Lecture 17: Cone square Function V

Centred planks

Recall, last time we introduced plank decompositions of the dyadic truncates $\Gamma(h)$ of Γ_{whole}, which here we will denote by

$$\mathcal{S}(h) := \{ \theta(h,\nu) : \nu \in \mathbb{Z}_{h^{1/2}R^{1/2}} \}$$

In particular, when $h=1$, $\mathcal{S} := \mathcal{S}(1)$ is our original plank decomposition of $\Gamma = \Gamma(1)$.

Given $\theta \in \mathcal{S}$, we also considered the centred plank $\bar{\Theta}$ which contains the Minkowski difference $\theta - \theta$. The rationale for this is that

$$\operatorname{supp} (|f_\theta|^2)^\wedge \subseteq \bar{\Theta}. \qquad (1)$$

The next step is to consider centred planks at all scales h. In particular, let

$$\bar{\Theta}(h,\nu) := \{ \xi \in \hat{\mathbb{R}}^3 : |\langle \vec{c}(\nu), \xi \rangle| \le \frac{h}{2}, \ |\langle \vec{n}(\nu), \xi \rangle| \le 2R^{-1},$$
$$|\langle \vec{t}(\nu), \xi \rangle| \le 2 h^{1/2} R^{-1/2} \}$$

so that $\theta(h,\nu) - \theta(h,\nu) \subseteq \bar{\Theta}(h,\nu)$. The collection of scale h centred planks is denoted

$$\mathbb{CP}_h := \{ \bar{\Theta}(h,\nu) : \nu \in \mathbb{Z}_{h^{1/2}R^{1/2}} \}.$$

We will investigate relationships between these centred planks at different scales.

Geometric observation :-

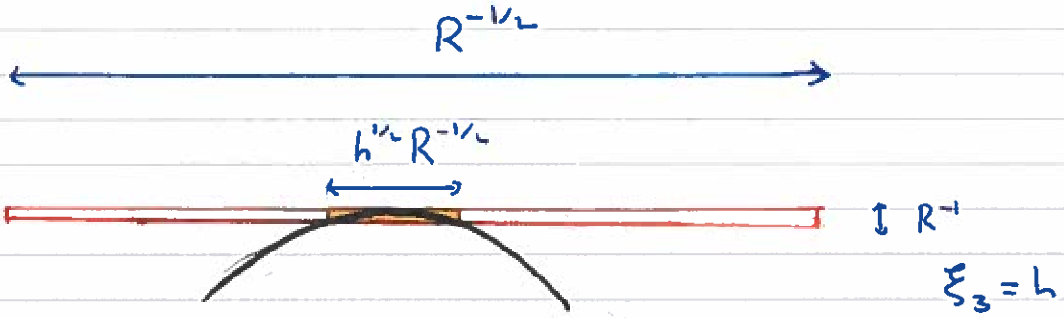
a) Given $\bar{\Theta}(1,\nu) \in \mathbb{CP}_1$, there exists $\bar{\Theta}(h,\nu_h) \in \mathbb{CP}_h$ such that

$$\bar{\Theta}(h,\nu_h) \subseteq 2 \cdot \bar{\Theta}(1,\nu)$$

b) If $\tilde{\nu} \in \mathbb{Z}_{R^{1/2}}$ satisfies $|\nu - \tilde{\nu}| \gtrsim h^{-1/2}_{,} P$

then $2\cdot\Theta(h,\nu_h) \cap 2\cdot\overline{\Theta}(1,\tilde{\nu}) = \emptyset$.

Proof :- a) This is easy to see by considering slices at $\xi_3 = \pm h$.



On this slice, the large centred plate $\Theta(1,\nu)$ intersects $N_{1/R}\,\Gamma(h)$ only on a small portion, corresponding to a block of dimension $h^{1/2}R^{-1/2} \times R^{-1}$

This $h^{1/2}R^{-1/2} \times R^{-1}$ block will intersect some $\Theta(h,\nu_h) \cap \{\xi_3 = h\}$ for some $\nu_h \in \mathcal{Z}_{h^{1/2}R^{1/2}}$. By symmetry & convexity, it follows that

$$\Theta(h,\nu_h) \subseteq 2\cdot\Theta(1,\nu)$$

as required.

