Lecture 18 : Cone Square Function VI

Geometric lemmas

To conclude the proof of the L^4 square function bound, it remains to verify the two key geometric lemmas:-

Lemma 1 : If $\left|\sum_{\theta \in \mathcal{S}_{\bar{\Theta}}} (|f_\theta|^2)^\wedge(\xi)\right| \neq 0$ for some

$$\xi \in \Omega_{\leq h} \quad \text{and} \quad \bar{\Theta} \in \mathbb{CP}_h$$

then $\xi \in 4 \cdot \bar{\Theta}$.

Lemma 2 :- If $\xi \in \Omega_h$, then

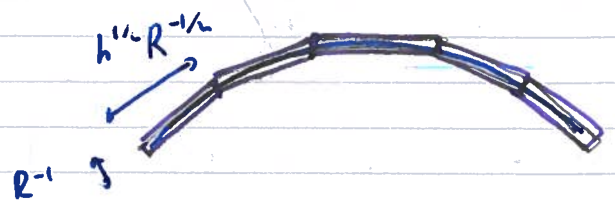
$$\#\{\bar{\Theta} \in \mathbb{CP}_h : \xi \in 4 \cdot \bar{\Theta}\} \lesssim 1.$$

This is a fairly straightforward unpacking of the definitions.

Geometry of the $\Omega_{\leq h}$ and Ω_h sets.

Recall: $\Omega_{\leq h} := \bigcup_{\bar{\Theta} \in \mathbb{CP}_h} \bar{\Theta}$. We consider cross-sections $\Omega_{\leq h} \cap \{\xi_3 = t\}$ for $|t| \leq h$.

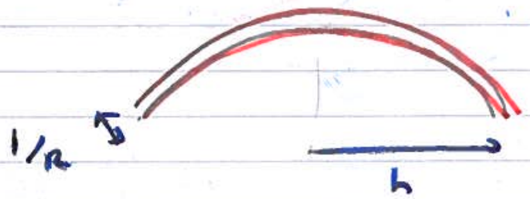
$$\left\{ \begin{pmatrix} s^2/2t \\ s \\ t \end{pmatrix} : |s| \leq t \right\}$$



Recall, for $t \sim h$ the $\bar{\Theta} \in \mathbb{CP}_h$ were defined so that $\bar{\Theta} \cap \{\xi_3 = t\}$ forms a plate / slab decomposition of $\Gamma_{\text{whole}} \cap \{\xi_3 = t\}$

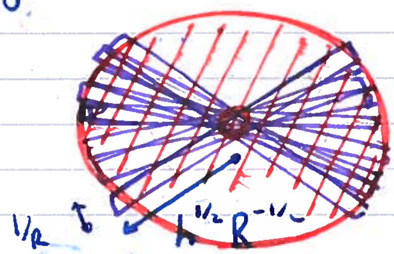
(the $h^{1/2} R^{-1/2}$ length of the plank is chosen to "compensate" for the curvature $\sim h^{-1}$ of $\Gamma_{\text{whole}} \cap \{\xi_3 = t\}$).

$|t| \sim h$ Thus, in this case $\Omega_{\le h} \cap \{\xi_3 = t\}$ looks like a R^{-1}-neighborhood of $h \cdot \left\{ \begin{pmatrix} s^2/2h^2 \\ s/h \\ 1 \end{pmatrix} : |s| \le 1 \right\}$.



$t \sim h$

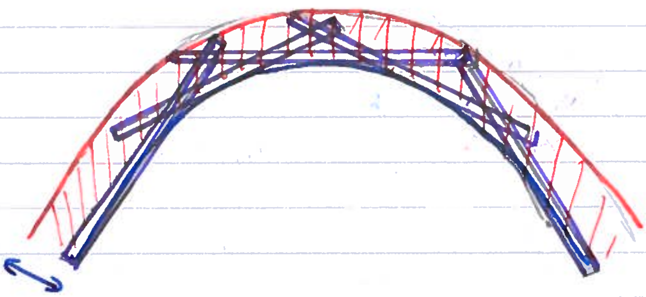
At the other extreme, if $|t| \lesssim 1/R$, then all the slabs in $\Omega_{\le h} \cap \{\xi_3 = t\}$ are focused at the origin



$|t| \lesssim h^{1/2}/R^{1/2}$.

