Lecture 2

The Uncertainty Principle

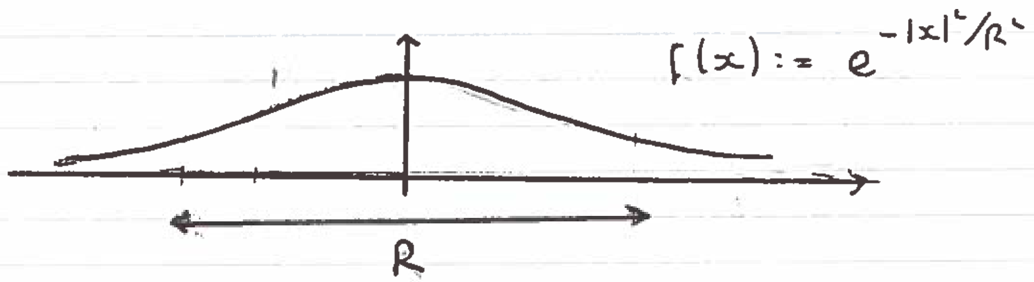
Much of the geometry of the Fourier transform is dictated by the uncertainty principle. For our purposes, this is the following heuristic:-

Uncertainty principle: If f is spatially concentrated at scale R, then $\hat{f}$ is locally constant at scale $1/R$.

Example:- i) For $R \geqslant 1$ let f_R be the centred Gaussian

$$f_R(x) := e^{-|x|^2/R^2}$$

with 'variance' R. Hence f_R is 'spatially concentrated' on $B(0,R)$. For instance, for $\varepsilon > 0$, if $|x| \geqslant R^{1+\varepsilon}$, then $|f(x)| \leqslant C_{\varepsilon,N} R^{-N}$ for all $N \in \mathbb{N}$.



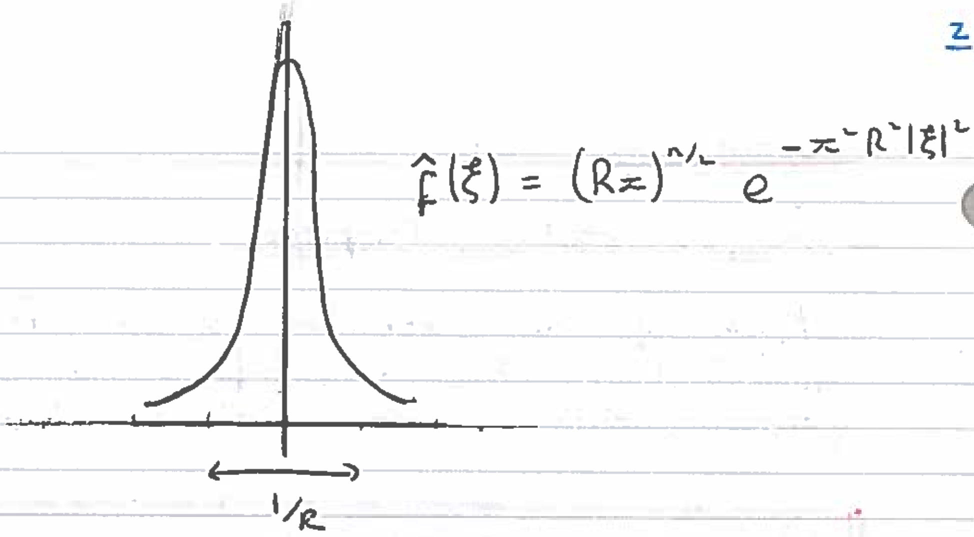
Note that $\hat{f}_R(0) = \int_{\mathbb{R}^n} f = (\pi R)^{n/2}$ so, by the uncertainty principle, we expect

$$\hat{f}_R(\xi) \sim (\pi R)^{n/2} \quad \text{for } |\xi| \leqslant R^{-1}.$$

Indeed, by the well-known formula

$$\hat{f}_R(\xi) = (\pi R)^{n/2} e^{-\pi^2 R^2 |\xi|^2}$$

so this is the case. Furthermore, for $\varepsilon > 0$, if $|\xi| \geqslant R^{-1+\varepsilon}$, then $|\hat{f}_R(\xi)| \leqslant C_{\varepsilon,N} R^{-N}$ for all $N \in \mathbb{N}$ so $\hat{f}_R$ is 'essentially 0' on all other R^{-1} balls.



Thus f_R 'respects the uncertainty principle'.

ii) More generally, let $\phi \in \mathcal{S}(\mathbb{R}^n)$ and define

$$\phi_R(x) := \phi(R^{-1}x).$$

Then $(\phi_R)\hat{}(\xi) = R^n \hat{\phi}(R\xi)$ and it follows that

- ϕ_R is 'concentrated' on $B(0,R)$
- $\hat{\phi}_R$ is 'concentrated' on $B(0,1/R)$.

