Lecture 22: More tools for proving decoupling inequalities.

Bilinear reduction.

In order to apply the observations of the previous lecture, it is necessary to "bilinearise" the decoupling setup.

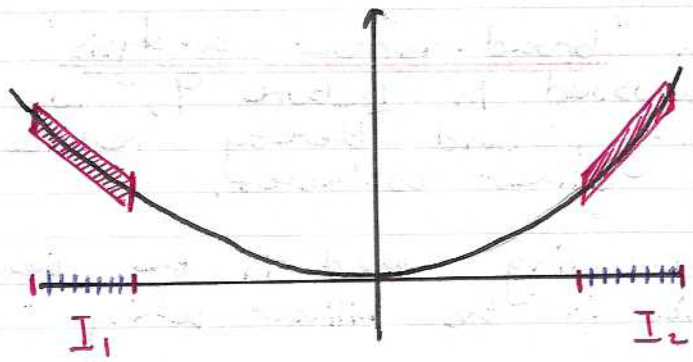
Def^{n} (Bilinear decoupling constants) Let $B_{k,p}(\delta)$ denote the infimum over all $C \geq 1$ for which

$$\Big\| \prod_{r=1}^{2} \Big| \sum_{J_r \in \mathcal{P}(I_r;\delta)} f_{J_r} \Big|^{1/2} \Big\|_{L^p(\mathbb{R}^k)} \qquad (1)$$

$$\leq C \cdot \prod_{r=1}^{2} \Big(\sum_{J_r \in \mathcal{P}(I_r;\delta)} \| f_{J_r} \|^2_{L^p(\mathbb{R}^k)} \Big)^{1/4}.$$

holds for any $I_1, I_2 \in \mathcal{P}(1/4)$ with $\mathrm{dist}(I_1, I_2) \geq 1/4$ and any tuple

$$(f_J)_{J \in \mathcal{P}(I_1;\delta) \cup \mathcal{P}(I_2;\delta)} \quad \text{with } \mathrm{supp}\, \hat{f}_J \subseteq \Theta_J.$$



Writing $f_{I_r} := \sum_{J_r \in \mathcal{P}(I_r;\delta)} f_{J_r}$, in the $k=2$ and $p = \bar{p}_2 = 2 \cdot 3 = 6$ case, the 6-power of the left-hand norm is

$$\int_{\mathbb{R}^2} |f_{I_1}|^3 |f_{I_2}|^3$$

which should be compared with the 'asymmetric' expression

$$\int_{\mathbb{R}^2} |f_{I_1}|^2 |f_{I_2}|^4$$

relevant to the previous discussion in Lecture 21. We will an 'asymmetric' setup later in this lecture, and so the bilinear estimates (1) should be viewed as a stepping stone towards this.

By Cauchy-Schwarz,

$$\Big\| \prod_{r=1}^{2} \Big| \sum_{J_r \in \mathcal{P}(I_r;\delta)} f_{J_r} \Big|^{1/2} \Big\|_{L^p(\mathbb{R}^n)}$$

$$\leq \prod_{r=1}^{2} \Big\| \sum_{J_r \in \mathcal{P}(I_r;\delta)} f_{J_r} \Big\|_{L^p(\mathbb{R}^n)}^{1/2}$$

$$\lesssim D_{k,p}(\delta) \prod_{r=1}^{2} \Big(\sum_{J_r \in \mathcal{P}(I_r;\delta)} \| f_{J_r} \|_{L^p(\mathbb{R}^n)}^2 \Big)^{1/4}$$

and so $B_{k,p}(\delta) \lesssim D_{n,p}(\delta)$.

Using a 'broad-narrow analysis' of the type discussed in Lecture 9, we shall reverse this inequality, and thereby reduce the problem to studying bilinear estimates.

The rescaling used in the linear inequalities also applies in the bilinear setting

Lemma (Scaling): Let $0 < \delta \leq \eta \leq 1$ and $I_1, I_2 \in \mathcal{P}(\eta)$ with

$$\mathrm{dist}(I_1, I_2) \geq \eta. \quad \text{Then}$$

$$\Big\| \prod_{r=1}^{2} \Big| \sum_{J_r \in \mathcal{P}(I_r;\delta)} f_{J_r} \Big|^{1/2} \Big\|_{L^p(\mathbb{R}^n)}$$

$$\lesssim B_{k,p}\Big(\frac{\delta}{\eta}\Big) \prod_{r=1}^{2} \Big(\sum_{J_r \in \mathcal{P}(I_r;\delta)} \| f_{J_r} \|_{L^p(\mathbb{R}^n)}^2 \Big)^{1/4}$$

holds whenever $(f_J)_{J \in \mathcal{P}(I_1;\delta) \cup \mathcal{P}(I_2;\delta)}$ satisfies
$\operatorname{supp} \hat{f}_J \subseteq \Theta_J$. $\square$.