$|t| \lesssim 1/R$ Thus, in this case $\Omega_{\le h} \cap \{\xi_3 = t\}$ fills up a portion of the ball $B(0, h^{1/2} R^{-1/2})$

$1/R \lesssim |t| \lesssim h$ <u>Intermediate cases</u>:– In general, the slabs in $\Omega_{\le h} \cap \{\xi_3 = t\}$ for $|t| \ll h$ are <u>not</u> well adapted to the $\Gamma_{\text{whole}} \cap \{\xi_3 = t\}$ curve:–

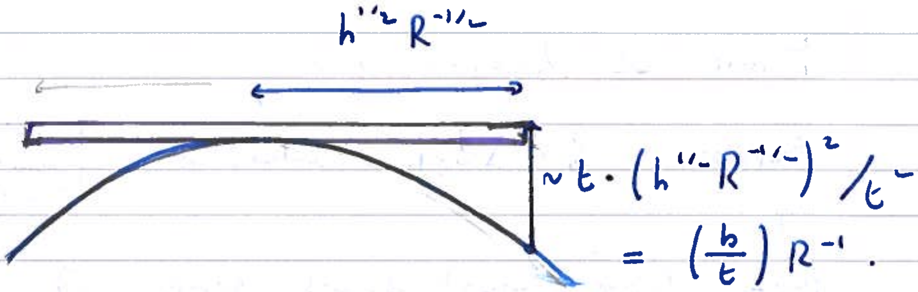


$\left\{ \begin{pmatrix} s^2/2t \\ s \\ t \end{pmatrix} : |s| \le t \right\}$

In particular, their union fills out a neighborhood of

$$\Gamma_{\text{whole}} \cap \{\xi_3 = t\} = t \cdot \left\{ \begin{pmatrix} s^2/t^2 \\ s/t \\ 1 \end{pmatrix} : |s| \le 1 \right\}$$

of size $\left(\frac{h}{t}\right) R^{-1}$.



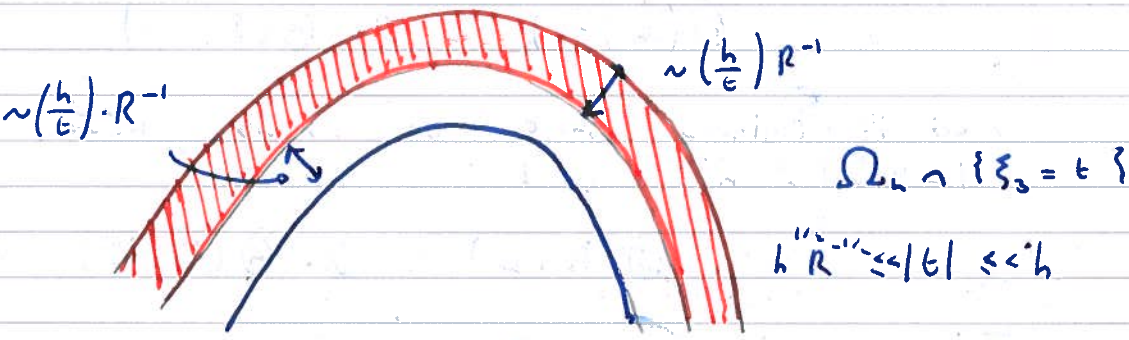
We also obtain a picture for the Ω_h sets.

Recall: $\Omega_h := \Omega_{\le h} \setminus \Omega_{\le h/2}$.