One manifestation of the uncertainty principle is:

Lemma (Bernstein inequality) For any $1 \le p \le q \le \infty$, if $f \in \mathcal{S}(\mathbb{R}^n)$ (say) with $\operatorname{supp} \hat{f} \subseteq B(0,R)$, then

$$\|f\|_{L^q(\mathbb{R}^n)} \lesssim R^{n(1/p - 1/q)} \|f\|_{L^p(\mathbb{R}^n)}.$$

Idea / heuristic: Since $\operatorname{supp} \hat{f} \subseteq B(0,R)$, by the uncertainty principle f is locally constant on R^{-1}-balls. In particular,

$$f \sim \sum_{k \in \mathbb{Z}^n} c_k \chi_{B(R^{-1}k, R^{-1})}$$

for some $(c_k)_{k \in \mathbb{Z}}$ a complex sequence (as this is just a heuristic, we won't worry about the ball overlaps).

Thus, $$\|f\|_{L^q(\mathbb{R}^n)} \sim \Big(\sum_{k \in \mathbb{Z}^n} |c_k|^q \, |B(R^{-1}k, R^{-1})|\Big)^{1/q}$$

$$\sim R^{-n/q} \, \|(c_k)\|_{\ell^q(\mathbb{Z}^n)}$$

$$\lesssim R^{n(1/p - 1/q)} R^{-n/p} \|(c_n)\|_{\ell^p(\mathbb{Z}^n)}$$

$$\sim R^{n(1/p-1/q)} \|f\|_{L^p(\mathbb{R}^n)}$$

by the nesting of the ℓ^p-spaces. □.

Proof :- It is possible to implement an argument via ℓ^p as in the heuristic, but here we give a cleaner proof.

Fix $\phi \in \mathcal{S}(\hat{\mathbb{R}}^n)$ with $\phi(\xi) = 1$ for $|\xi| \leq 1$ and let $\phi_R(\xi) := \phi(R^{-1}\xi)$. Then

$$\hat{f}(\xi) = \hat{f}(\xi) \cdot \phi_R(\xi)$$

by the support condition. Consequently,

$$f(x) = f * (\phi_R)^\vee (x)$$

We now bound $\|f\|_{L^q(\mathbb{R}^n)}$ using the classical Young's inequality. In particular, if

$$1 - 1/r = 1/p - 1/q, \quad \text{then}$$

$$\|f\|_{L^q(\mathbb{R}^n)} \leq \|(\phi_R)^\vee\|_{L^r(\mathbb{R}^n)} \cdot \|f\|_{L^p(\mathbb{R}^n)}$$

$$\sim R^{n(1-1/r)} \|f\|_{L^p(\mathbb{R}^n)}$$

$$= R^{n(1/p-1/q)} \|f\|_{L^p(\mathbb{R}^n)}$$

since $(\phi_R)^\vee(x) = R^n(\phi)^\vee(Rx)$. □.

We can also prove a local version of this result, where the $L^{q/p}$ norms are taken over a ball B_r, provided the scale r respects the uncertainty principle.

Def$^{\underline{n}}$:- For $B_r = B(\bar{x}, r)$ an r-ball in $\mathbb{R}^n$ and $N \in \mathbb{N}$ define the weight

$$w_{B_r}^N(x) := (1 + r|x - \bar{x}|)^{-N}.$$

Note that $\chi_{B_r} \leq w_{B_r}^N$ - we think of the weight as an 'approximate characteristic function'.

Corollary (Local Bernstein Inequality):- Let $1 \le p \le q \le \infty$ and $R^{-1} \le r$. If $f \in \mathcal{S}(\mathbb{R}^n)$ with $\operatorname{supp} \hat{f} \subseteq B(0,R)$, then

$$\|f\|_{L^q(B_r)} \lesssim_N R^{n(1/p - 1/q)} \|f\|_{L^p(w_{B_r}^N)}.$$

Remark:- This is an example of an estimate with 'Schwartz tails' — we are unable to perfectly localise on each side of the inequality.

For example, let $g \in \mathcal{S}(\mathbb{R}^1)$ with

$g(0) = 1$ and $\operatorname{supp} \hat{g} \subseteq [-1,1]$.

For $N \in \mathbb{N}$ define $f_N(x) := (1-x^2)^N g(x)$ so that each $\hat{f}_N$ is a linear comb. of derivatives of $\hat{g}$ and so $\operatorname{supp} \hat{f}_N \subseteq [-1,1]$ for all $N \in \mathbb{N}$.