With this, we can implement a 'broad-narrow' dichotomy to prove the following:-

Lemma (Bilinear reduction) For all $\varepsilon > 0$, $p \geq 2$,
$$\mathcal{D}_{k,p}(\delta) \lesssim_\varepsilon \delta^{-\varepsilon} \mathcal{B}_{k,p}(\delta).$$

Proof :- Let $0 < \delta < 1$ and $(f_J)_{J \in \mathcal{P}(\delta)}$ with $\operatorname{supp} \hat{f}_J \subseteq \Theta_J$. Fix $\varepsilon > 0$.

We will work with an additional scale $\delta \ll \eta \ll 1$. Later in the proof we will choose $\eta \sim_\varepsilon 1$, so one may think of η as relatively large.

As usual, write $f_I := \sum_{J \in \mathcal{P}(I;\delta)} f_J$, $I \in \mathcal{P}(\eta)$

so that
$$\sum_{J \in \mathcal{P}(\delta)} f_J = \sum_{I \in \mathcal{P}(\eta)} f_I.$$

For each $x \in \mathbb{R}^k$ fix an interval $I_x \in \mathcal{P}(\eta)$ so that
$$\Big|\sum_{I \in \mathcal{P}(\eta)} f_I(x)\Big| \leq \sum_{I \in \mathcal{P}(\eta)} |f_I(x)|$$
$$= \sum_{\substack{I \in \mathcal{P}(\eta) \\ \operatorname{dist}(I, I_x) < \eta}} |f_I(x)| + \sum_{\substack{I \in \mathcal{P}(\eta) \\ \operatorname{dist}(I, I_x) \geq \eta}} |f_I(x)|.$$

The use of the triangle inequality here appears brutal, but since η is relatively large there are few terms in the sum and it will not hurt us too badly.

Now we choose I_x to satisfy

$$|f_{I_x}(x)| = \max_{I \in \mathcal{P}(\eta)} |f_I(x)|$$

so that

$$\Big|\sum_{I \in \mathcal{P}(\eta)} f_I(x)\Big| \leq 3 \cdot |f_{I_x}(x)| + \sum_{\substack{I \in \mathcal{P}(\eta) \\ \mathrm{dist}(I, I_x) \geq \eta}} |f_I(x)|^{1/2} |f_{I_x}(x)|^{1/2}$$

The dependence of I_x on x is awkward so we relax the bound to

$$\Big|\sum_{I \in \mathcal{P}(\eta)} f_I(x)\Big| \leq 3\Big(\sum_{I \in \mathcal{P}(\eta)} |f_I(x)|^p\Big)^{1/p}$$

$$+ \sum_{\substack{I_1, I_2 \in \mathcal{P}(\eta) \\ \mathrm{dist}(I_1, I_2) \geq \eta}} \prod_{r=1}^{2} |f_{I_r}(x)|^{1/2}$$

The first term on the right-hand side is called the "narrow" part, whilst the second term is the "broad" part of our function.

We now apply L^p-norms in x to deduce that

$$\Big\|\sum_{J \in \mathcal{P}(\delta)} f_J\Big\|_{L^p(\mathbb{R}^n)} = \Big\|\sum_{I \in \mathcal{P}(\eta)} f_I\Big\|_{L^p(\mathbb{R}^n)}$$

$$\leq 3\Big(\sum_{I \in \mathcal{P}(\eta)} \|f_I\|^p_{L^p(\mathbb{R}^n)}\Big)^{1/p}$$

$$+ \sum_{\substack{I_1, I_2 \in \mathcal{P}(\eta) \\ \mathrm{dist}(I_1, I_2) \geq \eta}} \Big\|\prod_{r=1}^{2} |f_{I_r}|^{1/2}\Big\|_{L^p(\mathbb{R}^k_n)}$$

Estimating the narrow part.

