bilinear reduction lemma we have

$$\mathcal{D}_{2,6}(\delta) \lesssim_{\varepsilon} \delta^{-\frac{Nb}{100}\varepsilon} \mathcal{B}_{2,6}(\delta). \qquad (9)$$

Furthermore, by the crude comparison between symmetric and asymmetric decoupling constants

$$\mathcal{B}_{2,6}(\delta) \lesssim \delta^{-O(b)} \mathcal{B}_1(\delta, 2b, b).$$

In particular, we can choose N large, depending only on ε, so that

$$\mathcal{B}_{2,6}(\delta) \lesssim_{\varepsilon} \delta^{-\frac{Nb}{100}\varepsilon} \mathcal{B}_1(\delta; 2b, b) \qquad (10)$$

Combining (8), (9) and (10) yields an iterative inequality for the linear decoupling norm:

$$\mathcal{D}_{2,6}(\delta) \lesssim_{\varepsilon} \delta^{-\frac{\varepsilon}{10}Nb} \prod_{j=1}^{N} \mathcal{D}_{2,6}(\delta^{1-2^{j-1}b})^{1/2^j}. \qquad (11)$$

We now use (11) to complete the argument, arguing by induction-on-scale.
As usual, the desired bound trivially holds for
$\delta_0(\varepsilon) \leq \delta \leq 1$, which serves as a base case for the induction

Inductive step: Suppose whenever $1 \geq \delta' \geq 2\delta$ we have

$$\mathcal{D}_{2,6}(\delta') \leq \overline{C}_{\varepsilon}(\delta')^{-\varepsilon}$$

for some fixed constant $\overline{C}_{\varepsilon} \geq 1$

Once again $\delta_0(\varepsilon)$ and $\bar{C}_\varepsilon$ are chosen in accordance to the requirements of the forthcoming argument.

We can apply the induction hypothesis directly to (11) to give

$$D_{2,b}(\delta) \lesssim_\varepsilon \bar{C}_\varepsilon \, \delta^{-\varepsilon/10Nb} \prod_{j=1}^{N} \delta^{-2^{-j}(1-2^{j-1}b)\varepsilon}$$

The δ-exponent arising from the product is

$$-\sum_{j=1}^{N} 2^{-j}\varepsilon + \frac{1}{2} N \cdot b\varepsilon = -\varepsilon + 2^{-N}\varepsilon + \frac{1}{2} Nb\varepsilon.$$

Thus, we have

$$D_{2,b}(\delta) \lesssim C_\varepsilon \delta^{-\frac{\varepsilon}{2}Nb} \bar{C}_\varepsilon \delta^{-\varepsilon}$$

so provided $0 < \delta \leq \delta_0(\varepsilon)$ for $\delta_0(\varepsilon)$ sufficiently small, depending only on ε,

$$D_{2,b}(\delta) \leq \bar{C}_\varepsilon \delta^{-\varepsilon}$$

and the induction closes. $\square$