Lecture 16: Cone square function IV

To conclude our discussion of local smoothing and the cone square function, it suffices to prove :-

Lemma (Kakeya-type bound): Let $R \geq 1$ and suppose $\mathrm{supp}\, \hat{f} \subseteq N_{1/R}\, \Gamma$. Then

$$\Big\| \Big(\sum_{d(\theta) = R^{-1/2}} |f_\theta|^2 \Big)^{1/2} \Big\|_{L^4(\mathbb{R}^3)}^4 \lesssim \qquad (*)$$

$$\sum_{\substack{R^{-1/2} \leq s \leq 1 \\ s \in 2^{\mathbb{Z}}}} \sum_{d(\tau) = s} \sum_{U \| U_{\tau,R}} |U|^{-1} \| S_U f \|_{L^2(U)}^4 .$$

This is the main innovation in the Guth-Wang-Zhang paper. The proof is, nevertheless, elementary and combines three ingredients :-

- Geometry of Γ
- Uncertainty principle
- L^2-orthogonality (Plancherel's theorem)

in a simple, direct manner.

Fourier support conditions.

Begin by writing the left-hand side of $(*)$ as $\|g\|_{L^2(\mathbb{R}^3)}^2$ where $g := \sum_{d(\theta) = R^{-1/2}} |f_\theta|^2$.

Since we are working with an L^2-norm of g it is natural to investigate the Fourier transform $\hat{g}$ and, in particular, support conditions of the constituent functions $(|f_\theta|^2)^\wedge$.

Writing $|f_\theta|^2 = f_\theta \cdot \overline{f_\theta}$ it follows that $(|f_\theta|^2)^\wedge = \hat{f}_\theta * \hat{\bar{f}}_\theta$ and since each f_θ satisfies

$\operatorname{supp} \hat{f}_\theta \subseteq \Theta$, it follows that

$$\operatorname{supp} (|f_\theta|^2)^\wedge \subseteq \Theta - \Theta.$$

Thus, $\hat{g}$ is a superposition of functions $(|f_\theta|^2)^\wedge$ with Fourier supports in the Minkowski differences $\Theta - \Theta$.

In order to understand these set differences, it is useful to work with a more formal setup for our plate decomposition.

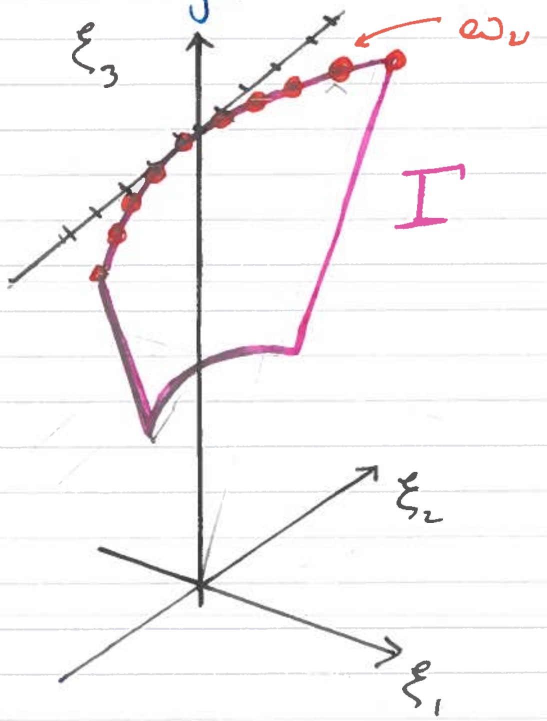
Let $\mathbb{Z}_{R^{1/2}} := \{ \nu \in \mathbb{Z} : |\nu| \leq R^{1/2} \}$ and