By the linear rescaling lemma, for $I \in \mathcal{P}(\eta)$

$$\|f_I\|_{L^p(\mathbb{R}^n)} = \Big\| \sum_{J \in \mathcal{P}(I;\delta)} f_J \Big\|_{L^p(\mathbb{R}^n)}$$

$$\lesssim D_{n,p}(\delta/\eta) \Big(\sum_{J \in \mathcal{P}(I;\delta)} \|f_J\|^2_{L^p(\mathbb{R}^n)} \Big)^{1/2}$$

Taking the ℓ^p sum and using the fact $p \geq 2$,

$$\Big(\sum_{I \in \mathcal{P}(\eta)} \|f_I\|^p_{L^p(\mathbb{R}^n)} \Big)^{1/p} \leq \Big(\sum_{I \in \mathcal{P}(\eta)} \|f_I\|^2_{L^p(\mathbb{R}^n)} \Big)^{1/2}$$

$$\lesssim D_{n,p}(\delta/\eta) \Big(\sum_{J \in \mathcal{P}(\delta)} \|f_J\|^2_{L^p(\mathbb{R}^n)} \Big)^{1/2},$$

which provides a good bound for the narrow part.

Estimating the broad part.

By the bilinear rescaling lemma,

$$\sum_{\substack{I_1, I_2 \in \mathcal{P}(\eta) \\ \mathrm{dist}(I_1, I_2) \geq \eta}} \Big\| \prod_{r=1}^{2} |f_{I_r}|^{1/2} \Big\|_{L^p(\mathbb{R}^n)}$$

$$\lesssim B_{n,p}\Big(\frac{\delta}{\eta}\Big) \sum_{\substack{I_1, I_2 \in \mathcal{P}(\eta) \\ \mathrm{dist}(I_1, I_2) \geq \eta}} \prod_{r=1}^{2} \Big(\sum_{J_r \in \mathcal{P}(I_r;\delta)} \|f_{J_r}\|^2_{L^p(\mathbb{R}^n)} \Big)^{\frac{1}{4}}$$

We bound $B_{n,p}(\delta/\eta) \lesssim B_{n,p}(\delta)$ and relax the summation by dropping the separation condition to bound

$$\lesssim B_{n,p}(\delta)\, \eta^{-1} \sum_{I \in \mathcal{P}(\eta)} \Big(\sum_{J \in \mathcal{P}(I,\delta)} \|f_J\|^2_{L^p(\mathbb{R}^n)} \Big)^{1/2}$$

$$\lesssim \eta^{-3/2} B_{n,p}(\delta) \Big(\sum_{J \in \mathcal{P}(\delta)} \|f_J\|^2_{L^p(\mathbb{R}^n)} \Big)^{1/2}$$

by Cauchy - Schwarz.

Combining the bounds an iterating the estimate

From the above,

$$\Big\| \sum_{J \in \mathcal{D}(\delta)} f_J \Big\|_{L^p(\mathbb{R}^n)} \lesssim \left[D_{k,p}(\delta/\eta) + \eta^{-3/2} B_{n,p}(\delta) \right] \cdot \Big(\sum_{J \in \mathcal{D}(\delta)} \| f_J \|^2_{L^p(\mathbb{R}^n)} \Big)^{1/2}$$

and so, by definition,

$$D_{k,p}(\delta) \leq C\, D_{k,p}(\delta/\eta) + C \cdot \eta^{-3/2} B_{n,p}(\delta)$$

for some constant $C \geq 1$.

The above inequality can be iterated to yield

$$D_{k,p}(\delta) \leq C^N \cdot D_{k,p}(\delta/\eta^N) + N \cdot C^N \cdot \eta^{-3/2} B_{n,p}(\delta).$$

for $N \in \mathbb{N}$.

We choose N so that $\frac{\delta}{\eta^N} \geq \frac{1}{100}$, say. In particular, take

$$N := \left\lceil \frac{\log \delta^{-1}}{\log \eta^{-1}} \right\rceil.$$

Then

$$C^N \sim \delta^{-\frac{\log C}{\log \eta^{-1}}} \leq \delta^{-\varepsilon/2}$$

provided η is chosen sufficiently small, depending only on ε and k.

Thus,

$$D_{n,p}(\delta) \lesssim_\varepsilon \delta^{-\varepsilon/2} D_{n,p}\left(\tfrac{1}{100}\right) + \delta^{-\varepsilon/2} N \cdot \eta^{-3/2} B_{n,p}(\delta)$$

$$\lesssim_\varepsilon \delta^{-\varepsilon} B_{n,p}(\delta) \quad , \text{ as required}$$

since $D_{n,p}\left(\tfrac{1}{100}\right) \lesssim 1$ and $N \lesssim_\varepsilon \delta^{-\varepsilon/2}$; $\eta^{-3/2} \lesssim_\varepsilon 1$.

