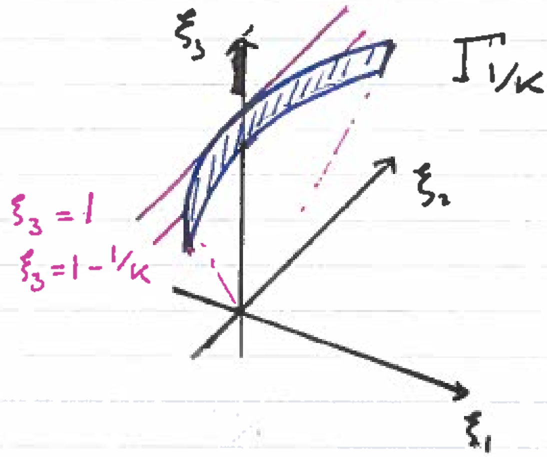
Lecture 15: Cone square function III

Truncated cones:–

Let $K \geq 2$ and consider the truncated cone

$$\Gamma_{1/K} := \{ \xi \in \Gamma : 1 - 1/K \leq \xi_3 \leq 1 \}$$



Key observation: If $K = R$, then $N_{1/R}\Gamma_{1/K} = N_{1/K}\Gamma_{1/K}$ is (essentially) a $1/K$-neighbourhood of the parabola $P' \times \{1\} = \{ \begin{pmatrix} t \\ t^2/2 \\ 1 \end{pmatrix} : |t| \leq 1 \}$.

From Lecture 6 we know how to prove square function bounds for parabolae in the plane; the same argument works in the current setting.

Proposition 1: If $K \geq 1$ and $\operatorname{supp} \hat{f} \subseteq N_{1/K}\Gamma_{1/K}$, then

$$\|f\|_{L^4(\mathbb{R}^3)} \lesssim \Big\| \Big(\sum_{d(\theta) = K^{-1/2}} |f_\theta|^2 \Big)^{1/2} \Big\|_{L^4(\mathbb{R}^3)}.$$

Proof: This is just the Cordoba–Fefferman argument. See Lecture 6.

Motivated by this observation, we will extend the setup introduced in the previous lectures to take into account truncations.

Def^{n}:- Given $K \geq 2$ and $1 \leq r \leq R$ let $S_K(r,R)$ denote the infemum over all $C \geq 1$ for which the inequality

$$\sum_{Q_r \in \mathcal{Q}_r} |Q_r|^{-1} \|S_{Q_r} f\|^4_{L^4(Q_r)} \leq C \cdot \sum_{\substack{s \in 2^{\mathbb{Z}} \\ R^{-1/2} \leq s \leq 1}} \sum_{d(\tau)=s} \sum_{U \| U_{\tau,R}} |U|^{-1} \|S_U f\|^4_{L^4(U)} \quad (*)$$

holds whenever $f \in \mathcal{S}(\mathbb{R}^3)$ with $\operatorname{supp} \hat{f} \subseteq N_{1/R} \Gamma_{1/K}$.

By definition, $S(r,R) = S_2(r,R)$ and $S_K(r,R) \lesssim S(r,R)$ for all $K \geq 2$.

A reverse inequality also holds.

Lemma 1:- $S(r,R) \lesssim K^{O(1)} S_K(r,R)$.

Proof (Idea):- This follows from the observation that the full conic neighbourhood $N_{1/R}\Gamma$ can be covered by the union of $O(K)$ affine images of the neighborhood of the truncated portion $N_{1/R}\Gamma_{1/K}$.

Roughly speaking, one can apply the estimate on each set in the union and then sum together. □

- See Guth-Wang-Zhang §3 for details.

Recall from the previous lecture that our goal was to show

$$\boxed{\begin{array}{l}\textbf{Goal:-}\ \text{For all } \varepsilon > 0, \\ \qquad S(r,R) \lesssim_\varepsilon (R/r)^\varepsilon \\ \text{whenever } 1 \leq r \leq R.\end{array}}$$

In view of Lemma 1:-

New Goal:- For all $\varepsilon > 0$, there exists some $K = K_\varepsilon \geq 2$ such that
$$S_K(r, R) \lesssim_\varepsilon (R/r)^\varepsilon$$
whenever $1 \leq r \leq R$.