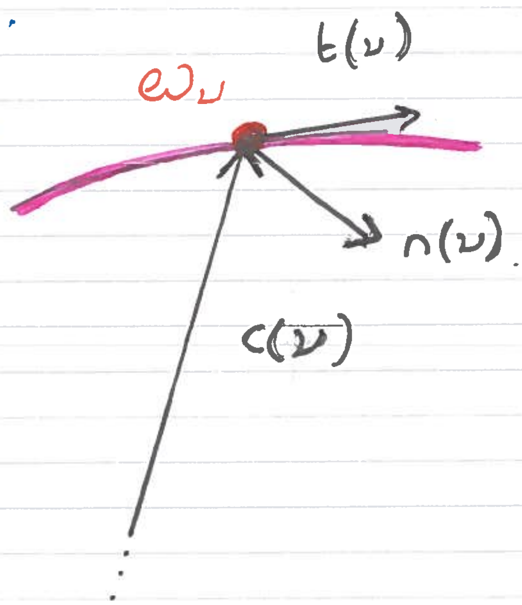
$$\omega_\nu := \begin{pmatrix} R^{-1}\nu^2/2 \\ R^{-1/2}\nu \\ 1 \end{pmatrix} \quad \text{for } \nu \in \mathbb{Z}_R.$$

Thus, the $\{\omega_\nu\}_{\nu \in \mathbb{Z}_{R^{1/2}}}$ are $R^{-1/2}$-separated points along the parabola

$$\left\{ \begin{pmatrix} t^2/2 \\ t \\ 1 \end{pmatrix} : t \in \mathbb{R},\ |t| \leq 1 \right\}$$

which generates Γ.





To define an $R^{-1/2}$-plank through the point ω_ν, introduce the following basis of $\mathbb{R}^3$:

$\{ \vec{c}(\nu), \vec{t}(\nu), \vec{n}(\nu) \}$ where :-

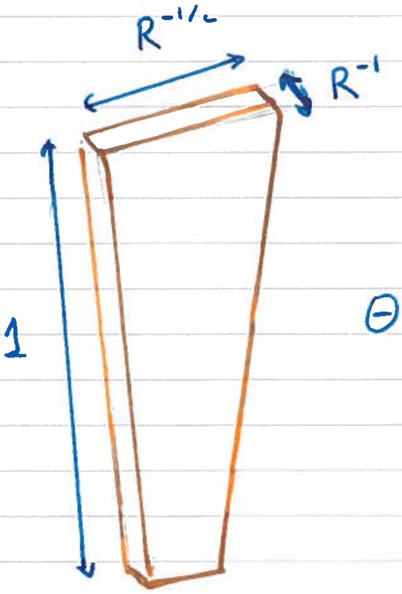
- $\vec{c}(\nu) := \omega_\nu = \begin{pmatrix} R^{-1}\nu^2/2 \\ R^{-1/2}\nu \\ 1 \end{pmatrix}$ is the "direction of the core line" of the plank.

- $\vec{t}(\nu) := \begin{pmatrix} R^{-1/2}\nu \\ 1 \\ 0 \end{pmatrix}$ is the tangent direction of the generating curve at ω_ν.

- $\vec{n}(\nu) := \begin{pmatrix} -1 \\ R^{-1/2}\nu \\ -R^{-1}\nu^2/2 \end{pmatrix}$ is the normal to the cone at ω_ν.

With these definitions, our plank is given by

$$\Theta(\nu) := \{ \xi \in \hat{\mathbb{R}}^3 : \tfrac{1}{2} \le \langle \vec{c}(\nu), \xi \rangle \le 1, \ |\langle \vec{n}(\nu), \xi \rangle| \le R^{-1}$$
$$|\langle \vec{t}(\nu), \xi \rangle| \le R^{-1/2} \}.$$



Θ a plank.

Since each $\Theta(\nu)$ is convex, forming the difference set $\Theta(\nu) - \Theta(\nu)$ doesn't distort the geometry of $\Theta(\nu)$ very much, but clearly $0 \in \Theta(\nu) - \Theta(\nu)$.

Roughly speaking, $\Theta(\nu) - \Theta(\nu)$ can be thought of as a (scaled by an $O(1)$ factor) copy of $\theta(\nu)$ centred at 0. More precisely, from (1) we clearly have

$$\theta(\nu) - \theta(\nu) \subseteq \bar{\Theta}(\nu) := \{\xi \in \mathbb{R}^3 : |\langle \vec{c}(\nu), \xi \rangle| \leq 1/2,$$
$$|\langle \vec{n}(\nu), \xi \rangle| \leq 2R^{-1}, \quad |\langle \vec{t}(\nu), \xi \rangle| \leq 2R^{-1/2}\}$$