$\square$

Asymmetric bilinear decoupling

Def$^{\underline{n}}$: (Asymmetric bilinear decoupling constants).
Let $\vec{\alpha} = (\alpha_1, \alpha_2)$, $\vec{a} = (a_1, a_2) \in [0,1]^2$ with $\alpha_1 + \alpha_2 = 1$.

Let $B^{\vec{\alpha}}_{k,p}(\delta; \vec{a})$ denote the infimum over all constants $C \geq 1$ for which

$$\Big\| \prod_{r=1}^{2} \Big| \sum_{J_r \in \mathcal{P}(I_r;\delta)} f_{J_r} \Big|^{\alpha_r} \Big\|_{L^p(\mathbb{R}^k)} \leq C \cdot \prod_{r=1}^{2} \Big(\sum_{J_r \in \mathcal{P}(I_r;\delta)} \| f_{J_r} \|^2_{L^p(\mathbb{R}^k)} \Big)^{\alpha_r/2}$$

whenever:

- $I_r \in \mathcal{P}(\delta^{a_r})$ for $r = 1, 2$
- $\mathrm{dist}(I_1, I_2) \geq 1/4$.
- $\mathrm{supp}\, \hat{f}_J \subseteq \Theta_J$ for all $J \in \mathcal{P}(I_1;\delta) \cup \mathcal{P}(I_2;\delta)$.

Examples:-

- If $\vec{\alpha} = (0,1)$ and $\vec{a} = (a_1, 0)$, then we recover the linear decoupling constant:-

$$B^{(0,1)}_{k,p}(\delta; a_1, 0) = D_{k,p}(\delta)$$

- More generally, by the linear rescaling lemma,

$$B^{(0,1)}_{k,p}(\delta; a_1, a_2) \lesssim D_{k,p}(\delta^{1-a_2}).$$

- If $\vec{\alpha} = (\tfrac{1}{2}, \tfrac{1}{2})$ and $\vec{a} = (0,0)$, then we recover the standard (symmetric) decoupling constant:

$$B^{(\frac{1}{2},\frac{1}{2})}_{k,p}(\delta; 0, 0) \lesssim B_{k,p}(\delta).$$

Remark:-

- The choice of $\vec{\alpha}$ parameter will allow us to consider expressions of the form

$$\int_{\mathbb{R}^2} |f|^2 |g|^4$$

as in the discussion from the previous lecture.

• The choice of $\vec{a} = (a_1, a_2)$ will allow us to work with functions with suitable Fourier localisation properties, again as in the discussion from the previous lecture (recall, there we wanted the Fourier support of f to be much larger than g).

To conclude this lecture we will demonstrate a general comparison result relating asymmetric and symmetric bilinear decoupling constants. This comparison will be rather crude, but nevertheless sufficient for our purposes.

Lemma (Asymmetric vs. Symmetric)

Suppose $p \cdot \min\{\alpha_1, \alpha_2\} \geq 2$. Then

$$B_{k,p}(\delta) \lesssim \delta^{-a_1(p\alpha_1 - 1)/p - a_2(p\alpha_2 - 1)/p} B^{\vec{\alpha}}_{k,p}(\delta; \vec{a})$$

Proof :- Fix $I_1, I_2 \in \mathcal{P}(1/4)$ with $\mathrm{dist}(I_1, I_2) \geq \frac{1}{4}$.

Let $(f_J)_{J \in \mathcal{P}(I_1;\delta) \cup \mathcal{P}(I_2;\delta)}$ satisfy $\operatorname{supp} \hat{f}_J \subseteq \Theta_J$.

and write $f_{I_r} := \sum_{J \in \mathcal{P}(I_r;\delta)} f_J$.

Then for $(\tilde{I}_1, \tilde{I}_2) = (I_2, I_1)$ we have

$$\Big\| \prod_{r=1}^{2} \Big| \sum_{J_r \in \mathcal{P}(I_r;\delta)} f_{J_r} \Big|^{1/2} \Big\|_{L^p(\mathbb{R}^k)}$$

$$= \Big\| \Big(\prod_{r=1}^{2} |f_{I_r}|^{\alpha_r} \Big)^{1/2} \Big(\prod_{r=1}^{2} |f_{\tilde{I}_r}|^{\alpha_r} \Big)^{1/2} \Big\|_{L^p(\mathbb{R}^k)}$$

$$\leq \Big\| \prod_{r=1}^{2} |f_{I_r}|^{\alpha_r} \Big\|^{1/2}_{L^p(\mathbb{R}^k)} \Big\| \prod_{r=1}^{2} |f_{\tilde{I}_r}|^{\alpha_r} \Big\|^{1/2}_{L^p(\mathbb{R}^k)}. \qquad (2)$$