Recall from the previous lecture that
$$\sum_{\substack{R^{-1/2} \leq s \leq 1 \\ s \in 2^{\mathbb{Z}}}} \sum_{d(\tau) = s} \sum_{U \parallel U_{\tau,R}} |U|^{-1} \| S_U f \|_{L^2(U)}^4$$
$$\lesssim \log R \cdot \Big\| \Big(\sum_{d(\theta) = R^{-1/2}} |f_\theta|^2 \Big)^{1/2} \Big\|_{L^4(\mathbb{R}^3)}^4 ;$$

indeed, this is a simple consequence of the Cauchy-Schwarz inequality.

The key ingredient in the proof of the new goal is the following lemma, which says that the above estimate is essentially reversible.

Lemma 2 (Kakeya-type bound):- Let $R \geq 1$ and suppose $\operatorname{supp} \hat{f} \subseteq N_{1/R} \Gamma$. Then

$$\Big\| \Big(\sum_{d(\theta) = R^{-1/2}} |f_\theta|^2 \Big)^{1/2} \Big\|_{L^4(\mathbb{R}^3)}^4 \lesssim \qquad (1)$$
$$\sum_{\substack{R^{-1/2} \leq s \leq 1 \\ s \in 2^{\mathbb{Z}}}} \sum_{d(\tau) = s} \sum_{U \parallel U_{\tau,R}} |U|^{-1} \| S_U f \|_{L^2(U)}^4 .$$

We will discuss this key lemma in detail in the upcoming lectures.

It is remarked that the terminology 'Kakeya-type bound' is motivated by the uncertainty principle and the fact that square functions appear on both sides of (1). In particular, heuristically,
$$\sum_{d(\theta) = R^{-1/2}} |f_\theta|^2 \sim \sum_{T \in \mathbb{T}} c_T^2 \chi_T$$

where $(C_T)_{T\in\mathbb{T}}$ are constants and $\mathbb{T}$ is a family of 'plates' (more precisely, each $|f_\theta|^2$ is heuristically given by

$$|f_\theta|^2 \sim \sum_{T \parallel \theta^*} C_T^2 \chi_T$$

where the sum is over a tessellation by planks parallel to θ^*).

In this lecture we will see how Lemma 2 can be combined with an induction-on-scale argument to obtain our new goal.

Consequences of the Kakeya-type bound.

Comparing (1) with the desired inequality (*), we see the key difference is the form of the left-hand sides. In particular, we have

$$\text{LHS } (*) : \qquad \sum_{Q_r \in \mathcal{Q}_r} |Q_r|^{-1} \| S_{Q_r} f \|^4_{L^4(Q_r)}$$

$$\text{LHS } (1) : \qquad \Big\| \Big(\sum_{d(\theta) = R^{-1/2}} |f_\theta|^2 \Big)^{1/2} \Big\|^4_{L^4(\mathbb{R}^3)}.$$

In certain situations we can compare these two expressions and consequently use Lemma 2 to prove bounds for $S_K(r,R)$. There are two different scenarios :-

1. Approximation by parabola

If $R = K$, then Lemma 2 can be combined with Proposition 1 to prove :-

Lemma 3 (Approximation by parabola) :- If $K \geq 2$ and $1 \leq r \leq K$, then for all $\varepsilon > 0$,

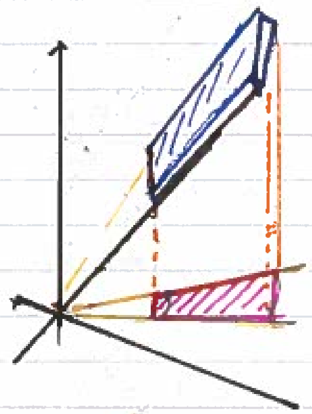
$$S_K(r, K) \lesssim_\varepsilon K^\varepsilon.$$

Proof :- Let $f \in \mathcal{S}(\mathbb{R}^3)$ be such that $\operatorname{supp} \hat{f} \subseteq N_{1/K} \Gamma_{1/K}$.
By Cauchy-Schwarz,

$$\sum_{Q_r \in \mathcal{Q}_r} |Q_r|^{-1} \| S_{Q_r} f \|^4_{L^2(Q_r)} \leq \sum_{Q_r \in \mathcal{Q}_r} \| S_{Q_r} f \|^4_{L^4(Q_r)}$$

$$= \Big\| \Big(\sum_{d(\sigma) = r^{-1/2}} |f_\sigma|^2 \Big)^{1/2} \Big\|^4_{L^4(\mathbb{R}^3)} . \qquad (2)$$

Each f_σ is Fourier supported on an $r^{-1/2} \times r^{-1} \times 1$ plate σ on the light cone. The projection of this plate onto the xy-plane is a (truncated) angular sector of aperture $r^{-1/2}$.