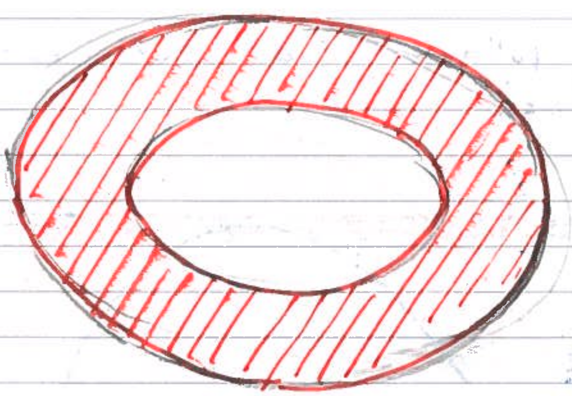
The slices of Ω_h agree with $\Omega_{\le h}$ at heights $\frac{h}{4} \le |t| \le \frac{h}{2}$, say.

For smaller heights $|t| \le h/4$,

$\Omega_h \cap \{\xi_3 = t\}$ essentially consists of $\Omega_{\le h} \cap \{\xi_3 = t\}$ with the inner half portion removed:-



$\left\{\begin{pmatrix} s^2/2t \\ s \\ t \end{pmatrix} : |s| \le t\right.$



$|t| \sim h^{1/2}R^{-1/2}$.

(Schematic!).

Proof of Lemma 1 :-

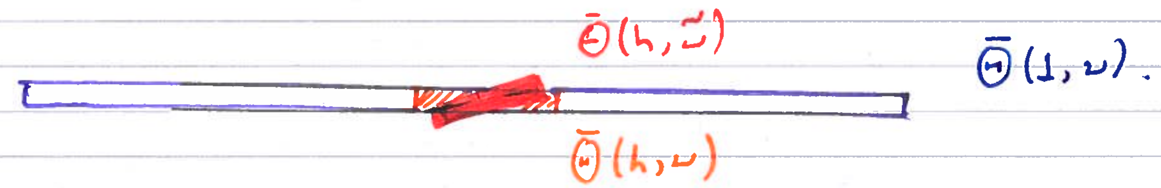
Suppose $\xi \in \Omega_{\leq h}$ satisfies $\left|\sum_{\theta \in S_{\bar{\Theta}}} (|f_\theta|^2)^\wedge(\xi)\right| \neq 0$.

Then there must exist some $\theta = \theta(1, \nu) \in S_{\bar{\Theta}}$ such that

$$\xi \in \theta - \theta \subseteq \bar{\Theta}(1, \nu).$$

Let our fixed $\bar{\Theta} \in \mathbb{CP}_h$ be given by $\bar{\Theta} := \bar{\Theta}(h, \tilde{\nu})$ so that, by definition,

$$\bar{\Theta}(h, \tilde{\nu}) \subseteq 2 \cdot \bar{\Theta}(1, \nu)$$



On the other hand,

$$\bar{\Theta}(h, \nu) \subseteq \bar{\Theta}(1, \nu)$$

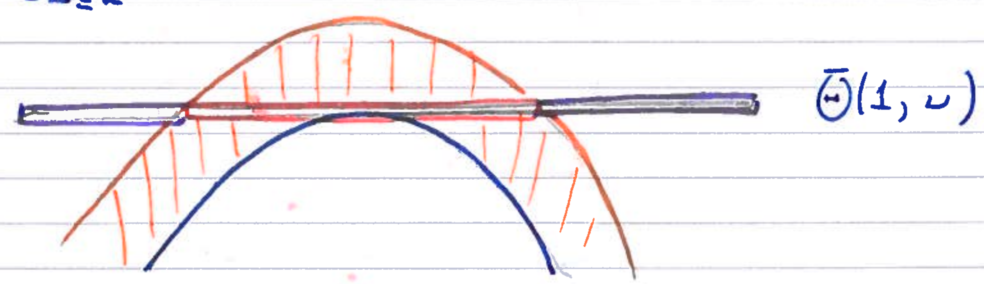
and it follows that we must have $|\nu - \tilde{\nu}| \lesssim h^{-1/2}$ and

$$\bar{\Theta}(h, \nu) \subseteq 2 \cdot \bar{\Theta}(h, \tilde{\nu}) = 2 \cdot \bar{\Theta}$$