b) Let $\frac{h}{2} \leq h' \leq h$. The block

$$\begin{cases} \overline{\Theta}(1,\nu) \cap \{\xi_3 = h'\} \text{ is centred at } h'\cdot\begin{pmatrix} R^{-1}\nu^2/2 \\ R^{-1/2}\nu \end{pmatrix} \\ \Theta(1,\tilde{\nu}) \cap \{\xi_3 = h'\} \text{ is centred at } h'\cdot\begin{pmatrix} R^{-1}\tilde{\nu}^2/2 \\ R^{-1/2}\tilde{\nu} \end{pmatrix} \end{cases}$$

Note that

$$\left| h'\begin{pmatrix} R^{-1}\nu^2/2 \\ R^{-1/2}\nu \end{pmatrix} - h'\begin{pmatrix} R^{-1}\tilde{\nu}^2/2 \\ R^{-1/2}\tilde{\nu} \end{pmatrix} \right| \geq h' R^{-1/2}|\nu - \tilde{\nu}|$$

$$\geq \frac{1}{2}\, h\cdot R^{-1/2}|\nu - \tilde{\nu}|$$

so that, if $|\nu - \tilde{\nu}| \geq 2\cdot C\cdot h^{-1/2}$, then the centres are separated by

$$C\cdot h^{1/2}R^{-1/2}.$$

If C is chosen sufficiently large, then the desired conclusion follows. $\square$

For each h, we can therefore partition the $\Theta \in \mathbb{S}$ as follows:-

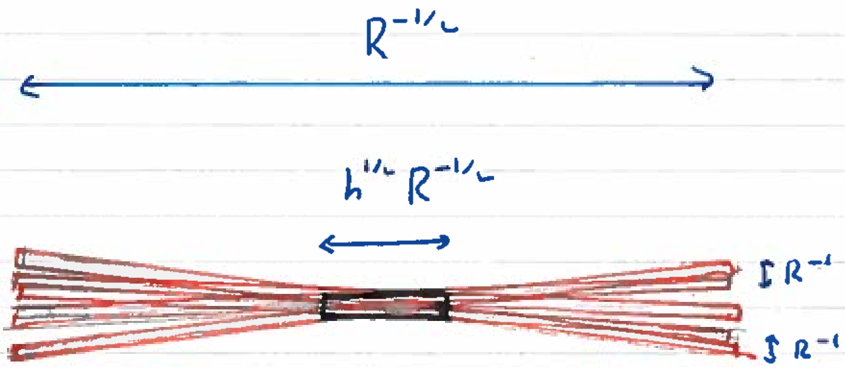
$$\mathbb{S} := \bigcup_{\bar{\Theta} \in \mathbb{CP}_h} \mathbb{S}_{\bar{\Theta}} \qquad (2)$$

where

$$\mathbb{S}_{\bar{\Theta}_h} := \{ \theta(1,\nu) \in \mathbb{S} : \bar{\Theta}_h \subseteq 2 \cdot \bar{\Theta}(1,\nu) \}.$$

It follows that (2) is a 'partition' of $\mathbb{S}$ into finitely-overlapping families of roughly even size:-

$$\# \mathbb{S}_{\Theta} \sim \frac{\# \mathbb{S}}{\# \mathbb{CP}_h} \sim \frac{R^{1/2}}{h^{1/2} R^{1/2}} = h^{-1/2}.$$



The family $\mathbb{S}_{\bar{\Theta}}$ consists of plates $\theta(1,\nu)$ such that the slices

$$\bar{\Theta}(1,\nu) \cap \{ \xi_3 = h \}$$

'essentially all agree' with some fixed $R^{-1} \times h^{1/2} R^{-1/2}$ block

Alternative definition:- Alternatively, if we decompose $N_{1/R}\Gamma$ into a finitely-overlapping family of planks τ with

$$d(\tau) = s, \qquad s = h^{1/2} R^{-1/2}$$

then it follows each S_{Θ} is a set of the form

$$\{ \theta \in S : \theta \leq \tau \} \qquad (3)$$

for some choice of $\tau = \tau_{\Theta}$.

(Note :- the number of such τ is $h^{1/2} R^{1/2}$ which coincides with $\# \mathbb{CP}_h$).