We will show that

$$\Big\|\prod_{r=1}^{2} |f_{I_r}|^{\alpha_r}\Big\|_{L^p(\mathbb{R}^n)} \lesssim \tag{3}$$

$$\Big(\prod_{r=1}^{2} \delta^{-a_r(p\alpha_r-1)/p}\Big)\cdot \mathcal{B}^{\vec{\alpha}}_{k,p}(\delta;\vec{a})\cdot \prod_{r=1}^{2}\Big(\sum_{J_r\in\mathcal{P}(I_r;\delta)} \|f_{J_r}\|^2_{L^p(\mathbb{R}^n)}\Big)^{\frac{\alpha_r}{2}}$$

By symmetry, it then follows that

$$\Big\|\prod_{r=1}^{2} |f_{\tilde{I}_r}|^{\alpha_r}\Big\|_{L^p(\mathbb{R}^n)} \lesssim$$

$$\Big(\prod_{r=1}^{2} \delta^{-a_r(p\alpha_r-1)/p}\Big)\cdot \mathcal{B}^{\vec{\alpha}}_{k,p}(\delta;\vec{a})\cdot \prod_{r=1}^{2}\Big(\sum_{J_r\in\mathcal{P}(I_r;\delta)} \|f_{J_r}\|^2_{L^p(\mathbb{R}^n)}\Big)^{\frac{1-\alpha_r}{2}}$$

and plugging these bounds into (2) yields the desired bound.

To show (3), for $K_r \in \mathcal{P}(I_r;\delta^{a_r})$ let

$$f_{K_r} := \sum_{J_r\in\mathcal{P}(K_r;\delta)} f_{J_r} \quad \text{so that}$$

$$f_{I_r} = \sum_{K_r\in\mathcal{P}(I_r;\delta^{a_r})} f_{K_r}.$$

By Hölder,

$$\Big\|\prod_{r=1}^{2} |f_{I_r}|^{\alpha_r}\Big\|_{L^p(\mathbb{R}^n)} \lesssim$$

$$\prod_{r=1}^{2} \#\mathcal{P}(I_r;\delta^{a_r})^{\alpha_r-1/p}\ \Big\|\prod_{r=1}^{2}\Big(\sum_{K_r\in\mathcal{P}(I_r;\delta^{a_r})} |f_{K_r}|^{\alpha_r p}\Big)^{1/p}\Big\|_{L^p(\mathbb{R}^n)}$$

$$\approx \prod_{r=1}^{2} \delta^{-a_r(p\alpha_r-1)/p}\Big(\sum_{\substack{K_1\in\mathcal{P}(I_1;\delta^{a_1})\\ K_2\in\mathcal{P}(I_2;\delta^{a_2})}} \Big\|\prod_{r=1}^{2} |f_{K_r}|^{\alpha_r}\Big\|^p_{L^p(\mathbb{R}^n)}\Big)^{1/p} \tag{4}$$

By definition,

$$\Big\|\prod_{r=1}^{2} |f_{K_r}|^{\alpha_r}\Big\|_{L^p(\mathbb{R}^n)}$$

$$\leq \mathcal{B}^{\vec{\alpha}}_{k,p}(\delta;\vec{a})\cdot \prod_{r=1}^{2}\Big(\sum_{J_r\in\mathcal{P}(K_r;\delta)} \|f_{J_r}\|^2_{L^p(\mathbb{R}^n)}\Big)^{\alpha_r/2}$$

Plugging this into the $\ell^p L^p$ expression in (4) gives $B^{\vec{\alpha}}_{k,p}(\delta; a)$ times

$$\Big[\sum_{\substack{\underline{K}_1 \in P(I_1, \delta^{a_1}) \\ \underline{K}_2 \in P(I_2; \delta^{a_2})}} \prod_{r=1}^{2} \Big(\sum_{J_r \in P(\underline{K}_r; \delta)} \| f_{J_r} \|^2_{L^p(\mathbb{R}^k)} \Big)^{\frac{\alpha_r p}{2}} \Big]^{\frac{\alpha_r}{\alpha_r p}}$$

$$\leqslant \prod_{r=1}^{2} \Big(\sum_{J_r \in P(I_r; \delta)} \| f_{J_r} \|^2_{L^p(\mathbb{R}^k)} \Big)^{\alpha_r/2}$$

since $\alpha_r p \geqslant 2$ for $r = 1, 2$.

Combining these observations yields (3).

$\square$