Thus, it suffices to show if

$$\xi \in \Omega_{\leq h} \cap \bar{\Theta}(1, \nu), \text{ then } \xi \in 2 \cdot \bar{\Theta}(h, \nu).$$

But this follows from our earlier observations regarding $\Omega_{\leq h}$



Indeed, for any $|t| \leq h$, $\bar{\Theta}(1, \nu) \cap \{\xi_3 = t\}$ will essentially intersect $\Omega_{\leq k} \cap \{\xi_3 = t\}$ along a

$R^{-1} \times h^{1/2} R^{-1/2}$ rectangle which essentially corresponds to $\bar{\Theta}(h, \nu) \cap \{\xi_3 = t\}$. $\square$

Proof of Lemma 2:-

If we consider the portions of the planks

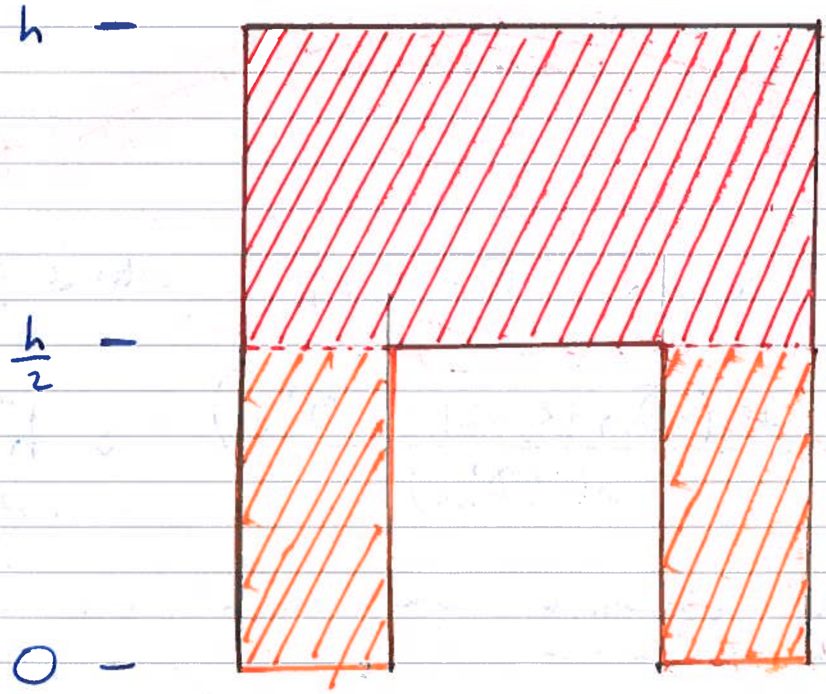
$$\bar{\Theta} \cap \mathbb{R}^2 \times [\tfrac{h}{2}, h] \qquad \bar{\Theta} \in \mathbb{CP}_h,$$

these sets form a plank decomposition of $\Gamma(h)$ and so are essentially disjoint.