Note $\|f_N\|_{L^\infty([-1,1])} \geq |f_N(0)| = |g(0)| = 1$

but $f_N(x) \to 0$ as $N \to \infty$ for all $x \in [-1,1] \setminus \{0\}$. Thus, there can be no uniform estimate

$$\|f_N\|_{L^\infty([-1,1])} \lesssim \|f_N\|_{L^p([-1,1])}$$

for $1 \le p < \infty$. $\square$.

Proof (of local Bernstein):- Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ satisfy

$$\operatorname{supp} \hat{\phi} \subseteq B(0,1) \quad \text{and} \quad |\phi(x)| \gtrsim 1 \text{ for } |x| \le 1$$

(Exercise: show that such a ϕ exists!)

Define $\phi_r(x) := \phi(r^{-1}(x - \bar{x}))$ so that

$\chi_{B_r}(x) \lesssim |\phi_r(x)|^q$. (here $B_r = B(\bar{x}, r)$).

Hence $\|f\|_{L^q(B_r)} \lesssim \|f \cdot \phi_r\|_{L^q(\mathbb{R}^n)}$.

Now $(f \cdot \phi_r)^\wedge = \hat{f} * (\phi_r)^\wedge$ and

$$\operatorname{supp} \hat{f} * (\phi_r)^\wedge \subseteq \operatorname{supp} \hat{f} + \operatorname{supp} (\phi_r)^\wedge \subseteq B(0,R) + B(0, r^{-1}) \subseteq B(0, 2R).$$

Hence, by Bernstein's inequality,

$$\|f \cdot \phi_r\|_{L^q(\mathbb{R}^n)} \lesssim R^{n(1/p - 1/q)} \|f \cdot \phi_r\|_{L^p(\mathbb{R}^n)}$$

Since ϕ_r is rapidly decaying away from B_r we have

$$|\phi_r|^p \lesssim_N w_{B_r}^N$$ and the desired

estimate follows. $\square$.

Affine extensions:-

More general versions of the uncertainty principle deal with "anisotropic" localisation.

> **(Affine) Uncertainty Principle**: If f is spatially concentrated on a centred ellipse E, then $\hat{f}$ is locally constant on translates of the dual ellipse E^*.

Here an ellipse E is a set of the form

$$E := \left\{ x \in \mathbb{R}^n : \sum_{j=1}^{n} \left(\frac{|\langle x, \vec{e}_j \rangle|}{r_j} \right)^2 \leq 1 \right\} + x_0 \qquad (1)$$

where:
- $x_0 \in \mathbb{R}^n$ is the centre of E
- $\{\vec{e}_1, \dots, \vec{e}_n\}$ forms an ON basis of $\mathbb{R}^n$ and are called the axes of E
- $r_1, \dots, r_n$ are the axis lengths

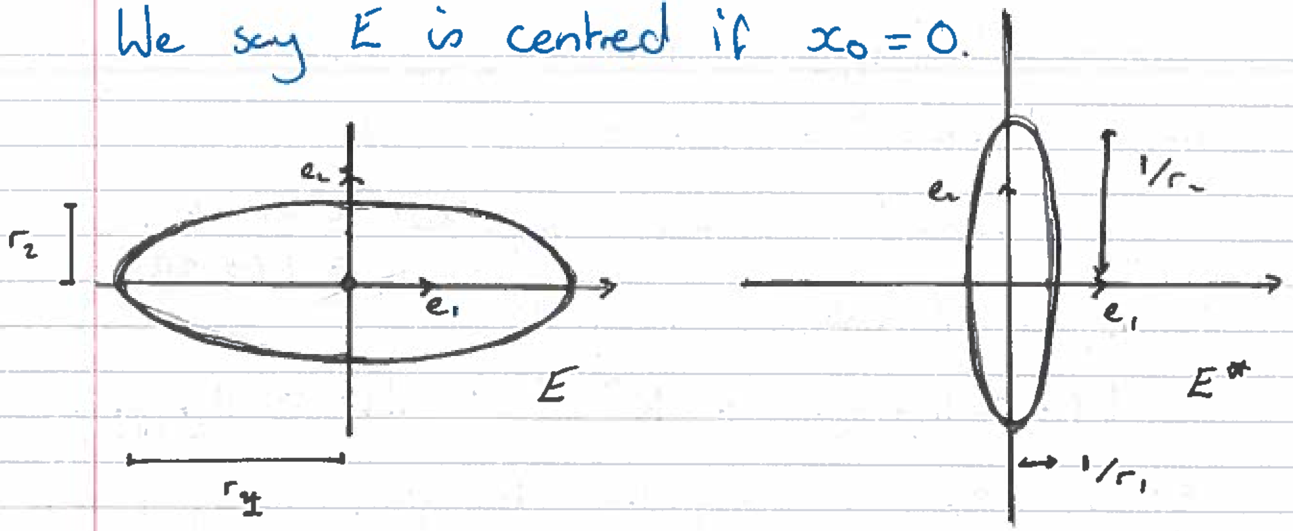
The dual ellipse E^* is defined by

$$E^* := \left\{ \xi \in \mathbb{R}^n : \sum_{j=1}^{n} (r_j \langle x, e_j \rangle)^2 \leq 1 \right\}$$

so E^* has:-
- centre 0
- axes $\{\vec{e}_1, \dots, \vec{e}_n\}$
- axis lengths $1/r_1, \dots, 1/r_n$.