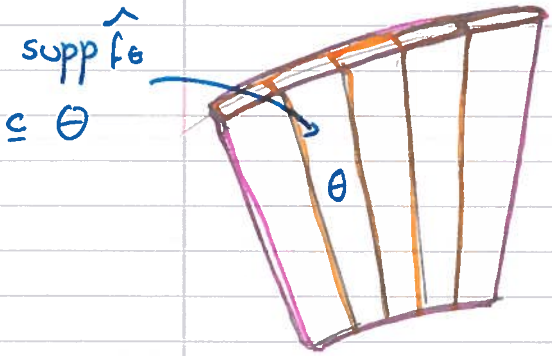
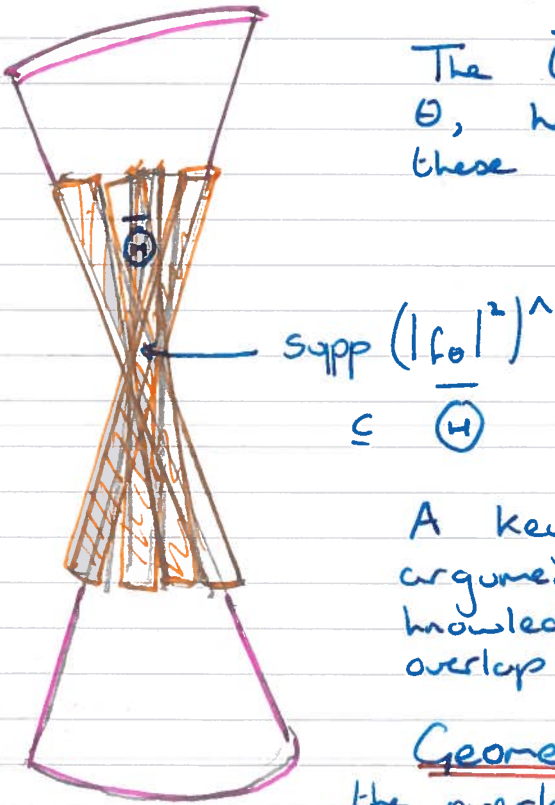


Fig a

Thus, we see on the one hand, the $\hat{f}_\theta$ are supported in (essentially) disjoint plank regions θ which cover Γ... (Fig a)

... but on the other hand, the $(|\hat{f}_\theta|^2)^\wedge$ are supported in translates of the planks θ centred at the origin, and denoted $\bar{\Theta}$.

The $\bar{\Theta}$, in contrast with the θ, heavily overlap. Indeed, all these translated plates overlap at 0.



A key ingredient in the argument is a detailed knowledge of how the $\bar{\Theta}$ overlap one another.

Geometry of Γ :- Analysing the overlap of the $\bar{\Theta}$.

The next step is to analyse the overlap of the $\bar{\Theta}$. This is achieved by forming a series of auxillary plate decompositions at small scales.

Consider the "whole" (one-sided) light cone

$$\Gamma_{whole} := \{ \xi \in \widehat{\mathbb{R}}^3 : \xi_1 = \xi_2^2 / 2\xi_3 , \ |\xi_2/\xi_3| \leq 1, \ \xi_3 > 0 \}.$$

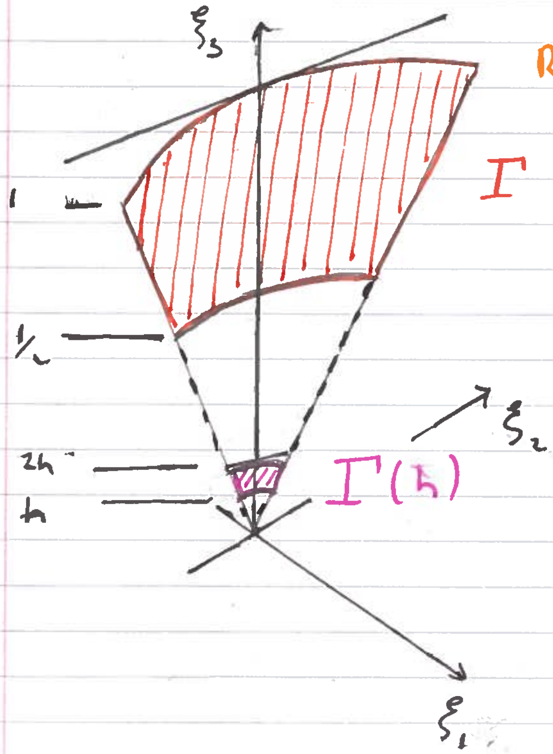
so that our truncated piece Γ is given by