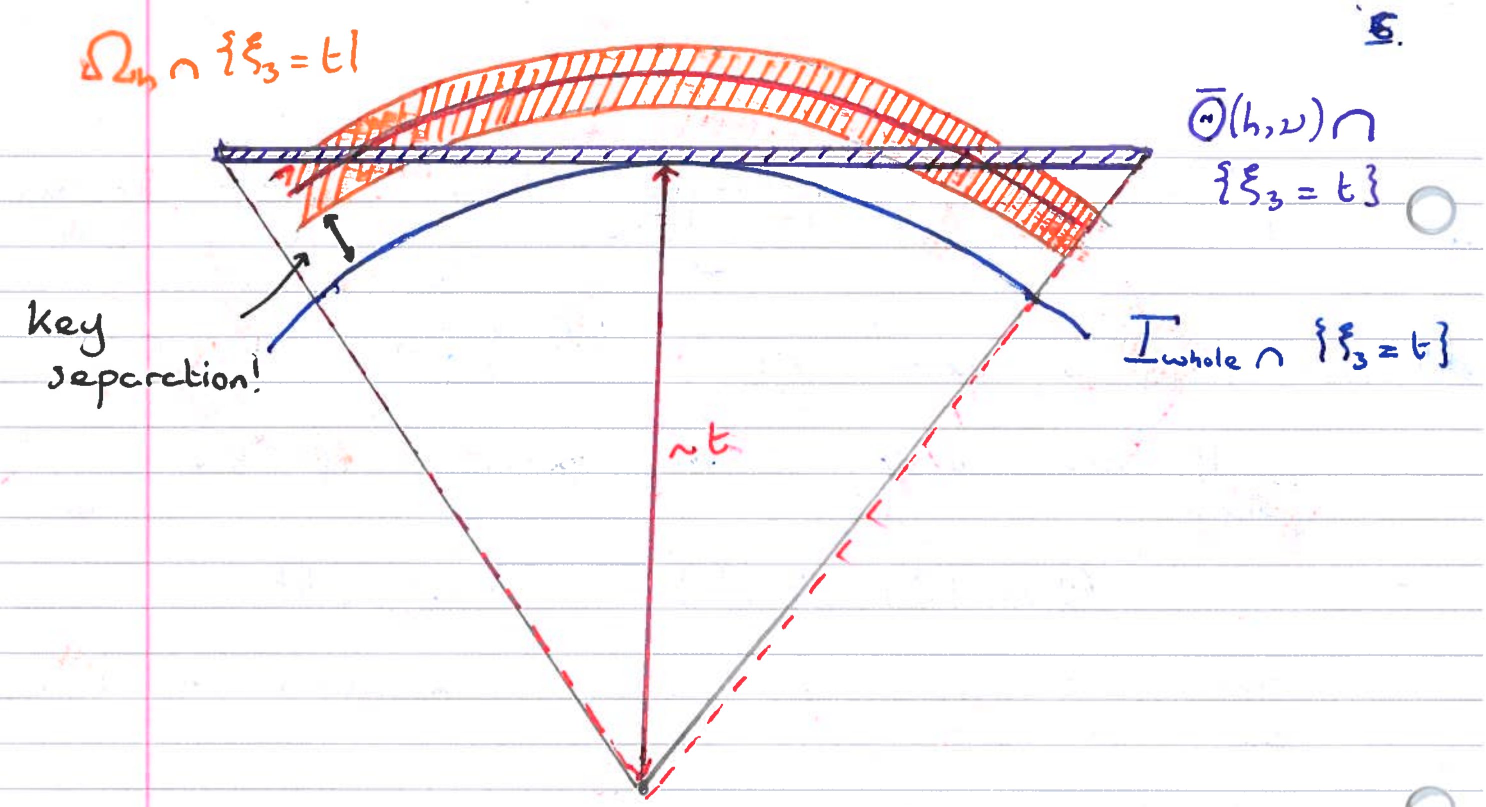
It therefore remains to show the sets

$$\bar{\Theta} \cap \mathbb{R}^2 \times [0, \tfrac{h}{2}] \qquad \bar{\Theta} \in \mathbb{CP}_h$$

are essentially disjoint on Ω_h. The restriction to $\mathbb{R}^2 \times [0, \frac{h}{2}]$ corresponds to considering the "leg" regions in the schematic for Ω_h :-