μ

We say E is centred if $x_0 = 0$.



- Every ellipse is given by the image of the unit ball under an affine transformation :-

Let $O := \begin{bmatrix} - & \vec{e}_1 & - \\ & \vdots & \\ - & \vec{e}_n & - \end{bmatrix}$ (so O is orthogonal)

$$D := \begin{bmatrix} 1/r_1 & & 0 \\ & \ddots & \\ 0 & & 1/r_n \end{bmatrix}$$

and note that $A := D \cdot O$ satisfies

$$|Ax|^2 = \sum_{j=1}^{n} \left(\frac{|\langle x, e_j \rangle|}{r_j} \right)^2 . \qquad (2)$$

Thus, E defined in (1) is given by

$$E = A^{-1} B(0,1) + x_0 .$$

On the other hand,

$$E^* = (A^*)^{-1} B(0,1)$$

where $A^* = D^{-1} \cdot O$. Note that $A^* = A^{-T}$ is the inverse of the transpose (or transpose of the inverse) of A since

$$A^* = D^{-1} O = (O^{-1} D)^{-1} = (O^T D)^{-1} = (D\,O)^{-T}$$
$$= A^{-T}$$

since $O^{-1} = O^T$ by O orthogonal and
$D^T = D$ by D diagonal.

- On the other hand, if $f \in \mathcal{J}(\mathbb{R}^n)$, then for any $A \in GL(n, \mathbb{R})$

$$(f \circ A)^\wedge(\xi) = \frac{1}{|\det A|} \hat{f} \circ A^*(\xi)$$

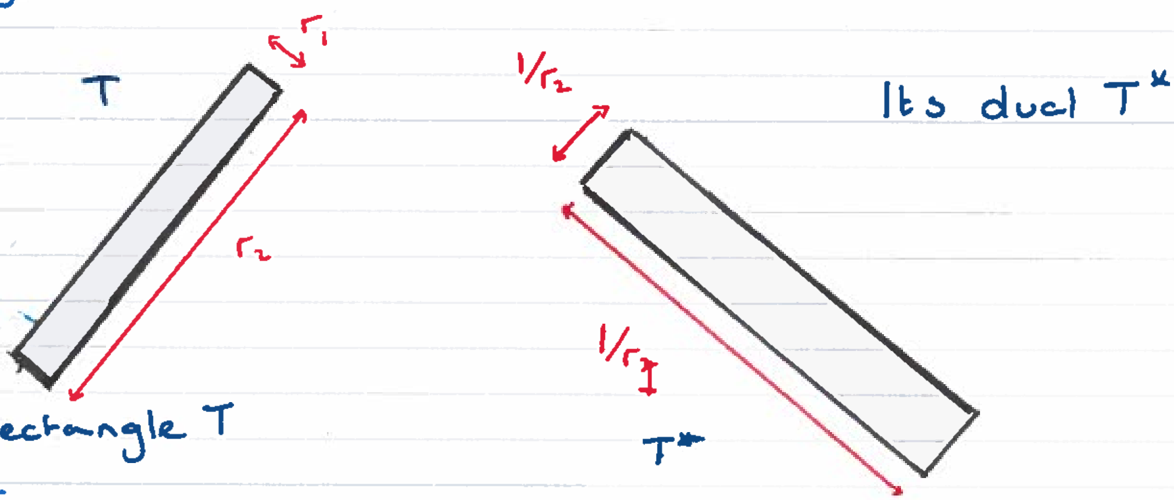
where $A^* = A^{-T}$.

In particular, if $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\phi(0) = 1$ and A is the matrix featured from (2), then

- $\phi \circ A$ is concentrated on $A^{-1}(B(0,1)) = E$ (for $x_0 = 0$)
- $(\phi \circ A)^\wedge = |\det A|^{-1} \hat{\phi} \circ A^*$ is concentrated on $(A^*)^{-1} B(0,1) = E^*$.

__Exercise:-__ Use these observations to state and prove an appropriate affine extension of the Bernstein inequalities above.

__Remark:-__ We'll often apply the uncertainty principle to rectangles / cylinders rather than ellipses, with duality interpreted in the obvious way



A rectangle T

From a convex geometry perspective, ellipses are perhaps more natural, but rectangles can be thought of as arising from the 1-d uncertainty principle and tensorising.