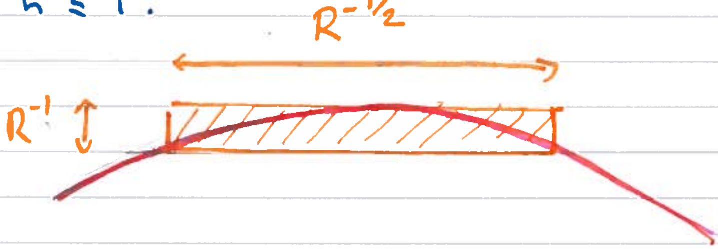
$$\Gamma = \Gamma_{whole} \cap (\mathbb{R}^2 \times [1/2, 1]).$$

More generally, given a dyadic $h \in 2^{\mathbb{Z}}$ define

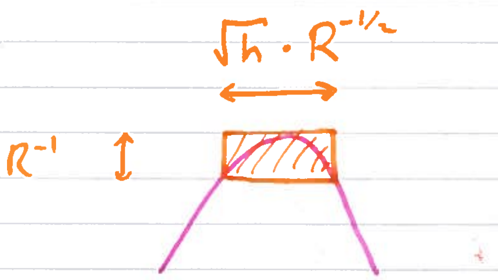
$$\Gamma(h) := \Gamma_{whole} \cap (\mathbb{R}^2 \times [h/2, h]).$$

We will consider plate decompositions for the $\Gamma(h)$ where $R^{-1} \leq h \leq 1$.





Curvature ~ 1



Curvature $\sim 1/h$

The key difference between $\Gamma(h)$ and Γ is that the non-zero principal curvature of $\Gamma(h)$ is much larger.

In particular, consider the "cross-sectional curve"

$$\Gamma(h) \cap \{ \xi_3 = h \}$$

which is given by $\left\{ \begin{pmatrix} t^2/2h \\ t \\ 1 \end{pmatrix} : |t| \leq 1 \right\}$.

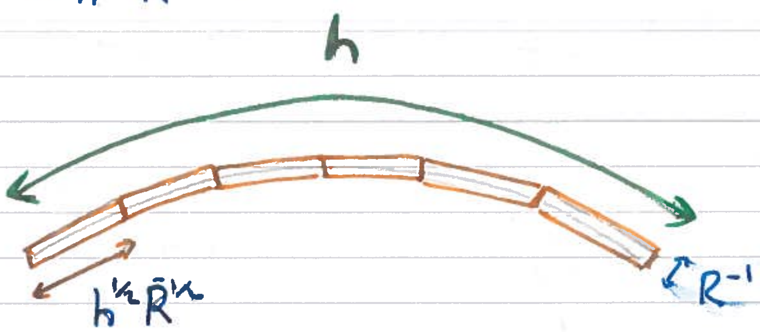
This has curvature $\sim \frac{1}{h}$.

Consequently, if we wish to cover a $1/R$-neighbourhood of $\Gamma(h)$ with plates, then the plates should be chosen to have size $(hR^{-1})^{1/2}$ in the tangential direction to the defining curve (we'll refer to this simply as the "tangential direction" below).

In view of this, let

$$\Theta(h,\nu) := \left\{ \xi \in \widehat{\mathbb{R}}^3 : \frac{h}{2} \leq \langle \vec{c}(\nu), \xi \rangle \leq h,\ |\langle \vec{n}(\nu), \xi \rangle| \leq R^{-1},\ |\langle \vec{t}(\nu), \xi \rangle| \leq h^{1/2} R^{-1/2} \right\}.$$

for $\nu \in \mathcal{Z}_{h^{1/2}R^{1/2}}$.



Note that the planks at scale h are shorter (than those at scale 1): — they have length $h^{1/2}R^{-1/2}$ in the "tangential" direction.

However, the length of the cross-curve at scale h is **much** shorter — the whole curve has length h here.

Thus, at scale h there are **fewer** planks in our decomposition — only $h^{1/2}R^{1/2}$.

(This is, of course, a manifestation of the "quadratic nature" of curvature).