Thus, we can trivially extend the Cordoba square function for sectorial frequency projections to $\mathbb{R}^3$ and apply this to the (f_σ) to deduce

$$\Big\| \Big(\sum_{d(\sigma) = r^{-1/2}} |f_\sigma|^2 \Big)^{1/2} \Big\|^4_{L^4(\mathbb{R}^3)} \lesssim_\varepsilon r^\varepsilon \cdot \| f \|^4_{L^4(\mathbb{R}^3)} \qquad (3)$$

Proposition 1 implies

$$\| f \|^4_{L^4(\mathbb{R}^3)} \lesssim \Big\| \Big(\sum_{d(\theta) = K^{-1/2}} |f_\theta|^2 \Big)^{1/2} \Big\|^4_{L^4(\mathbb{R}^3)} \qquad (4)$$

and, finally, by Lemma 2 we have

$$\Big\| \Big(\sum_{d(\theta) = K^{-1/2}} |f_\theta|^2 \Big)^{1/2} \Big\|^4_{L^4(\mathbb{R}^3)} \qquad (5)$$

$$\lesssim \sum_{\substack{s \in 2^{\mathbb{Z}} \\ K^{-1/2} \leq s \leq 1}} \sum_{d(\tau) = s} \sum_{U \parallel U_{\tau,R}} |U|^{-1} \| S_U f \|^4_{L^2(U)} .$$

Combining (2), (3), (4), (5), we deduce that

$$\sum_{Q_r \in \mathcal{Q}_r} |Q_r|^{-1} \| S_{Q_r} f \|^4_{L^2(Q_r)} \lesssim_\varepsilon K^\varepsilon .$$

$$\sum_{\substack{s \in 2^{\mathbb{Z}} \\ K^{-1/2} \leq s \leq 1}} \sum_{d(\tau)=s} \sum_{U \| U_{\tau,K}} |U|^{-1} \| S_U f \|^4_{L^2(U)} ,$$

for $\operatorname{supp} \hat{f} \subseteq N_{1/K} I_{1/K}$,

but this is precisely the statement

$S_K(r, K) \lesssim_\varepsilon K^\varepsilon$, as desired. □

2. Local orthogonality :-

If $R^{1/2} \leq r \leq R$, then Lemma 2 can be combined with basic L^2-orthogonality to prove :-

Lemma 4 (L^2 local orthogonality) :- If $R^{1/2} \leq r \leq R$, then

$$S(r, R) \lesssim 1 .$$

Proof :- Here we heavily exploit the 'L^2 nature' of (*). Fix f with $\operatorname{supp} \hat{f} \subseteq N_{1/R} I$ and consider

$$\sum_{Q_r \in \mathcal{Q}_r} |Q_r|^{-1} \| S_{Q_r} f \|^4_{L^2(Q_r)} .$$

Clearly, by definition,

$$\| S_{Q_r} f \|^2_{L^2(Q_r)} = \sum_{d(\sigma) = r^{-1/2}} \| f_\sigma \|^2_{L^2(Q_r)}$$

where $f_\sigma = \sum_{\substack{d(\theta) = R^{-1/2} \\ \theta \subseteq \sigma}} f_\theta$.