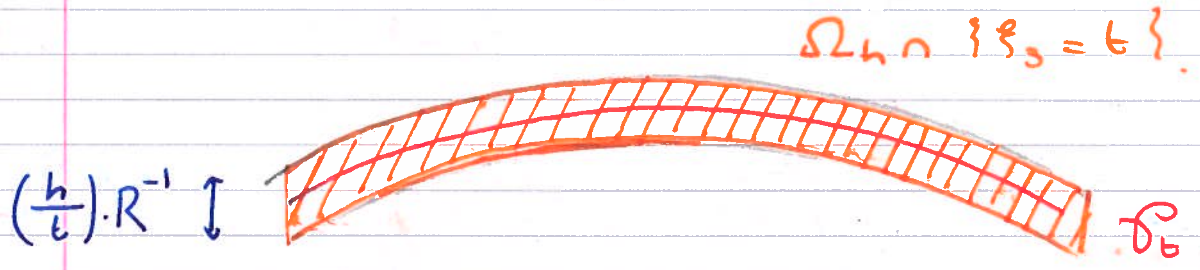


Moreover, it corresponds to the case where the t-cross-section of Ω_h is seperated from $\Gamma_{whole} \cap \{\xi_3 = t\}$:-



The set $\Omega_h \cap \{\xi_3 = t\}$ can essentially be foliated into parabolae $\mathcal{P}_{t,e} = \mathcal{P}_t + e\,\vec{e}_1$ where

- e varies over an interval I_t of length $\sim \left(\frac{h}{t}\right)R^{-1}$
- $\mathcal{P}_t$ has length $\sim \max\{|t|, h^{1/2}R^{-1/2}\}$



Claim:- Given $\bar{\Theta} \in CP_h$, $|t| \leq \frac{h}{2}$

$$\frac{\mathcal{H}^1(4\bar{\Theta} \cap \{\xi_3 = t\} \cap \mathcal{P}_{t,e})}{\mathcal{H}^1(\mathcal{P}_{t,e})} \lesssim h^{-1/2}R^{-1/2} \quad \text{for } e \in I_t$$

Once we establish the claim, we may argue as follows:-

Integrating in $e \in I_t$,

$$|4\bar{\Theta} \cap \{\xi_3 = t\} \cap \Omega_h| \lesssim h^{-1/2}R^{-1/2}\,|\Omega_h \cap \{\xi_3 = t\}|$$

and integrating in $|t| \leq h/2$

$$|4\bar{\Theta} \cap \mathbb{R}^2 \times [0, h/2] \cap \Omega_h| \lesssim h^{-1/2} R^{-1/2} |\Omega_h|$$

Write $\bar{\Theta}_T := \bar{\Theta} \cap \mathbb{R}^2 \times [0, h/2]$ for these truncations.

Thus

$$\int_{\Omega_h} \sum_{\bar{\Theta} \in \mathbb{CP}_h} \chi_{4\bar{\Theta}_T} = \sum_{\bar{\Theta} \in \mathbb{CP}_h} |4\bar{\Theta}_T \cap \Omega_h|$$

$$\lesssim |\Omega_h|$$

since $\# \mathbb{CP}_h \sim h^{+1/2} R^{+1/2}$ (recall:- planks have length $h^{1/2} R^{-1/2}$ and they have to cover a curve of length h => there are $h^{1/2} R^{1/2}$ planks).

Consequently,

$$\frac{1}{|\Omega_h|} \int_{\Omega_h} \sum_{\bar{\Theta} \in \mathbb{CP}_h} \chi_{4.\bar{\Theta}_T} \lesssim 1$$

so on average, for $\xi \in \Omega_h$

$$\# \{ \bar{\Theta} \in \mathbb{CP}_h : \xi \in 4.\bar{\Theta}_T \} \lesssim 1$$

But, by the symmetry of the setup the function

$$\sum_{\bar{\Theta} \in \mathbb{CP}_h} \chi_{4.\bar{\Theta}_T} \quad \text{should be roughly}$$