Using (2), we write for any h,

$$|\hat{g}(\xi)| = \Big| \sum_{\theta \in S} (|f_\theta|^2)^\wedge(\xi) \Big|$$

$$\leq \sum_{\bar{\Theta} \in \mathbb{CP}_h} \Big| \sum_{\theta \in S_{\bar{\Theta}}} (|f_\theta|^2)^\wedge(\xi) \Big|$$

Using the alternative definition from (3), this is the same as:

$$|\hat{g}(\xi)| \leq \sum_{d(\tau)=s} \Big| \sum_{\theta \leq \tau} (|f_\theta|^2)^\wedge(\xi) \Big| .$$

Partitioning the frequency support.

Define

$$\Omega := \bigcup_{\bar{\Theta} \in \mathbb{CP}_1} \bar{\Theta}$$

so that, by (1), $\operatorname{supp} \hat{g} \subseteq \Omega$.

We form a partition of Ω as follows:

For $h \in 2^{\mathbb{Z}}$ let

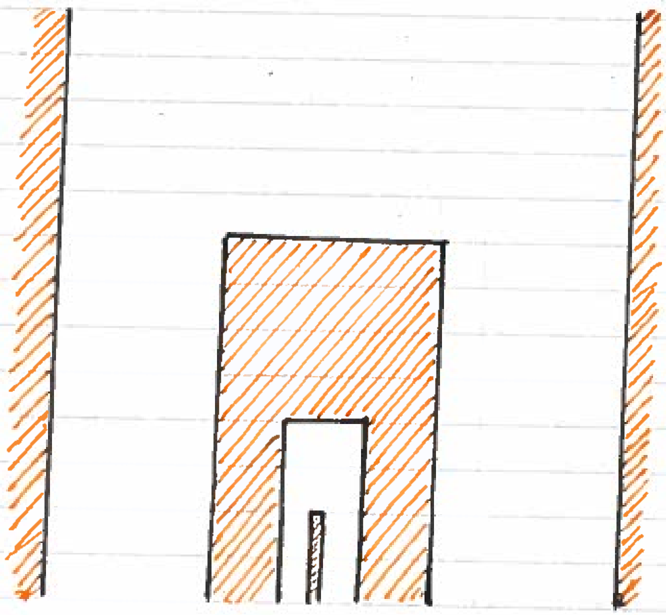
$$\Omega_{\leq h} := \bigcup_{\bar{\Theta} \in \mathbb{CP}_h} \bar{\Theta}$$

and define

$$\Omega_h := \begin{cases} \Omega_{\leq h} \setminus \Omega_{\leq h/2} & \text{if} \quad R^{-1} < h \leq 1 \\ \Omega_{\leq R^{-1}} & \text{if} \quad h = R^{-1} \end{cases}$$

Thus,

$$\Omega = \bigcup_{\substack{h \in 2^{\mathbb{Z}} \\ R^{-1} \leq h \leq 1}} \Omega_h .$$



A schematic for the $\{\Omega_h\}$ decomposition.

Key geometric lemmas:-

Lemma 1: If $\left| \sum_{\theta \in \mathbb{S}_{\bar{\Theta}}} (|f_\theta|^2)^\wedge(\xi) \right| \neq 0$ for some $\xi \in \Omega_{\leq h}$ and $\bar{\Theta} \in \mathbb{CP}_h$, then $\xi \in 4 \cdot \bar{\Theta}$.

Lemma 2: If $\xi \in \Omega_h$, then

$$\# \{ \bar{\Theta} \in \mathbb{CP}_h : \xi \in 4 \cdot \bar{\Theta} \} \lesssim 1 .$$

Assuming these lemmas allows us to write

$$|\hat{g}(\xi)|^2 \lesssim \Big(\sum_{\substack{\bar{\Theta} \in \mathbb{CP}_h \\ \xi \in 4\cdot\bar{\Theta}}} \Big| \sum_{\theta \in S_{\bar{\Theta}}} (|f_\theta|^2)^\wedge(\xi) \Big| \Big)^2$$

$$\lesssim \sum_{\substack{\bar{\Theta} \in \mathbb{CP}_h \\ \xi \in 4\cdot\bar{\Theta}}} \Big| \sum_{\theta \in S_{\bar{\Theta}}} (|f_\theta|^2)^\wedge(\xi) \Big|^2 \qquad \text{if } \xi \in \Omega_h$$

by Cauchy-Schwarz.