By L^2 orthogonality we have

$$\|f_\sigma\|^2_{L^2(\mathbb{R}^3)} = \sum_{\substack{d(\theta)=R^{-1/2} \\ \theta\le\sigma}} \|f_\theta\|^2_{L^2(\mathbb{R}^3)} .$$

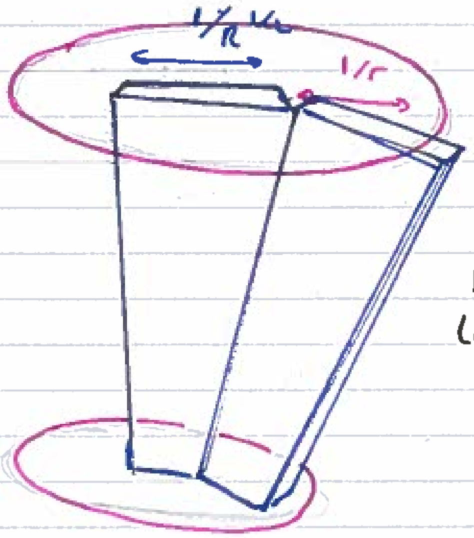
We can essentially 'upgrade' this to a localised version

$$\|f_\sigma\|^2_{L^2(Q_r)} \sim \sum_{\substack{d(\theta)=R^{-1/2} \\ \theta\le\sigma}} \|f_\theta\|^2_{L^2(Q_r)} \qquad (6)$$

provided $r \gtrsim R^{1/2}$. Indeed, the idea here is that spatial localisation to scale r causes frequency uncertainty at scale r^{-1} and, consequently, we can think of the frequency supports θ being fattened up to

$N_{1/r}(\theta)$'s by the spatial localisation.

The key point is that $1/r \lesssim 1/R^{1/2}$, where $1/R^{1/2}$ is the width of the plates, so the $N_{1/r}(\theta)$'s are still essentially disjoint :-



Fattening the plates by $1/r$ preserves disjointedness (essentially).

Thus, we still have L^2 orthogonality.
It is not difficult to make this heuristic precise (the trick is to dominate χ_{Q_r} by a suitable smooth approximant η_{Q_r}). More precisely, rather than (6) one rigorously works with a bound

$$\|f_\sigma\|^2_{L^2(Q_r)} \lesssim \sum_{\substack{d(\theta)=R^{-1/2} \\ \theta\le\sigma}} \|f_\theta\|^2_{L^2(wQ_r)}$$

for a rapidly decaying weight w_{Q_r} concentrated on Q_r. We will omit these technicalities.

Thus, from (6) we see

$$\| S_{Q_r} f \|^2_{L^2(Q_r)} \sim \sum_{d(\sigma)=r^{-1/2}} \sum_{\substack{d(\theta)=R^{-1/2} \\ \theta \subseteq \sigma}} \| f_\theta \|^2_{L^2(Q_r)}$$

$$\sim \Big\| \Big(\sum_{d(\theta)=R^{-1/2}} |f_\theta|^2 \Big)^{1/2} \Big\|^2_{L^2(Q_r)} \qquad (7)$$

Thus,

$$\sum_{Q_r \in \mathcal{Q}_r} |Q_r|^{-1} \| S_{Q_r} f \|^4_{L^2(Q_r)}$$

$$\lesssim \sum_{Q_r \in \mathcal{Q}_r} \Big\| \Big(\sum_{d(\theta)=R^{-1/2}} |f_\theta|^2 \Big)^{1/2} \Big\|^4_{L^4(Q_r)}$$

$$= \Big\| \Big(\sum_{d(\theta)=R^{-1/2}} |f_\theta|^2 \Big)^{1/2} \Big\|^4_{L^4(\mathbb{R}^3)}$$

by (7) and Cauchy - Schwarz. Applying Lemma 2,

$$\sum_{Q_r \in \mathcal{Q}_r} |Q_r|^{-1} \| S_{Q_r} f \|^4_{L^2(Q_r)} \lesssim$$

$$\sum_{\substack{s \in 2^{\mathbb{Z}} \\ R^{-1/2} \le s \le 1}} \sum_{d(\tau)=s} \sum_{U \| \mathcal{U}_{\tau,R}} |U|^{-1} \| S_U f \|^4_{L^2(U)}$$

for $\operatorname{supp} \hat{f} \subseteq N_{1/R}\Gamma$,

which is precisely the statement

$$S(r, R) \lesssim 1 \, , \quad \text{as required.} \qquad \square .$$

Induction-on-scale.

To conclude this lecture, we show how Lemmas 3 and 4 can be combined to prove the desired bound, using an induction-on-scale argument.