constant on Ω_h, so we conclude the desired result. $\square$

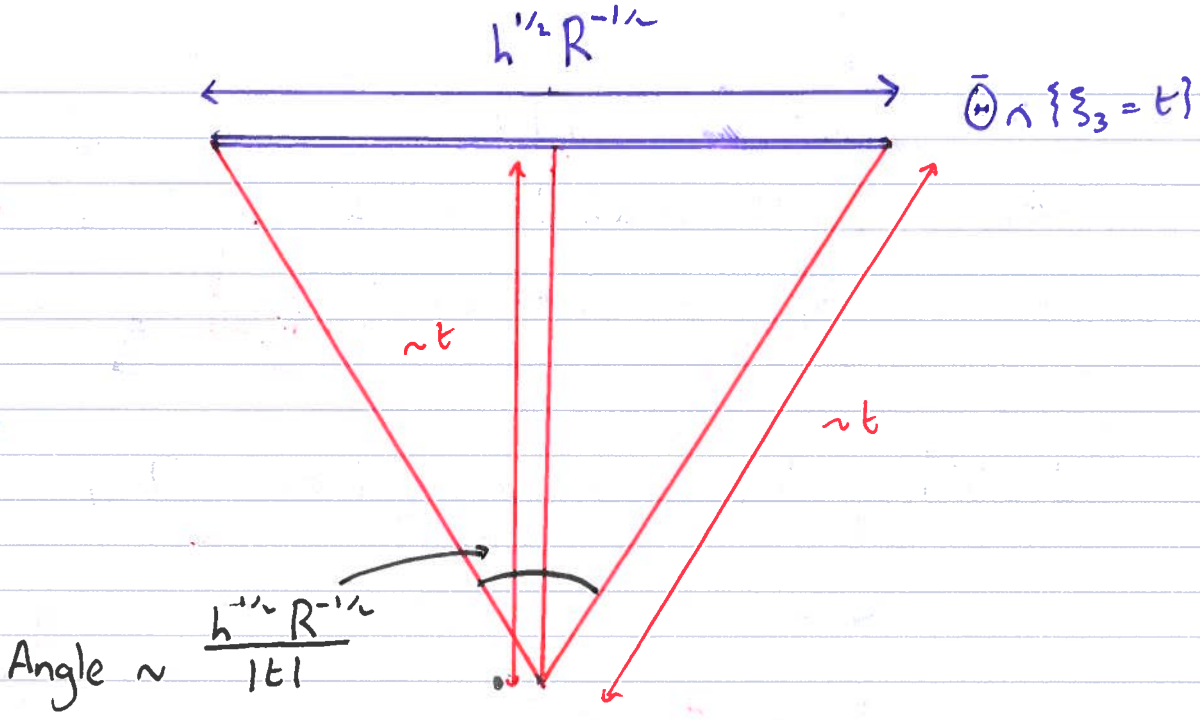
Proof of Claim:-

Case 1:- $\frac{h}{3} \geq |t| \geq h^{1/2} R^{-1/2}$.

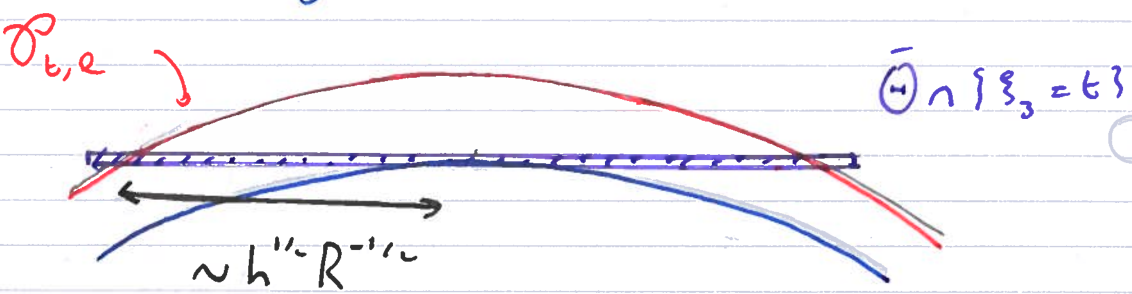
In this case, if we fix $\bar{\Theta} \in \mathbb{CP}_h$ and consider two points at the ends of the rectangle