For each $\bar{\Theta} \in \mathbb{CP} := \bigcup_{\substack{R^{-1/2} \le h \le 1 \\ h \in 2^{\mathbb{Z}}}} \mathbb{CP}_h$ let $\eta_\Theta \in \mathcal{S}(\mathbb{R}^3)$ be such that $(\eta_\Theta)^\wedge$ is a smooth approximation of $\chi_{4\bar{\Theta}}$. Thus, we can write

$$|\hat{g}(\xi)|^2 \cdot \chi_{\Omega_h}(\xi) \lesssim \sum_{\bar{\Theta} \in \mathbb{CP}_h} \Big| \widehat{\eta_\Theta}(\xi) \sum_{\theta \in S_\Theta} (|f_\theta|^2)^\wedge(\xi) \Big|^2$$

and thus, summing over all $R^{-1} \le h \le 1$,

$$|\hat{g}(\xi)|^2 \lesssim \sum_{\bar{\Theta} \in \mathbb{CP}} \Big| \widehat{\eta_\Theta}(\xi) \cdot \sum_{\theta \in S_\Theta} (|f_\theta|^2)^\wedge(\xi) \Big|^2$$

Integrating in ξ and applying Plancherel,

$$\Big\| \Big(\sum_{d(\theta) = R^{-1/2}} |f_\theta|^2 \Big)^{1/2} \Big\|_{L^4(\mathbb{R}^3)}^4 \qquad (4)$$

$$= \|g\|_{L^2(\mathbb{R}^3)}^2 \lesssim \sum_{\bar{\Theta} \in \mathbb{CP}} \Big\| \eta_\Theta * \sum_{\theta \in S_\Theta} |f_\theta|^2 \Big\|_{L^2(\mathbb{R}^3)}^2$$

To conclude the argument we use locally-constant properties of the right-hand functions, based on the uncertainty principle.

Since η_Θ is Fourier localised to Θ, by the uncertainty principle this function is

L^1 normalized and spatially concentrated in the dual plank Θ^*.

Note $\bar{\Theta}$ has dimensions $R^{-1} \times h^{1/2} R^{-1/2} \times h$ so the dual plank $\bar{\Theta}^*$ has dimensions $R \times h^{-1/2} R^{1/2} \times h^{-1}$.

The function $|\eta_{\Theta} * \sum_{\theta \in S_\Theta} |f_\theta|^2|$ is essentially constant on translates $B \parallel \Theta^*$ and, moreover, roughly satisfies

$$|\eta_\Theta * \sum_{\theta \in S_\Theta} |f_\theta|^2 (\xi)| \sim \fint_B \sum_{\theta \in S_\Theta} |f_\theta|^2 \quad (5).$$

for $\xi \in B$.

Substituting (5) into (4),

$$\|(\sum_{d(\theta) = R^{-1/2}} |f_\theta|^2)^{1/2}\|^4_{L^4(\mathbb{R}^3)}$$

$$\lesssim \sum_{\bar{\Theta} \in \mathbb{CP}} \sum_{B \parallel \bar{\Theta}^*} \|\eta_{\bar{\Theta}} * \sum_{\theta \in S_{\bar{\Theta}}} |f_\theta|^2\|^2_{L^2(B)}$$

$$\sim \sum_{\bar{\Theta} \in \mathbb{CP}} \sum_{B \parallel \bar{\Theta}^*} |B|^{-1} \|(\sum_{\theta \in S_\Theta} |f_\theta|^2)^{1/2}\|^4_{L^2(B)}. \quad (6)$$

Finally, it remains to rewrite this estimate in the notation of the Lemma statement.

- Recall, each S_Θ corresponds to a family of plates satisfying $\theta \subseteq \tau$ for some s-plate τ where $s = h^{-1/2} R^{-1/2}$.
- The B's satisfying $B \parallel \bar{\Theta}^*$ have size $R^{+1} \times h^{1/2} R^{+1/2} \times h^{-1} = R \times Rs \times Rs^2$ for s as above.
- Thus, we can write RHS of (6) as

$$\sum_{R^{-1/2} \leq s \leq 1} \sum_{d(\tau) = s} \sum_{U \parallel U_{\tau,R}} |U|^{-1} \|S_U f\|^4_{L^2(U)},$$

as required.