We first note that the Lorentz rescaling lemma from the previous lecture extends to the $S_K(r,R)$:-

Lemma 5 (Lorentz rescaling, revisited) If $r_1 \leq r_2 \leq r_3$, then, for all $K \geq 2$,

$$S_K(r_1, r_3) \lesssim \log r_2 \cdot S_K(r_1, r_2) \max_{\substack{r_2^{-1/2} \leq s \leq 1 \\ s \in 2^{\mathbb{Z}}}} S_K(s^2 r_2, s^2 r_3) \tag{8}$$

Proof :- This is clear from the argument used to bound the $S(r,R)$'s - indeed the Lorentz rescalings fix the ξ_3 coordinate where the truncation takes place. □

We'll work with the 'moral' version of (8)

$$S_K(r_1, r_3) \lesssim S_K(r_1, r_2) S_K(r_2, r_3) \tag{8'}$$

which is perhaps conceptually a little simpler. It is very easy to adapt the forthcoming arguments to work with the rigorous inequality (8).

Theorem :- For all $\varepsilon > 0$ there exists some $K = K_\varepsilon \geq 2$ such that

$$S_K(r,R) \lesssim_\varepsilon (R/r)^\varepsilon$$

whenever $1 \leq r \leq R$.

Proof (assuming Lemma 2). We induct on the ratio R/r. It is not difficult to show (c.f. the previous lecture) that :-

$$S(r,R) \lesssim_\varepsilon 1 \qquad \text{for } 1 \leq R/r \lesssim_\varepsilon 1,$$

which serves as a base case.

To state the induction hypothesis, fix $\varepsilon > 0$ and $\bar{C}_\varepsilon \geq 1$, $K = K_\varepsilon \geq 1$ sufficiently large for the forthcoming purposes of the proof.

Induction hypothesis:- If $1 \le r \le R$ satisfy $1 \le R/r \le e/2$, then

$$S_K(r,R) \le \overline{C}_\varepsilon (R/r)^\varepsilon .$$

To prove the inductive step, fix $1 \le r \le R$ satisfying $1 \le R/r \le e$. We consider two cases:-

• Large r case:- $K^{1/2} \le r \le R$.

In this case, we further consider two subcases:-

• $K^{1/2} \le r \le R^{1/2}$ Then, by (8'), we have

$$\begin{aligned} S_K(r,R) &\le S_K(r,r^2) \cdot S_K(r^2,R) \\ &\lesssim C S_K(r^2,R) && \text{by Lemma 4} \\ &\le C \cdot \overline{C}_\varepsilon (R/r^2)^\varepsilon && \text{by ind. hypothesis} \\ &\le (C\, K^{-\varepsilon/2})\, \overline{C}_\varepsilon (R/r)^\varepsilon \end{aligned}$$

since $r \geq K^{1/2}$. Thus,

$$S_K(r,R) \le (C K^{-\varepsilon/2})\, \overline{C}_\varepsilon (R/r)^\varepsilon \le \overline{C}_\varepsilon (R/r)^\varepsilon$$

provided K is sufficiently large depending only on ε. This concludes the proof of the inductive step in this case.

• $R^{1/2} \le r \le R$. In this case we simply apply Lemma 4 to deduce that

$$S_K(r,R) \le C \le \overline{C}_\varepsilon (R/r)^\varepsilon ,$$

provided $\overline{C}_\varepsilon$ is chosen sufficiently large. This concludes the proof of the inductive step in this case.

• Small r case : $1 \leq r \leq K^{1/2}$.

By (8') we have

$$S_K(r,R) \leq S_K(r,K) \cdot S_K(K,R)$$

$$\lesssim C_\varepsilon K^{\varepsilon/4} \; S_K(K,R) \quad \text{by Lemma 3}$$

$$\lesssim C_\varepsilon K^{\varepsilon/4} \; \overline{C}_\varepsilon (R/K)^\varepsilon \quad \text{by the ind. hypothesis.}$$

Here we use $R/K \leq \frac{1}{K^{1/2}} \frac{R}{r} \leq \frac{1}{2} \frac{R}{r}$, so one may invoke the induction hypothesis.

Note that we therefore obtain

$$S_K(r,R) \leq (C_\varepsilon K^{-\varepsilon/4}) \cdot \overline{C}_\varepsilon (R/r)^\varepsilon \leq \overline{C}_\varepsilon (R/r)^\varepsilon$$

provided K is chosen sufficiently large, depending only on ε. This concludes the proof of the inductive step in this case.