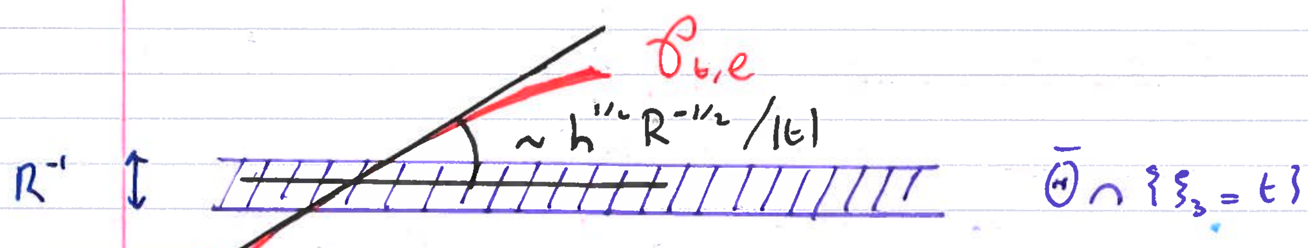
$$\bar{\Theta} \cap \{ \xi_3 = t \}$$

then the angle these points make with the origin is $\sim \frac{h^{1/2} R^{-1/2}}{|t|}$.



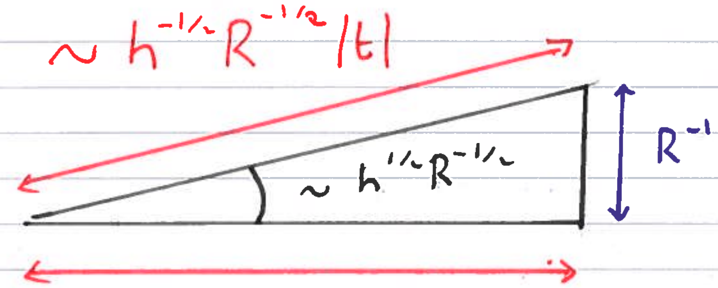
Because of the key separation between $I_{whole} \cap \{\xi_3 = t\}$ and $\Omega_h \cap \{\xi_3 = t\}$ for $|t| \le \frac{h}{2}$, it follows that the tangent to $\mathcal{P}_{t,e}$ makes an angle of $\sim \frac{h^{1/2}R^{-1/2}}{|t|}$ with the rectangle $\bar{\Theta} \cap \{\xi_3 = t\}$.





Thus, $\mathcal{H}^1(\mathcal{P}_{t,e} \cap 4\bar{\Theta} \cap \{\xi_3 = t\}) \lesssim \frac{R^{-1}|t|}{h^{1/2}R^{-1/2}}$

$= h^{-1/2}R^{-1/2}|t|$.



On the other hand, $\mathcal{H}^1(\mathcal{P}_{t,e}) \sim |t|$ and so

$$\frac{\mathcal{H}^1(\mathcal{P}_{t,e} \cap 4\bar{\Theta} \cap \{\xi_3 = t\})}{\mathcal{H}^1(\mathcal{P}_{t,e})} \lesssim h^{-1/2} R^{-1/2},$$

as required.

Case 2: $\frac{h}{2} \gtrsim |t|$ and $h^{1/2} R^{-1/2} \gtrsim |t|$.

In this case $\mathcal{P}_{t,e}$ intersects $\bar{\Theta} \cap \{\xi_3 = t\}$ at an angle ~ 1 and so

$$\mathcal{H}^1(\mathcal{P}_{t,e} \cap 4\cdot\bar{\Theta} \cap \{\xi_3 = t\}) \lesssim R^{-1}$$

giving a bound of

$$\frac{\mathcal{H}^1(\mathcal{P}_{t,e} \cap 4\cdot\bar{\Theta} \cap \{\xi_3 = t\})}{\mathcal{H}^1(\mathcal{P}_{t,e})} \lesssim (h^{1/2}R^{-1/2})^{-1} R^{-1} = h^{-1/2} R^{-1/2},$$

as required. Here we used the fact that

$$\mathcal{H}^1(\mathcal{P}_{t,e}) \sim h^{1/2}R^{-1/2} \text{ in this case.}$$

$\square$